WEAK UNIFORM NORMAL STRUCTURE AND ITERATIVE
FIXED POINTS OF NONEXPANSIVE MAPPINGS

BY

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0. Introduction. This paper is concerned with weak uniform normal structure and iterative fixed points of nonexpansive mappings. Precisely, in Section 1, we show that the geometrical coefficient $\beta(X)$ for a Banach space $X$ recently introduced by Jimenez-Melado [8] is exactly the weakly convergent sequence coefficient $WCS(X)$ introduced by Bynum [1] in 1980. We then show in Section 2 that all kinds of James’ quasi-reflexive spaces have weak uniform normal structure. Finally, in Section 3, we show that in a space $X$ with weak uniform normal structure, every nonexpansive self-mapping defined on a weakly sequentially compact convex subset of $X$ admits an iterative fixed point.

1. Weak uniform normal structure. Let $X$ be a Banach space which is not Schur (i.e., the weak and strong convergence for sequences in $X$ do not coincide). Then Bynum [1] defined the weakly convergent sequence coefficient of $X$ as the number

$$WCS(X) := \inf \left\{ \frac{A(\{x_n\})}{\inf \{\limsup_{n \to \infty} \|x_n - y\| : y \in \text{co}(\{x_n\})\} } \right\},$$

where the first infimum is taken over all weakly (not strongly) convergent sequences $\{x_n\}$ in $X$, $\text{co}(A)$ denotes the closure of the convex hull of the subset $A \subset X$, and $A(\{x_n\})$ is the asymptotic diameter of $\{x_n\}$, i.e., the number

$$\lim_{n \to \infty} \left( \sup \{\|x_i - x_j\| : i, j \geq n\} \right).$$

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It is easy to see that $1 \leq \text{WCS}(X) \leq 2$ and that if $\text{WCS}(X) > 1$, then $X$ has weak normal structure. This means that any weakly compact convex subset $C$ of $X$ consisting of more than one point has a nondiametral point, i.e., an $x \in C$ such that
\[
\sup\{\|x - y\| : y \in C\} < \text{diam}(C).
\]
Following [2], we shall say that a Banach space $X$ has \textit{weak uniform normal structure} provided $\text{WCS}(X) > 1$.

Recently Jimenez-Melado [8] introduced the generalized Gossez–Lami Dozo property (GGLD in short) for a Banach space $X$ as follows: $X$ is said to have GGLD provided
\[
D[(x_n)] > 1
\]
whenever $\{x_n\}$ is a weakly null sequence such that $\lim_{n \to \infty} \|x_n\| = 1$, where
\[
D[(x_n)] = \limsup_n \|x_n - x_m\|.
\]
He also defined the coefficient $\beta(X)$ of $X$ by
\[
\beta(X) = \inf \{D[(x_n)] : x_n \to 0 \text{ and } \|x_n\| \to 1\}.
\]
(Here “$\to$” denotes weak convergence.) We shall show that the GGLD is exactly the weakenization of the weak uniform normal structure and that the coefficient $\beta(X)$ is equal to Bynum’s coefficient $\text{WCS}(X)$.

**Theorem 1.** Suppose $X$ is a Banach space which is not Schur. Then

(i) $X$ has the GGLD property if and only if
\[
\limsup_n \|x_n - x_\infty\| < A(\{x_n\})
\]
whenever $\{x_n\}$ is a weakly (not strongly) convergent sequence in $X$ with limit $x_\infty$.

(ii) $\beta(X) = \text{WCS}(X)$.

**Proof.** (i) Assume the space $X$ has the GGLD and let $\{x_n\}$ be a weakly (not strongly) convergent sequence in $X$ with limit $x_\infty$. Define $a = \limsup_n \|x_n - x_\infty\|$. Choose a subsequence $\{x_n'\}$ of $\{x_n\}$ such that $a = \lim_{n' \to \infty} \|x_n' - x_\infty\|$ and set $z_n' = (x_n' - x_\infty)/a$. Then $z_n' \to 0$ and $\|z_n'\| \to 1$.

