1. Introduction. In this paper we consider rational subspaces of the plane. A rational space is a space which has a basis of open sets with countable boundaries. In the special case where the boundaries are finite, the space is called rim-finite.

G. Nöbeling [8] has proved that the family of all rim-finite spaces does not contain a universal element. The same is true even for the family of planar rim-finite spaces. This fact is included in a wider result (see [1] and [4]) concerning some families of planar rim-scattered spaces.

S. Iliadis [3] (see also [7]) proved that there exists a universal rational space. Therefore there exists a rational space which contains topologically all rational compacta.

In [6] J. Mayer and E. Tymchatyn constructed a planar continuum of rim-type $\alpha + 1$ which is a containing space for all planar compacta of rim-type $\leq \alpha$, where $\alpha$ is a countable ordinal.

In this paper we give a simple, direct and visualized example of a planar rational connected and locally connected space which is a containing space for all planar rational compacta. This provides an affirmative answer to Problem 5(2) of [2].

2. Definitions and notations. Let $E^2$ be the plane with a system $Oxy$ of orthogonal coordinates. By a simple closed curve we mean a subset of $E^2$ which is homeomorphic to the set $\{(x, y) : x^2 + y^2 = 1\}$, and by a disk a subset of $E^2$ homeomorphic to $\{(x, y) : x^2 + y^2 \leq 1\}$. An arc is a subset $A$ of $E^2$ for which there exists a homeomorphism $h$ of $I \equiv [0, 1]$ onto $A$. The points $h(0)$ and $h(1)$ are the endpoints of the arc and the set $A \setminus h(\{0, 1\})$ is its interior.

Let $G \subseteq D \subseteq E^2$. By $\text{Cl}_D(G)$, $\text{Int}_D(G)$ and $\text{Bd}_D(G)$ we denote the closure, interior, and boundary of $G$, respectively, in $D$. We omit the subscript “$D$” if $D = E^2$. For each $\varepsilon > 0$ we denote by $N(G, \varepsilon)$ the set of all points of $E^2$ whose distance from $G$ is less than $\varepsilon$. By $\omega$ we denote the set $\{0, 1, 2, \ldots\}$ of all non-negative integers.

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A space $Y$ is called a containing space for a family $\mathcal{F}$ of spaces if for every $X \in \mathcal{F}$, there exists a homeomorphism of $X$ onto a subset of $Y$. If in addition $Y \in \mathcal{F}$, then $Y$ is called a universal space for the family $\mathcal{F}$.

We denote by $L_n$, $n = 1, 2, \ldots$, the set of all ordered $n$-tuples $i_1 \ldots i_n$ where $i_t = 0$ or $1$, for every $t = 1, \ldots, n$, and by $L_0$ the set $\{0\}$. By $I_i$, where $i = i_1 \ldots i_n \in L_n$, $n \geq 1$, we denote the set of all points of $I \equiv [0, 1]$ for which the $k$th digit of the dyadic expansion, $k = 1, \ldots, n$, coincides with $i_k$. Also we set $I_\emptyset = I$.

Let $W_n = \{I_i \times I_j : i, j \in L_n\}$, $n \in \omega$. Obviously for every $n \in \omega$ the family $W_n$ is a finite closed covering of $I^2$. If $a$ is an endpoint of $I_i$ and $b$ is an endpoint of $I_j$, then the sets $\{a\} \times I_j$ and $I_i \times \{b\}$ are called edges and the point $(a, b) \in E^2$ is called a vertex of $W_n$. The sets of all edges and of all vertices of $W_n$ are denoted by $E(W_n)$ and $V(W_n)$, respectively. We set $\text{Bd}(W_n) = \bigcup \{\text{Bd}(F) : F \in W_n\} = \bigcup \{e : e \in E(W_n)\}$.

