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WEAK UNIFORM NORMAL STRUCTURE AND ITERATIVE FIXED POINTS OF NONEXPANSIVE MAPPINGS

BY

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0. Introduction. This paper is concerned with weak uniform normal structure and iterative fixed points of nonexpansive mappings. Precisely, in Section 1, we show that the geometrical coefficient $\beta(X)$ for a Banach space X recently introduced by Jimenez-Melado [8] is exactly the weakly convergent sequence coefficient $\mathrm{WCS}(X)$ introduced by Bynum [1] in 1980. We then show in Section 2 that all kinds of James' quasi-reflexive spaces have weak uniform normal structure. Finally, in Section 3, we show that in a space X with weak uniform normal structure, every nonexpansive self-mapping defined on a weakly sequentially compact convex subset of X admits an iterative fixed point.

1. Weak uniform normal structure. Let X be a Banach space which is not Schur (i.e., the weak and strong convergence for sequences in X do not coincide). Then Bynum [1] defined the *weakly convergent sequence coefficient* of X as the number

$$\mathrm{WCS}(X) := \inf\left\{\frac{A(\{x_n\})}{\inf\{\limsup_{n\to\infty}\|x_n - y\| : y \in \overline{\mathrm{co}}\{x_n\}\}}\right\},$$

where the first infimum is taken over all weakly (not strongly) convergent sequences $\{x_n\}$ in X, $\overline{\mathrm{co}}(A)$ denotes the closure of the convex hull of the subset $A \subset X$, and $A(\{x_n\})$ is the *asymptotic diameter* of $\{x_n\}$, i.e., the number

$$\lim_{n\to\infty}(\sup\{\|x_i - x_j\| : i, j \ge n\}).$$

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It is easy to see that $1 \le \mathrm{WCS}(X) \le 2$ and that if $\mathrm{WCS}(X) > 1$, then X has weak normal structure. This means that any weakly compact convex subset C of X consisting of more than one point has a nondiametral point, i.e., an $x \in C$ such that

$$\sup\{\|x-y\| : y \in C\} < \mathrm{diam}(C).$$

Following [2], we shall say that a Banach space X has *weak uniform normal structure* provided $\mathrm{WCS}(X) > 1$.

Recently Jimenez-Melado [8] introduced the generalized Gossez–Lami Dozo property (GGLD in short) for a Banach space X as follows: X is said to have GGLD provided $D[(x_n)] > 1$ whenever $\{x_n\}$ is a weakly null sequence such that $\lim_{n\to\infty}\|x_n\| = 1$, where

$$D[(x_n)] = \limsup_m (\limsup_n \|x_n - x_m\|).$$

He also defined the coefficient $\beta(X)$ of X by

$$\beta(X) = \inf\{D[(x_n)] : x_n \rightharpoonup 0 \text{ and } \|x_n\| \to 1\}.$$

(Here "$\rightharpoonup$" denotes weak convergence.) We shall show that the GGLD is exactly the weakenization of the weak uniform normal structure and that the coefficient $\beta(X)$ is equal to Bynum's coefficient WCS(X).

THEOREM 1. *Suppose X is a Banach space which is not Schur. Then*

(i) *X has the GGLD property if and only if*

$$\limsup_n \|x_n - x_\infty\| < A(\{x_n\})$$

whenever $\{x_n\}$ is a weakly (not strongly) convergent sequence in X with limit x_∞.

(ii) $\beta(X) = \mathrm{WCS}(X)$.

Proof. (i) Assume the space X has the GGLD and let $\{x_n\}$ be a weakly (not strongly) convergent sequence in X with limit x_∞. Define $a = \limsup_n \|x_n - x_\infty\|$. Choose a subsequence $\{x_{n'}\}$ of $\{x_n\}$ such that $a = \lim_{n'} \|x_{n'} - x_\infty\|$ and set $z_{n'} = (x_{n'} - x_\infty)/a$. Then $z_{n'} \rightharpoonup 0$ and $\|z_{n'}\| \to 1$. It follows from the GGLD that

$$D[(z_{n'})] > 1.$$

Hence $a < \limsup_{m'}(\limsup_{n'} \|x_{n'} - x_{m'}\|) \le A(\{x_n\})$.