It follows from the GGLD that
\[
D[(z_n')] > 1.
\]
Hence $a < \limsup_{m'}(\limsup_{n'} \|x_n' - x_{m'}\|) \leq A(\{x_n\})$.

Next we show the converse implication. Given any weakly null sequence $\{x_n\}$ with $\|x_n\| \to 1$, using a diagonal method as in [3] and [4], we get a subsequence $\{x_n'\}$ of $\{x_n\}$ such that
\[
\lim_{n', m' \to \infty, n' \neq m'} \|x_n' - x_{m'}\|
\]
exists. It follows that
\[1 = \lim_{n'} \|x_{n'}\| < A([x_{n'}]) = \lim_{n', m', n' \neq m'} \|x_{n'} - x_{m'}\| = D([x_{n'}]) \leq D([x_n]),\]
which shows that \(X\) has the GGLD. This proves (i).

(ii) By Lemma 1.1 of [4], we have
\[(1) \quad \text{WCS}(X) = \inf\{ \lim_{n, m, n \neq m} \|x_n - x_m\| : x_n \rightharpoonup 0, \|x_n\| \to 1, \text{ and } \lim_{n, m, n \neq m} \|x_n - x_m\| \text{ exists}\}.\]
So given any \(\varepsilon > 0\), we have a sequence \(\{x_n\}\) in \(X\) with the properties:
\(x_n \rightharpoonup 0, \|x_n\| \to 1\) and \(\lim_{n, m, n \neq m} \|x_n - x_m\| \text{ exists}\), such that
\[\lim_{n, m, n \neq m} \|x_n - x_m\| < \text{WCS}(X) + \varepsilon.\]
It follows that \(\beta(X) \leq \text{WCS}(X)\).

Hence \(\beta(X) \leq \text{WCS}(X)\), since \(\varepsilon\) is arbitrary.

Finally, we show that \(\beta(X) \geq \text{WCS}(X)\). Given any sequence \(\{x_n\}\) in \(X\) such that \(x_n \rightharpoonup 0\) and \(\|x_n\| \to 1\), select a subsequence \(\{x_{n'}\}\) of \(\{x_n\}\) such that
\[\lim_{n', m', n' \neq m'} \|x_{n'} - x_{m'}\| \text{ exists}.\]
Then by (1), we obtain
\[D([x_n]) \geq D([x_{n'}]) = \lim_{n', m', n' \neq m'} \|x_{n'} - x_{m'}\| \geq \text{WCS}(X).\]
It follows that \(\beta(X) \geq \text{WCS}(X)\). This completes the proof.

**Corollary 1.** If \(X\) is a Banach space with the property that for any weakly (not strongly) convergent sequence \(\{x_n\}\) in \(X\),
\[\limsup_n \|x_n - x_\infty\| < A(\{x_n\}),\]
where \(x_\infty\) is the weak limit of \(\{x_n\}\), then \(X\) has weak normal structure.

2. **James’ quasi-reflexive space.** In this section we show that three kinds of James’ quasi-reflexive spaces have weak uniform normal structure. Recall that James’ quasi-reflexive space \(J\) consists of all null sequences \(x = \{x_n\} = \sum_{n=1}^{\infty} x_n e_n\) (\(\{e_n\}\) is the standard basis of \(c_0\)) for which the squared variation
\[(2) \quad \sup_m \left[ \sum_{j=2}^{m} (x_{p_{j-1}} - x_{p_j})^2 \right]^{1/2}
\]
is finite. According to different purposes, there are three kinds of equivalent norms on $J$. We denote by $\|x\|_1$ the norm of $x$ given by (2). The other two norms $\|\cdot\|_2$ and $\|\cdot\|_3$ are defined by

$$\|x\|_2 = \sup_{p_1 < \ldots < p_p} \left[ \sum_{j=1}^{m} (x_{p_{j-1}} - x_{p_j})^2 \right]^{1/2}$$

and

$$\|x\|_3 = \sup_{p_1 < \ldots < p_p} \left[ \sum_{j=2}^{m} (x_{p_{j-1}} - x_{p_j})^2 + (x_{p_m} - x_{p_1})^2 \right]^{1/2}.$$

**Theorem 2.** Each of James’ spaces $(J, \| \cdot \|_j)$ $(j = 1, 2, 3)$ has weak uniform normal structure; moreover, $\text{WCS}(J, \| \cdot \|_j) = \sqrt{2}$ for $j = 1, 2$ and $\text{WCS}(J, \| \cdot \|_3) = (3/2)^{1/2}$.

