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ON MÜNTZ RATIONAL APPROXIMATION IN MULTIVARIABLES

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The present paper shows that for any s sequences of real numbers, each with infinitely many distinct elements, $\{\lambda_n^j\}$, $j = 1, \ldots, s$, the rational combinations of $x_1^{\lambda_{m_1}^1} x_2^{\lambda_{m_2}^2} \ldots x_s^{\lambda_{m_s}^s}$ are always dense in C_{I^s}.

1. Introduction. Let $C_{[0,1]}$ be the class of all real continuous functions in $[0,1]$. For $f \in C_{[0,1]}$,

$$\omega(f,t) = \max_{0<h<t,x\in[0,1-h]} |f(x+h) - f(x)|,$$
$$\|f\| = \max_{x\in[0,1]} |f(x)|.$$

Given a subspace S of $C_{[0,1]}$, let

$$R(S) = \{P(x)/Q(x) : P(x) \in S,\ Q(x) \in S,\ Q(x) > 0,\ x \in (0,1]\},$$

where we assume that $\lim_{x\to 0+} P(x)/Q(x) = P(0)/Q(0)$ is finite in the case $Q(0) = 0$. For a sequence of real numbers $\Lambda = \{\lambda_n\}_{n=0}^{\infty}$, write

$$R(\Lambda) = R(\mathrm{span}\{x^{\lambda_n}\}).$$

From Müntz's theorem (cf. [2]), it is well-known that the combinations of x^{λ_n} for

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots \tag{1}$$

are dense in $C_{[0,1]}$ if and only if

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty.$$

As to the rational case, in 1976, Somorjai [6] showed a beautiful result that under (1), $R(\Lambda)$ is always dense in $C_{[0,1]}$. In 1978, Bak and Newman [1] proved that if λ_n is a sequence of distinct positive numbers, then $R(\Lambda)$ is dense in $C_{[0,1]}$ as well. Recently, our work [7] showed that the above result

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also holds for any sequence of real numbers with infinitely many distinct elements.

On the other hand, S. Ogawa and K. Kitahara [5] gave a generalization of Müntz's theorem to multivariable cases. They proved (1) that for two given positive monotone sequences $\{\alpha_i\}$, $\{\beta_j\}$, the set $\{1\}\cup\{x^{\alpha_i}\}\cup\{y^{\beta_j}\}$ is complete in C_{I^2} if and only if $\sum_{i=1}^{\infty} 1/\alpha_i$ and $\sum_{j=1}^{\infty} 1/\beta_j$ diverge, where

$$I^s = \{X = (x_1, \ldots, x_s) : 0 \le x_j \le 1,\ 1 \le j \le s\},$$

and C_{I^s} is the class of all continuous functions on I^s.

For many reasons, it is quite reasonable to conjecture that the conclusion corresponding to that of [7] will hold for Müntz rational approximation in the multivariable case, that is, for any s sequences of real numbers $\{\lambda_n^j\}$, $j = 1, \ldots, s$, each with infinitely many distinct elements, the rational combinations of $\{x_1^{\lambda^1_{m_1}} x_2^{\lambda^2_{m_2}} \ldots x_s^{\lambda^s_{m_s}}\}$ are always dense in C_{I^s}. Since rational combinations are not linear, it is not a trivial work.

The present paper will prove that this is true.

2. Result and proof

Theorem. *Let $\Lambda^j = \{\lambda_n^j\}$, $j = 1, \ldots, s$, be s sequences of real numbers, each with infinitely many distinct elements. Then $R(\Lambda^1 \times \ldots \times \Lambda^s)$ is dense in C_{I^s}.*

We need the following lemmas from the univariable case, the first two of which are due to Somorjai [6] and the author [7]. We will, however, give the sketch of proofs here for the sake of completeness.