Next we show the converse implication. Given any weakly null sequence $\{x_n\}$ with $\|x_n\| \to 1$, using a diagonal method as in [3] and [4], we get a subsequence $\{x_{n'}\}$ of $\{x_n\}$ such that

$$\lim_{n',m';n'\neq m'} \|x_{n'} - x_{m'}\|$$

exists. It follows that

$$1 = \lim_{n'} \|x_{n'}\| < A(\{x_{n'}\}) = \lim_{n',m';n'\neq m'} \|x_{n'} - x_{m'}\| = D[(x_{n'})] \leq D[(x_n)],$$

which shows that X has the GGLD. This proves (i).

(ii) By Lemma 1.1 of [4], we have

$$\text{(1)} \quad \mathrm{WCS}(X) = \inf\{\lim_{n,m;n\neq m} \|x_n - x_m\| : x_n \rightharpoonup 0,\ \|x_n\| \to 1, \text{ and } \lim_{n,m;n\neq m} \|x_n - x_m\| \text{ exists}\}.$$

So given any $\varepsilon > 0$, we have a sequence $\{x_n\}$ in X with the properties: $x_n \rightharpoonup 0$, $\|x_n\| \to 1$ and $\lim_{n,m;n\neq m} \|x_n - x_m\|$ exists, such that

$$\lim_{n,m;n\neq m} \|x_n - x_m\| < \mathrm{WCS}(X) + \varepsilon.$$

It follows that

$$\beta(X) \leq D[(x_n)] \leq \lim_{n,m;n\neq m} \|x_n - x_m\| < \mathrm{WCS}(X) + \varepsilon.$$

Hence $\beta(X) \leq \mathrm{WCS}(X)$, since ε is arbitrary.

Finally, we show that $\beta(X) \geq \mathrm{WCS}(X)$. Given any sequence $\{x_n\}$ in X such that $x_n \rightharpoonup 0$ and $\|x_n\| \to 1$, select a subsequence $\{x_{n'}\}$ of $\{x_n\}$ such that

$$\lim_{n',m';n'\neq m'} \|x_{n'} - x_{m'}\|$$

exists. Then by (1), we obtain

$$D[(x_n)] \geq D[(x_{n'})] = \lim_{n',m';n'\neq m'} \|x_{n'} - x_{m'}\| \geq \mathrm{WCS}(X).$$

It follows that $\beta(X) \geq \mathrm{WCS}(X)$. This completes the proof.

Corollary 1. *If X is a Banach space with the property that for any weakly (not strongly) convergent sequence $\{x_n\}$ in X,*

$$\limsup_n \|x_n - x_\infty\| < A(\{x_n\}),$$

where x_∞ is the weak limit of $\{x_n\}$, then X has weak normal structure.

2. James' quasi-reflexive space. In this section we show that three kinds of James' quasi-reflexive spaces have weak uniform normal structure. Recall that James' quasi-reflexive space J consists of all null sequences $x = \{x_n\} = \sum_{n=1}^{\infty} x_n e_n$ ($\{e_n\}$ is the standard basis of c_0) for which the squared variation

$$\text{(2)} \qquad \sup_{\substack{m \\ p_1 < \ldots < p_m}} \Big[\sum_{j=2}^{m} (x_{p_{j-1}} - x_{p_j})^2\Big]^{1/2}$$

is finite. According to different purposes, there are three kinds of equivalent norms on J. We denote by $\|x\|_1$ the norm of x given by (2). The other two norms $\|\cdot\|_2$ and $\|\cdot\|_3$ are defined by

$$\|x\|_2 = \sup_{\substack{m \\ p_1<\ldots<p_{2m}}} \Big[\sum_{j=1}^{m}(x_{p_{2j-1}} - x_{p_{2j}})^2\Big]^{1/2}$$

and

$$\|x\|_3 = \sup_{\substack{m \\ p_1<\ldots<p_m}} \Big[\sum_{j=2}^{m}(x_{p_{j-1}} - x_{p_j})^2 + (x_{p_m} - x_{p_1})^2\Big]^{1/2}.$$

THEOREM 2. *Each of James' spaces* $(J, \|\cdot\|_j)$ $(j = 1, 2, 3)$ *has weak uniform normal structure*; *moreover*, $\mathrm{WCS}(J, \|\cdot\|_j) = \sqrt{2}$ *for* $j = 1, 2$ *and* $\mathrm{WCS}(J, \|\cdot\|_3) = (3/2)^{1/2}$.