Let $D$ be a disk of the plane. A finite closed covering $\mathcal{V}$ of $D$ is said to be an $n$-subdivision (or subdivision) of $D$, where $n \in \omega$, if there exists a homeomorphism $h$ of $D$ onto $I^2$ such that $\mathcal{V} = \{h^{-1}(F) : F \in W_n\}$. Every such homeomorphism is called a $\mathcal{V}$-homeomorphism. The sets $h^{-1}(e)$, where $e \in E(W_n)$, are called edges of $\mathcal{V}$ and the points $h^{-1}(v)$, where $v \in V(W_n)$, are called vertices of $\mathcal{V}$. We denote by $E(\mathcal{V})$ and $V(\mathcal{V})$ the sets of all edges and of all vertices of $\mathcal{V}$, respectively. We set $\text{Bd}(\mathcal{V}) = \bigcup \{\text{Bd}(F) : F \in \mathcal{V}\} = \bigcup \{e : e \in E(\mathcal{V})\}$ and $\text{mesh}(\mathcal{V}) = \max\{\text{diam}(F) : F \in \mathcal{V}\}$. Obviously $\text{Bd}(\mathcal{V}) = h^{-1}(\text{Bd}(D))$. Also, for $G \subseteq D$ we set $\text{st}(G, \mathcal{V}) = \bigcup \{F \in \mathcal{V} : F \cap G \neq \emptyset\}$.

We say that a subdivision $\mathcal{V}$ of $D$ is rational with respect to a set $X \subseteq D$ if for every edge $e$ of $\mathcal{V}$ the set $e \cap X$ is a countable subset of the interior of $e$. Note that in this case no point of $X$ is a vertex of $\mathcal{V}$.

Let $n_1, n_2 \in \omega$, $n_1 \leq n_2$. We say that an $n_2$-subdivision $\mathcal{V}_2$ of $D$ is inscribed in an $n_1$-subdivision $\mathcal{V}_1$ of $D$ if: $(\alpha)$ each element of $\mathcal{V}_2$ is contained in some element of $\mathcal{V}_1$ and $(\beta)$ for every $F \in \mathcal{V}_1$ the set of all elements of $\mathcal{V}_2$ which are contained in $F$ is an $(n_2 - n_1)$-subdivision of the disk $F$. We observe that in this case $\text{Bd}(\mathcal{V}_1) \subseteq \text{Bd}(\mathcal{V}_2)$.

3. Containing space. Let $Q_\Delta = \{p/2^n \in I : \{0, 1\} : p, n \in \omega\}$, $Q_T = \{p/3^n \in I : p, n \in \omega\}$

and

$$Y = I^2 \setminus ((I \setminus Q_T) \times Q_\Delta) \cup (Q_\Delta \times (I \setminus Q_T)).$$

We shall prove that $Y$ is a containing space for the family of all planar rational compacta. It is easy to verify that $I^2 \setminus \bigcup \{\text{Bd}(W_n) : n \in \omega\} \subseteq Y$. 
We observe that this remains true if $Q_{\Delta}$ and $Q_T$ are replaced by any pair of disjoint countable dense subsets of $I$.

**4. Lemma.** The space $Y$ is rational, connected and locally connected.

**Proof.** We observe that the set $K \equiv (Q_T \times I) \cup (I \times Q_T)$ is connected and $K \subseteq Y \subseteq I^2 = \text{Cl}(K)$. So $Y$ is connected.

For $y \in Y$ and $i \in \omega$ we set

$$U_i(y) \equiv Y \cap \text{Int}_I(st(y, W_i)).$$

It is easy to verify (as for the space $Y$) that $U_i(y)$ is connected. Moreover, $\text{Bd}_Y(U_i(y))$ is countable. Also it is easy to see that $\{U_i(y) : i \in \omega\}$ is a basis of open neighbourhoods of $y$ in $Y$. Thus $Y$ is a planar rational connected and locally connected space.

**5. Lemma.** Let $D$ be a disk of the plane, $a, b \in \text{Bd}(D)$, $a \neq b$, and $X \subseteq D \setminus \{a, b\}$ be a rational compact space. Then there exists an arc $A \subseteq D$ with endpoints $a, b$ such that $A \cap X$ is countable.