Lemma 1 (Somorjai [6]). *Let $\{\lambda_n\}$ be a sequence of real numbers such that $\lambda_n \to +\infty$ as $n \to \infty$. Given $N \ge 1$, for any $f \in C_{[0,1]}$, there are an integer n_N and an operator*

$$\sum_{k=0}^{N} f\left(\frac{k}{N}\right)\frac{Z_k(x)}{Z(x)} =: r_N^1(f, x) \in R(\{\lambda_j\}_{j=0}^{n_N})$$

with

$$0 \le Z_k(x) \in \operatorname{span}\{x^{\lambda_j}\}_{j=0}^{n_N}, \qquad Z(x) = \sum_{k=0}^{N} Z_k(x)$$

such that

$$\|f - r_N^1(f)\| = O(\omega(f, N^{-1})).$$

Proof. We select a sequence $\{\lambda_{n_j}\}_{j=1}^{n_N}$ from Λ by induction. Let λ_{n_0} be any element from Λ, and $Z_0(x) = x^{\lambda_{n_0}}$. Choose $\lambda_{n_{j+1}}$ with the following properties:

(1) For convenience, we only state their result for two variables.

$$Z_{j+1}(x) = \left(\frac{N}{j+1}x\right)^{\lambda_{n_{j+1}}} \le N^{-1} Z_j(x) \quad \text{for } x < \frac{j}{N},$$

$$Z_{j+1}(x) > N Z_j(x) \quad \text{for } x > \frac{j+2}{N}.$$

Define

$$r_N^1(f,x) = \sum_{k=0}^{N} f\left(\frac{k}{N}\right) \frac{Z_k(x)}{\sum_{v=0}^{N} Z_v(x)}$$

for $f \in C_{[0,1]}$. Then by calculation

$$f(x) - r_N^1(f,x) = O(\omega(f, N^{-1})). \ ■$$

LEMMA 2 (Zhou [7]). *Let $\{\lambda_n\}$ be a sequence of real numbers such that $\lambda_n \to -\infty$ as $n \to \infty$. Given $N \ge 1$, for any $f \in C_{[0,1]}$, there are an integer n_N and an operator*

$$\sum_{k=0}^{N} f\left(\frac{k}{N}\right) \frac{C_k(x)}{C(x)} =: r_N^2(f,x) \in R(\{\lambda_j\}_{j=0}^{n_N})$$

with

$$0 \le C_k(x) \in \operatorname{span}\{x^{\lambda_j}\}_{j=0}^{n_N}, \qquad C(x) = \sum_{k=0}^{N} C_k(x)$$

such that

$$\|f - r_N^2(f)\| = O(\omega(f, N^{-1})).$$

Proof. Similar to Lemma 1, let λ_{n_1} be any element from Λ, and $C_1^*(x) = x^{\lambda_{n_1}}$. Choose $\lambda_{n_{j+1}}$ satisfying

$$C_{j+1}^*(x) = \left(\frac{N}{N-j}x\right)^{\lambda_{n_{j+1}}} \ge N C_j^*(x) \quad \text{for } x < \frac{N-j-1}{N},$$

$$C_{j+1}^*(x) < N^{-1} C_j^*(x) \quad \text{for } x > \frac{N-j+2}{N}.$$

For $f \in C_{[0,1]}$, define

$$r_N^2(f,x) = \sum_{k=1}^{N} f\left(\frac{N-k+1}{N}\right) \frac{C_k^*(x)}{\sum_{v=1}^{N} C_v^*(x)}.$$

Then the required result follows. ■

LEMMA 3. *Let $\{\lambda_n\}$ be a sequence of real numbers with infinitely many distinct elements such that $\lambda_n \to l$ as $n \to \infty$ with $-\infty < l < \infty$. Given $N \ge 1$ and $\varepsilon > 0$, there are an integer n_N and an operator*

$$\sum_{k=0}^{N} f(e^{1-N/k}) \frac{D_k(x)}{D(x)} =: r_N^3(f,x) \in R(\{\lambda_j\}_{j=0}^{n_N})$$

with $D_k(x), D(x) \in \operatorname{span}\{x^{\lambda_j}\}_{j=0}^{n_N}$ *such that*