**Proof.** First consider the cases $j = 1, 2$. For simplicity, we write $X$ for $(J, \| \cdot \|_j)$. It is then easily seen that if $u, v$ are two points in $X$ such that $\max \text{supp}(u) < \min \text{supp}(v)$, then

$$\|u + v\|_j \geq (\|u\|_j^2 + \|v\|_j^2)^{1/2}.$$ 

Here we write $\text{supp}(u)$ to denote the set $\{n : u_n \neq 0\}$ for $u = \{u_n\}$ in $J$. Now suppose $\{x_n\}$ is a sequence in $X$ such that $x_n \to 0$ weakly and $\|x_n\|_j \to 1$. Then by a routine argument as used in [8], there exists a subsequence $\{z_n\}$ of $\{x_n\}$ and a sequence $\{u_n\}$ in $X$ of successive blocks such that

$$\lim_{n \to \infty} \|z_n - u_n\|_j = 0.$$ 

By (3) we get

$$\|u_n - u_m\|_j \geq (\|u_n\|_j^2 + \|u_m\|_j^2)^{1/2}.$$ 

It follows from (4) that

$$D([x_n]) \geq D([z_n]) = D((u_n)) = \lim_{m} \sup_{n} \|u_n - u_m\|_j \geq \sqrt{2}.$$ 

Hence $\text{WCS}(X) = \beta(X) \geq \sqrt{2}$. On the other hand, consider the sequence $x_n = -e_n + e_{n+1}$. Then $x_n \to 0$ weakly in $X$, $D([x_n]) = \sqrt{2}$ for $j = 1, 2$ and hence $\text{WCS}(X) = \beta(X) \leq \sqrt{2}$. We thus conclude that $\text{WCS}(X) = \sqrt{2}$. The case $j = 3$ can be proved by combining Theorem 1(ii) and a result of [8]. The proof is complete.

**Remark 1.** It is remarkable that the nonreflexive James space $(J, \| \cdot \|_j)$ for $j = 1, 2$ has the same WCS value as a Hilbert space.

**Corollary 2.** Each of James’ spaces $(J, \| \cdot \|_j)$ for $j = 1, 2, 3$ has the fixed point property for nonexpansive mappings (FPP).

**Remark 2.** Khamsi [9] first proved that James’ space $(J, \| \cdot \|_3)$ has the FPP by using Maurey’s ultrapower technique which is, of course, non-
constructive. We shall see in the next section that in this James space, each nonexpansive mapping does admit an iterative fixed point.

3. Iterative fixed points of nonexpansive mappings. Kirk’s classical theorem [10] states that if $C$ is a weakly compact convex subset of a Banach space $X$ with normal structure, then every nonexpansive mapping $T : C \to C$ has a fixed point. Although there are proofs to this theorem which do not explicitly involve Zorn’s lemma (cf. [6]), it remains an open question whether there is an iterative fixed point for a nonexpansive mapping $T$ in such a space $X$, i.e., whether one can construct a sequence which converges strongly to a fixed point of $T$. In this section we show that this is true if the space $X$ has weak uniform normal structure.

Theorem 3. Suppose $X$ is a Banach space with weak uniform normal structure, $C$ is a weakly sequentially compact convex subset of $X$, and $T : C \to C$ is a nonexpansive mapping (i.e., $\|Tx - Ty\| \leq \|x - y\|$, $x, y \in C$). Then $T$ has an iterative fixed point.