**Proof.** Let $A_1$, $A_2$ be the arcs of $D$ with endpoints $a, b$ such that $A_1 \cup A_2 = \text{Bd}(D)$. It is clear that $X \cap A_1$ and $X \cap A_2$ are closed disjoint subsets of $X \cap D$. Thus there exists a closed countable subset $F$ of $X \cap D$ which separates (in $X \cap D$) the sets $X \cap A_1$ and $X \cap A_2$ (see [5], §51, IV, Th. 9).

Let $G_1$, $G_2$ be disjoint open subsets of $X \cap D$ such that $(X \cap D) \setminus F = G_1 \cup G_2$, $X \cap A_1 \subseteq G_1$ and $X \cap A_2 \subseteq G_2$.

Let $F_1 = \text{Cl}(G_1) \cup A_1$, $F_2 = \text{Cl}(G_2) \cup A_2$, $x \in A_1 \setminus \{a, b\}$ and $y \in A_2 \setminus \{a, b\}$. Since $F_1$ and $F_2$ are compact and $F_1 \cap F_2 \subseteq F \cup \{a, b\}$ is totally disconnected, there exists (see [9], p. 108, Th. (3.1)) a simple closed curve $J$ which separates the points $x$ and $y$ in the plane such that $J \cap (F_1 \cup F_2) \subseteq F \cup \{a, b\}$.

From the above it follows that $J \cap (A_1 \cup A_2) = \{a, b\}$. Since $J$ separates $x$ and $y$, the simple closed curve $J$ intersects the disk $D$ in an arc $A$ with endpoints $a, b$. We have $A \cap X \subseteq (J \cap D) \cap X \subseteq J \cap (F_1 \cup F_2 \cup F) \subseteq F \cup \{a, b\}$.

Hence $A \cap X$ is countable. Thus $A$ is the required arc.

**6. Theorem.** The space $Y$ is a containing space for all planar rational compacta.

**Proof.** Let $X$ be a planar rational compact space and $D$ be a disk of the plane such that $X \subseteq \text{Int}(D)$. We construct a homeomorphism $h : D \rightarrow I^2$ such that $h(X) \subseteq Y$. For every $i \in \omega$ we shall define by induction a natural number $n_i$, an $n_i$-subdivision $Y_i$ of $D$, rational with respect to $X$, and a $Y_i$-homeomorphism $h_i$ such that:

1. $Y_{i+1}$ is inscribed in $Y_i$,
2. $\text{mesh}(Y_{i+1}) < 1/2^{i+1}$,
3. $h_{i+1}|_{\text{bd}(Y_i)} = h_i|_{\text{bd}(Y_i)}$,
Let \( i = 0 \). We set \( n_0 = 0 \) and \( V_0 = \{ D \} \). Let \( h_0 \) be a homeomorphism of \( D \) onto \( I^2 \). Obviously \( h_0 \) is a \( V_0 \)-homeomorphism and (4) is satisfied for \( i = 0 \) because \( \text{Bd}(V_0) \cap X = \emptyset \). The other properties concern the case \( i > 0 \).

Suppose that for every \( i \leq k \) we have defined a natural number \( n_i \), an \( n_i \)-subdivision \( V_i \) of \( D \), rational with respect to \( X \), and a \( V_i \)-homeomorphism \( h_i \) such that (1)–(3) are satisfied if \( i + 1 \leq k \), and (4) is satisfied if \( i \leq k \).

We define a natural number \( n_{k+1} \), an \( n_{k+1} \)-subdivision \( V_{k+1} \) of \( D \), rational with respect to \( X \), and a \( V_{k+1} \)-homeomorphism \( h_{k+1} \) such that (1)–(3) are satisfied if \( i + 1 \leq k + 1 \), and (4) is satisfied if \( i \leq k + 1 \).