$$\|f - r_N^3(f)\| \le 2\omega(g, N^{-1/2}) + \|f\|\varepsilon,$$

where $g(u) = f(e^{1-1/u})$. *Precisely, we have*

$$\frac{D_k(x)}{D(x)} = G_k(x) + H_k(x), \tag{2}$$

$$G_k(x) = \binom{N}{k}\left(\frac{-1}{\ln(x/e)}\right)^k \left(1 + \frac{1}{\ln(x/e)}\right)^{N-k}, \tag{3}$$

and

$$|H_k(x)| \le \frac{\varepsilon}{N+1}. \tag{4}$$

Proof. There are two possibilities: (i) There is a subsequence $\{\lambda_{n_k}\}$ of $\{\lambda_n\}$ which strictly increases to $\lambda < +\infty$ (in symbols $\lambda_{n_k} \nearrow \lambda < +\infty$) as $k \to \infty$; (ii) there is a subsequence $\{\lambda_{n_k}\}$ which strictly decreases to $\lambda > -\infty$ (in symbols $\lambda_{n_k} \searrow \lambda > -\infty$) as $k \to \infty$. We will prove Lemma 3 in these two cases separately.

Case (i). For convenience, we still write λ_{n_k} as λ_n. So under the hypothesis, $\lambda_n \nearrow \lambda < +\infty$ as $n \to \infty$. Let $\alpha_0 < \alpha_1 < \ldots$, and let $P_k(x)$ denote the kth divided difference of $(x/e)^\alpha$ at $\alpha = \alpha_{2N-1}, \alpha_{2N-2}, \ldots, \alpha_{2N-k-1}$ for $k = 0, 1, \ldots, N-1$, that is,

$$P_0(x) = P_0(x, \alpha_{2N-1}) = (x/e)^{\alpha_{2N-1}},$$

$$P_1(x) = P_1(x, \alpha_{2N-1}, \alpha_{2N-2}) = \frac{(x/e)^{\alpha_{2N-1}} - (x/e)^{\alpha_{2N-2}}}{\alpha_{2N-1} - \alpha_{2N-2}},$$

in general,

$$\begin{aligned} P_k(x) &= P_k(x, \alpha_{2N-1}, \ldots, \alpha_{2N-k-1}) \\ &= \frac{P_{k-1}(x, \alpha_{2N-1}, \ldots, \alpha_{2N-k}) - P_{k-1}(x, \alpha_{2N-2}, \ldots, \alpha_{2N-k-1})}{\alpha_{2N-1} - \alpha_{2N-k-1}}, \\ &\qquad 0 \le k \le N-1, \end{aligned}$$

and

$$P_N(x) = P_N(x, \alpha_N, \ldots, \alpha_0).$$

By the mean value theorem

$$P_k(x) = \frac{(x/e)^{\eta_k} \ln^k(x/e)}{k!}, \quad \alpha_{2N-k-1} \le \eta_k \le \alpha_{2N-1},\ k = 0, 1, \ldots, N-1, \tag{5}$$

$$P_N(x) = \frac{(x/e)^{\eta_N} \ln^N(x/e)}{N!}, \qquad \alpha_0 \le \eta_N \le \alpha_N. \tag{6}$$

Now let $f \in C_{[0,1]}$. Then $g(u) = f(e^{1-1/u}) \in C_{[0,1]}$. Write

$$\begin{aligned} B_N(f,x) &= \sum_{k=0}^{N} f\left(\frac{k}{N}\right)\binom{N}{k} x^k (1-x)^{N-k} \\ &= \sum_{k=0}^{N} f\left(\frac{k}{N}\right)\binom{N}{k} \sum_{j=0}^{N-k} (-1)^j \binom{N-k}{j} x^{j+k} \\ &= \sum_{k=0}^{N} f\left(\frac{k}{N}\right)\binom{N}{k} \sum_{j=k}^{N} (-1)^{j-k} \binom{N-k}{j-k} x^j. \end{aligned}$$