Proof. Since one can easily construct (cf. [12]) a closed convex separable subset of $C$ that is invariant under $T$, we may assume $C$ itself is separable. Moreover, by considering the closed separable subspace $\text{span}(C)$ of $X$, we may further assume that the space $X$ itself is separable (in that case, the Hahn–Banach Theorem can be proved in a constructive way). Now fix a $\lambda \in (0, 1)$ and set

$$S = \lambda I + (1 - \lambda)T.$$  

(Here $I$ is the identity operator of $X$.) Then $S : C \to C$ is also nonexpansive with the same fixed point set of $T$. Moreover, $S$ is asymptotically regular on $C$ (see [7] and [5]), i.e.,

$$\lim_{n \to \infty} \|S^n x - S^{n+1} x\| = 0, \quad x \in C.$$  

The separability and weak sequential compactness of $C$ make it possible to choose a subsequence $\{n_k\}$ of positive integers such that for every $x \in C$,

$$\{S^{n_k} x\}$$  

converges weakly. Now we define a sequence $\{x_n\}_{n=1}^\infty$ in $C$ as follows:

$$x_0 \in C \text{ arbitrary}, \quad x_{m+1} = w-lim_{k \to \infty} S^{n_k} x_m, \quad m \geq 0.$$  

Note that since $S$ is asymptotically regular on $C$, it follows that $x_{m+1} = w-lim_{k \to \infty} S^{n_k + j} x_m$ for all integers $j \geq 0$. Write

$$R_m = \sup_{k \to \infty} \|S^{n_k} x_m - x_{m+1}\|.$$  

Then from Theorem 1, it follows that

$$R_m \leq \text{WCS}(X)^{-1} D[(S^{n_k} x_m)].$$
By the nonexpansiveness and asymptotic regularity of $S$ and the weak lower semicontinuity of the norm of $X$, we get for all $k > i$,

$$\|S^{n_k}x_m - S^{n_i}x_m\| \leq \liminf_{j \to \infty} \|S^{n_k-n_i}x_m - S^{n_j+(n_k-n_i)x_m-1}\|$$

$$\leq \liminf_{j \to \infty} \|x_m - S^{n_j}x_m-1\| \leq R_{m-1}.$$ 

It follows that $R_m \leq AR_{m-1} \leq \ldots \leq A^m R_0$, where $A = \text{WCS}(X)^{-1} < 1$.

Since

$$\|x_m - x_{m+1}\| \leq \limsup_{k \to \infty}(\|x_m - S^{n_k}x_m\| + \|x_{m+1} - S^{n_k}x_m\|)$$

$$\leq \limsup_{k \to \infty}(\liminf_{j \to \infty} \|S^{n_j+n_k}x_m-1 - S^{n_k}x_m\|) + R_m$$

$$\leq \liminf_{j \to \infty} \|S^{n_j}x_m-1 - x_m\| + R_m$$

$$\leq R_{m-1} + R_m \leq (A^{m-1} + A^m)R_0,$$

we see that $\{x_m\}$ is Cauchy. Let $x_\infty = \lim_{m \to \infty} x_m$. Then

$$\|S^{n_k}x_\infty - x_\infty\| \leq \|S^{n_k}x_\infty - S^{n_k}x_m\| + \|S^{n_k}x_m - x_m\| + \|x_m - x_\infty\|$$

$$\leq 2\|x_m - x_\infty\| + \liminf_{j \to \infty} \|S^{n_k}x_m - S^{n_j+n_k}x_m-1\|$$

$$\leq 2\|x_m - x_\infty\| + \liminf_{j \to \infty} \|x_m - S^{n_j}x_m-1\|$$

$$\leq 2\|x_m - x_\infty\| + R_{m-1} \to 0 \quad \text{as} \quad m \to \infty.$$ 

Therefore, $S^{n_k}x_\infty = x_\infty$ for all $k$ and the asymptotic regularity of $S$ shows that $x_\infty$ is a fixed point of $S$ and hence of $T$. This completes the proof.

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