There exists an integer \( j \in \omega \) such that \( \text{diam}(h_k^{-1}(F)) < 1/2^{k+1} \) for every \( F \in \mathcal{W}_{n_k+j} \). Since \( I^2 \setminus h_k(X) \) is a dense subset of \( I^2 \), \( V_k \) is rational with respect to \( X \) and since \( V(V_{n_k+j}) \cap Y = \emptyset \) there exists a \( V_k \)-homeomorphism \( h_k' \) such that \( h_k'|_{\text{Bd}(V_k)} = h_k|_{\text{Bd}(V_k)} \), \( h_k'(X) \cap V(V_{n_k+j}) = \emptyset \) and \( \text{diam}(h_k'^{-1}(F)) < 1/2^{k+1} \) for every \( F \in \mathcal{W}_{n_k+j} \). Let \( n_{k+1} = n_k + j \) and

\[
V'_{k+1} = \{(h_k')^{-1}(F) : F \in \mathcal{W}_{n_k+j}\}.
\]

Then \( V'_{k+1} \) is an \( n_{k+1} \)-subdivision of \( D \) with mesh\( (V_{k+1}'_k) < 1/2^{k+1} \), which is inscribed in \( V_k \). However, this subdivision is not, in general, rational with respect to \( X \). The \( n_{k+1} \)-subdivision \( V_{k+1} \) of \( D \) will be obtained by some modification of \( V'_{k+1} \).

For every edge \( e \in E(V'_{k+1} \setminus \text{Bd}(V_k)) \) we denote by \( D_e \) a disk such that:

(a) \( e \subseteq D_e \),
(b) \( D_e \cap \text{Bd}(V_k) \subseteq e \cap \text{Bd}(V_k) \),
(c) \( D_{e_1} \cap D_{e_2} \subseteq e_1 \cap e_2 \) if \( e_1 \neq e_2 \),
(d) \( \forall F \in \mathcal{V}_{k+1} \), \( \text{diam}(F \cup \bigcup \{D_e : e \subseteq F\}) < 1/2^{k+1} \).

For every \( e \in E(V'_{k+1}) \) we define an arc \( \tilde{e} \) as follows:
(a) if \( e \subseteq \text{Bd}(V_k) \), then \( \tilde{e} = e \),
(b) if \( e \not\subseteq \text{Bd}(V_k) \), then \( \tilde{e} \) is the arc \( A \) of Lemma 5, where \( D \) is the disk \( D_e \) and \( \tilde{e} \) is the union of arcs \( \tilde{e} \), where \( e \subseteq \text{Bd}(D) \).

Let \( \tilde{F} \) be the disk having as boundary the simple closed curve \( J_F \). We set

\[
V_{k+1} = \{\tilde{F} : F \in \mathcal{V}_{k+1}'\}.
\]

For every \( e \in E(V'_{k+1}) \) we define a homeomorphism \( h_{k+1}^e \) of \( \tilde{e} \) into \( \text{Bd}(\mathcal{W}_{n_{k+1}}) \) as follows:
(a) if \( \tilde{e} = e \subseteq \text{Bd}(V_k) \), then \( h_{k+1}^e = h_k|_e \),
(b) if \( e \not\subseteq \text{Bd}(V_k) \), then \( h_{k+1}^e \) is a homeomorphism of \( \tilde{e} \) onto \( h_k(e) \) such that

\[
\begin{align*}
\text{h}_{k+1}^e|_{(a,b)} &= h_k^e|_{(a,b)}, \\
\text{h}_{k+1}^e(\tilde{e} \cap X) &= Y \cap h_k(e),
\end{align*}
\]

where \( a, b \) are the endpoints of \( \tilde{e} \), and \( h_{k+1}^e \) is countable, and \( h_k^e(e) \cap Y \) is countable and dense in \( h_k(e) \). For every
\( \tilde{F} \in \mathcal{V}_{k+1} \) we denote by \( h_{k+1}^\tilde{F} \) a homeomorphism of \( \tilde{F} \) onto \( h'_k(F) \) such that \( h_{k+1}^\tilde{F}|_\tilde{F} = h_{k+1}^\tilde{F} \) for every \( \tilde{F} \subset \tilde{F} \). Let \( h_{k+1} \) be a homeomorphism of \( D \) onto \( I^2 \) for which \( h_{k+1}|_\tilde{F} = h_{k+1}^\tilde{F} \) for every \( \tilde{F} \in \mathcal{V}_{k+1} \).

It is easy to verify that \( \mathcal{V}_{k+1} \) is an \( n_{k+1} \)-subdivision of \( D \), rational with respect to \( X \), and \( h_{k+1} \) is a \( \mathcal{V}_{k+1} \)-homeomorphism with properties (1)–(4).