For given $N \geq 1$, the well-known Bernstein theorem implies that

$$\|g(u) - B_N(g,u)\| < \tfrac{3}{2}\omega(g, N^{-1/2}),$$

that is,

$$\|f(x) - B_N(g, -1/\ln(x/e))\| < \tfrac{3}{2}\omega(g, N^{-1/2}). \tag{7}$$

Choose sufficiently large m such that for $k \geq m$,

$$0 < \lambda - \lambda_k < \varepsilon/(4^N(N+1)).$$

Set $\alpha_k = \lambda_{m+k}$, $k = 0, 1, \ldots, 2N-1$. Define

$$r_N^3(f,x) = \sum_{k=0}^{N} f(e^{1-N/k})\binom{N}{k} \frac{\sum_{j=k}^{N}(-1)^{2j-k}(N-j)!\binom{N-k}{j-k}P_{N-j}(x)}{N!P_N(x)}.$$

Then $r_N^3(f,x)$ is a rational combination of $\{x^{\lambda_j}\}_{j=m}^{m+2N-1}$, and by (5), (6),

$$r_N^3(f,x) = \sum_{k=0}^{N} f(e^{1-N/k})\binom{N}{k} \sum_{j=k}^{N} (-1)^{j-k}\binom{N-k}{j-k}\left(\frac{-1}{\ln(x/e)}\right)^j (x/e)^{\eta_j^*}$$

with $\eta_0^* = 0$, $0 < \eta_j^* \leq \lambda_{m+N} - \lambda_m \leq \lambda - \lambda_m$, $j = 1, \ldots, N$. Now write

$$\begin{aligned} &\sum_{j=k}^{N} (-1)^{j-k}\binom{N-k}{j-k}\left(\frac{-1}{\ln(x/e)}\right)^j (x/e)^{\eta_j^*} \\ &\qquad = \sum_{j=k}^{N} (-1)^{j-k}\binom{N-k}{j-k}\left(\frac{-1}{\ln(x/e)}\right)^j \\ &\qquad\quad + \sum_{j=k}^{N} (-1)^{j-k}\binom{N-k}{j-k}\left(\frac{-1}{\ln(x/e)}\right)^j ((x/e)^{\eta_j^*} - 1) \\ &\qquad = \left(\frac{-1}{\ln(x/e)}\right)^k \left(1 + \frac{1}{\ln(x/e)}\right)^{N-k} + \Sigma_1. \end{aligned}$$

Since for $\eta > 0$,

$$\left\| \frac{1-(x/e)^{\eta}}{\ln(x/e)} \right\| \le \eta,$$

we have

$$\left\| \frac{1-(x/e)^{\eta_k^*}}{\ln^k(x/e)} \right\| \le \eta_k^* \le \frac{\varepsilon}{4^N(N+1)}$$

for $k \ge 1$. Consequently,

$$|\Sigma_1| \le 4^{-N}(N+1)^{-1}\varepsilon \sum_{j=k}^{N} \binom{N-k}{j-k} \le \frac{\varepsilon}{2^N(N+1)},$$

and thus (2)–(4) are proved. Now from (4), (7), together with $H_k(x) = \binom{N}{k}\Sigma_1$,

$$\begin{aligned}\|f(x)-r_N^3(f,x)\| &\le \|f(x)-B_N(g,-1/\ln(x/e))\| + \|f\|\sum_{k=0}^{N}|H_k(x)| \\ &\le \tfrac{3}{2}\omega(g,N^{-1/2}) + \|f\|\varepsilon,\end{aligned}$$

that is, Lemma 3 holds true in Case (i).