Furthermore, for every \( x \in X \) and \( i, j \in \omega \), \( j \geq i \), by the definition of \( \mathcal{V}_i \) we have

\[
(5) \quad h_i(st(x, \mathcal{V}_i)) = st(h_i(x), \mathcal{W}_{n_i}),
\]

by (1) it follows that

\[
(6) \quad st(x, \mathcal{V}_j) \subseteq st(x, \mathcal{V}_i),
\]

and by (3) we have

\[
(7) \quad h_j(st(x, \mathcal{V}_i)) = h_i(st(x, \mathcal{V}_i)).
\]

Now we define a map \( h : D \to I^2 \) setting for every \( x \in D \),

\[
h(x) = \bigcap \{h_i(st(x, \mathcal{V}_i)) : i \in \omega \}
\]

and prove that \( h \) is a homeomorphism such that \( h(X) \subseteq Y \).

First note that by (6) and (7), \( h_{i+1}(st(x, \mathcal{V}_{i+1})) \subseteq h_i(st(x, \mathcal{V}_i)) \) for every \( x \in D \) and \( i \in \omega \). On the other hand, by (5), \( \lim_{i \to \infty} \text{diam}(h_i(st(x, \mathcal{V}_i))) = 0 \). Hence \( \bigcap h_i(st(x, \mathcal{V}_i)) \) is a singleton. Thus \( h \) is well defined.

Let \( x_1, x_2 \in D \) and \( x_1 \neq x_2 \). By (2) there exists \( i \in \omega \) such that \( st(x_1, \mathcal{V}_i) \cap st(x_2, \mathcal{V}_i) = \emptyset \). Hence \( h_i(st(x_1, \mathcal{V}_i)) \cap h_i(st(x_2, \mathcal{V}_i)) = \emptyset \) and therefore \( h(x_1) \neq h(x_2) \), that is, \( h \) is one-to-one.

We prove that \( h \) is continuous. Let \( h(x) = y \) and \( U \) be an open neighbourhood of \( y \) in \( I^2 \). There exists \( i \in \omega \) such that \( st(y, \mathcal{W}_{n_i}) \subseteq U \). By (5), \( h_i(st(x, \mathcal{V}_i)) \subseteq U \). For the continuity of \( h \) it is sufficient to prove that \( h(\text{Int}(st(x, \mathcal{V}_i))) \subseteq U \). Let \( z \in \text{Int}(st(x, \mathcal{V}_i)) \). It is easy to see that \( st(z, \mathcal{V}_i) \subseteq st(x, \mathcal{V}_i) \). Hence \( h(z) \in h_i(st(z, \mathcal{V}_i)) \subseteq h_i(st(x, \mathcal{V}_i)) \subseteq U \). Thus \( h \) is continuous and therefore \( h \) is a homeomorphism.

To prove that \( h(X) \subseteq Y \), we observe that if \( x \in \text{Bd}(\mathcal{V}_i) \), then \( h_j(x) = h_i(x) \in h_j(st(x, \mathcal{V}_j)) \) for every \( j \geq i \). Thus \( h(x) = h_i(x) \). Since \( h_i(\text{Bd}(\mathcal{V}_i)) = \text{Bd}(\mathcal{W}_{n_i}) \) we see that if \( x \notin \bigcup \{\text{Bd}(\mathcal{V}_i) : i \in \omega \} \), then \( h(x) \notin \bigcup \{\text{Bd}(\mathcal{W}_{n_i}) : i \in \omega \} \).

Let \( x \in X \). If \( x \notin \bigcup \text{Bd}(\mathcal{V}_i) \), then \( h(x) \notin \bigcup \text{Bd}(\mathcal{W}_j) \). Since \( I^2 \setminus \bigcup \text{Bd}(\mathcal{W}_j) \subseteq Y \), we have \( h(x) \in Y \). If \( x \in \bigcup \text{Bd}(\mathcal{V}_i) \), then \( h(x) = h_i(x) \) for some \( i \in \omega \). By (4) it follows that \( h(x) \in Y \). Thus \( h(X) \subseteq Y \).
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