Case (ii). We may assume that $\lambda_n \searrow \lambda > -\infty$ as $n \to \infty$. Take

$$\begin{aligned}P_k(x) &= P_k(x, \lambda_m, \ldots, \lambda_{m+k}), \quad 0 \le k \le N-1, \\ P_N(x) &= P_N(x, \lambda_{m+N-1}, \ldots, \lambda_{m+2N-1}),\end{aligned}$$

and

$$r_N^3(f,x) = \sum_{k=0}^{N} f(e^{1-N/k}) \binom{N}{k} \frac{\sum_{j=k}^{N}(-1)^{2j-k}(N-j)!\binom{N-k}{j-k}P_{N-j}(x)}{N!P_N(x)}.$$

Similar to Case (i), for given $\varepsilon > 0$ and $N \ge 1$, we can prove (2)–(4) and for sufficiently large m,

$$\|f(x)-r_N^3(f,x)\| \le \tfrac{3}{2}\omega(g,N^{-1/2}) + \|f\|\varepsilon.$$

The proof of Lemma 3 is now complete. ∎

Proof of the Theorem. Given a sequence with infinitely many distinct elements $\{\lambda_n\}$, there are three possibilities: (i) $\{\lambda_n\}$ has at least one finite cluster point; (ii) one cluster point of $\{\lambda_n\}$ is $+\infty$; (iii) one cluster point of $\{\lambda_n\}$ is $-\infty$. Without loss of generality, we may assume

$$\begin{aligned}\lambda_n^j &\to +\infty \quad \text{as } n\to\infty,\ 1 \le j \le r, \\ \lambda_n^j &\to -\infty \quad \text{as } n\to\infty,\ r+1 \le j \le t,\end{aligned}$$

and

$$\lambda_n^j \to l \quad \text{as } n\to\infty,\ t+1 \le j \le s,\ -\infty < l < +\infty.$$

For given $N \geq 1$ and $\varepsilon > 0$, from Lemmas 1–3, we may select a common n_N such that

$$\|f(X) - r_N^1(f, x_j)\| = O(\omega_{x_j}(f, N^{-1})), \qquad 1 \leq j \leq r,$$
$$\|f(X) - r_N^2(f, x_j)\| = O(\omega_{x_j}(f, N^{-1})), \qquad r+1 \leq j \leq t,$$

and

$$\|f(X) - B_N(g, -1/\ln(x_j/e))\| = O(\omega_{x_j}(g, N^{-1/2})), \qquad t+1 \leq j \leq s,$$

hold at the same time, where

$$\omega_{x_j}(f, \delta) = \max_{0 \leq h \leq \delta} |f(x_1, \dots, x_j + h, x_{j+1}, \dots, x_s) - f(x_1, \dots, x_j, x_{j+1}, \dots, x_s)|.$$

Define

$$r_N(X) = \sum_{0 \leq j_1 \leq N} \cdots \sum_{0 \leq j_s \leq N} f\left(\frac{j_1}{N}, \dots, \frac{j_t}{N}, \dots, e^{1-N/j_{t+1}}, \dots, e^{1-N/j_s}\right) \times \frac{Z_{j_1}(x_1)}{Z(x_1)} \cdots \frac{Z_{j_r}(x_r)}{Z(x_r)} \frac{C_{j_{r+1}}(x_{r+1})}{C(x_{r+1})} \cdots \frac{C_{j_t}(x_t)}{C(x_t)} \times \frac{D_{j_{t+1}}(x_{t+1})}{D(x_{t+1})} \cdots \frac{D_{j_s}(x_s)}{D(x_s)}.$$

Evidently,

$$r_N(X) \in R(\mathrm{span}\{x^{\lambda_i^1}\}_{i=0}^{n_N} \times \mathrm{span}\{x^{\lambda_i^2}\}_{i=0}^{n_N} \times \dots \times \mathrm{span}\{x^{\lambda_i^s}\}_{i=0}^{n_N}).$$

From (2),

$$f(X) - r_N(X) = \sum_{0 \leq j_1 \leq N} \cdots \sum_{0 \leq j_s \leq N} \left(f(X) - f\left(\frac{j_1}{N}, \dots, \frac{j_t}{N}, e^{1-N/j_{t+1}}, \dots, e^{1-N/j_s}\right)\right) \times \frac{Z_{j_1}(x_1)}{Z(x_1)} \cdots \frac{Z_{j_r}(x_r)}{Z(x_r)} \frac{C_{j_{r+1}}(x_{r+1})}{C(x_{r+1})} \cdots \frac{C_{j_t}(x_t)}{C(x_t)} \times G_{j_{t+1}}(x_{t+1}) \dots G_{j_s}(x_s) + \Sigma_3 := \Sigma_2 + \Sigma_3,$$

where by Lemma 3,

$$|\Sigma_3| \leq 4^s \|f\| \varepsilon. \tag{8}$$

Because $f(X) \in C_{I^s}$, there is a $\delta > 0$ such that for $|X - Y| = \sqrt{\sum_{j=1}^s (x_j - y_j)^2} < \delta$,

$$|f(X) - f(Y)| < \varepsilon,$$

while for $|X-Y|\geq\delta$,

$$|f(X)-f(Y)|\leq 2\delta^{-2}\|f\|\sum_{j=1}^{s}(x_j-y_j)^2,$$

therefore in any case

$$|f(X)-f(Y)|\leq \varepsilon+2\delta^{-2}\|f\|\sum_{j=1}^{s}(x_j-y_j)^2.$$

Now

$$\begin{aligned}|\Sigma_2|&\leq \varepsilon+2\delta^{-2}\|f\|\bigg(\sum_{0\leq j_1\leq N}\dots\sum_{0\leq j_s\leq N}\sum_{i=1}^{t}\left(x_i-\frac{j_i}{N}\right)^2\\&\quad\times\sum_{i=t+1}^{s}(x_i-e^{1-N/j_i})^2\frac{Z_{j_1}(x_1)}{Z(x_1)}\dots\frac{Z_{j_r}(x_r)}{Z(x_r)}\\&\quad\times\frac{C_{j_{r+1}}(x_{r+1})}{C(x_{r+1})}\dots\frac{C_{j_t}(x_t)}{C(x_t)}G_{j_{t+1}}(x_{t+1})\dots G_{j_s}(x_s)\bigg)\\&=\varepsilon+2\delta^{-2}\|f\|\bigg(\sum_{i=1}^{r}\sum_{j_i=0}^{N}\left(x_i-\frac{j_i}{N}\right)^2\frac{Z_{j_i}(x_i)}{Z(x_i)}\\&\quad\times\sum_{i=r+1}^{t}\sum_{j_i=0}^{N}\left(x_i-\frac{j_i}{N}\right)^2\frac{C_{j_i}(x_i)}{C(x_i)}\sum_{i=t+1}^{s}\sum_{j_i=0}^{N}(x_i-e^{1-N/j_i})^2G_{j_i}(x_i)\bigg).\end{aligned}$$

Noting that

$$\sum_{k=0}^{N}\left(x-\frac{k}{N}\right)^2\frac{Z_k(x)}{Z(x)}=2x\bigg(x-\sum_{k=0}^{N}\frac{k}{N}\frac{Z_k(x)}{Z(x)}\bigg)-\bigg(x^2-\sum_{k=0}^{N}\left(\frac{k}{N}\right)^2\frac{Z_k(x)}{Z(x)}\bigg),$$

from Lemma 1 we deduce that

$$\sum_{k=0}^{N}\left(x-\frac{k}{N}\right)^2\frac{Z_k(x)}{Z(x)}=O(N^{-1/2}).$$

The same results also hold for $\sum_{k=0}^{N}(x-k/N)^2C_k(x)/C(x)$ and for $\sum_{k=0}^{N}(x-e^{1-N/k})^2G_k(x)$ by applying Lemmas 2 and 3. Altogether we have

$$|\Sigma_2|\leq\varepsilon+2\delta^{-2}\|f\|O(N^{-1/2}),\tag{9}$$

thus combining (8) and (9) we get

$$|f(X)-r_N(X)|=O(\varepsilon)+O(N^{-1/2}),$$

which is the required result. ∎

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