Proof. First consider the cases $j = 1, 2$. For simplicity, we write X for $(J, \|\cdot\|_j)$. It is then easily seen that if u, v are two points in X such that $\max \mathrm{supp}(u) < \min \mathrm{supp}(v)$, then

$$\|u + v\|_j \ge (\|u\|_j^2 + \|v\|_j^2)^{1/2}. \tag{3}$$

Here we write $\mathrm{supp}(u)$ to denote the set $\{n : u_n \neq 0\}$ for $u = \{u_n\}$ in J. Now suppose $\{x_n\}$ is a sequence in X such that $x_n \to 0$ weakly and $\|x_n\|_j \to 1$. Then by a routine argument as used in [8], there exists a subsequence $\{z_n\}$ of $\{x_n\}$ and a sequence $\{u_n\}$ in X of successive blocks such that

$$\lim_{n\to\infty} \|z_n - u_n\|_j = 0. \tag{4}$$

By (3) we get

$$\|u_n - u_m\|_j \ge (\|u_n\|_j^2 + \|u_m\|_j^2)^{1/2}.$$

It follows from (4) that

$$D[(x_n)] \ge D[(z_n)] = D[(u_n)] = \limsup_m (\limsup_n \|u_n - u_m\|_j) \ge \sqrt{2}.$$

Hence $\mathrm{WCS}(X) = \beta(X) \ge \sqrt{2}$. On the other hand, consider the sequence $x_n = -e_n + e_{n+1}$. Then $x_n \to 0$ weakly in X, $D[(x_n)] = \sqrt{2}$ for $j = 1, 2$ and hence $\mathrm{WCS}(X) = \beta(X) \le \sqrt{2}$. We thus conclude that $\mathrm{WCS}(X) = \sqrt{2}$. The case $j = 3$ can be proved by combining Theorem 1(ii) and a result of [8]. The proof is complete.

Remark 1. It is remarkable that the nonreflexive James space $(J, \|\cdot\|_j)$ for $j = 1, 2$ has the same WCS value as a Hilbert space.

COROLLARY 2. *Each of James' spaces* $(J, \|\cdot\|_j)$ *for* $j = 1, 2, 3$ *has the fixed point property for nonexpansive mappings* (FPP).

Remark 2. Khamsi [9] first proved that James' space $(J, \|\cdot\|_3)$ has the FPP by using Maurey's ultrapower technique which is, of course, nonconstructive. We shall see in the next section that in this James space, each nonexpansive mapping does admit an iterative fixed point.

3. Iterative fixed points of nonexpansive mappings. Kirk's classical theorem [10] states that if C is a weakly compact convex subset of a Banach space X with normal structure, then every nonexpansive mapping $T : C \to C$ has a fixed point. Although there are proofs to this theorem which do not explicitly involve Zorn's lemma (cf. [6]), it remains an open question whether there is an iterative fixed point for a nonexpansive mapping T in such a space X, i.e., whether one can construct a sequence which converges strongly to a fixed point of T. In this section we show that this is true if the space X has weak uniform normal structure.

THEOREM 3. *Suppose X is a Banach space with weak uniform normal structure, C is a weakly sequentially compact convex subset of X, and $T : C \to C$ is a nonexpansive mapping (i.e., $\|Tx - Ty\| \le \|x - y\|, x, y \in C$). Then T has an iterative fixed point.*

Proof. Since one can easily construct (cf. [12]) a closed convex separable subset of C that is invariant under T, we may assume C itself is separable. Moreover, by considering the closed separable subspace $\overline{\text{span}}(C)$ of X, we may further assume that the space X itself is separable (in that case, the Hahn–Banach Theorem can be proved in a constructive way). Now fix a $\lambda \in (0, 1)$ and set

$$S = \lambda I + (1 - \lambda)T.$$

(Here I is the identity operator of X.) Then $S : C \to C$ is also nonexpansive with the same fixed point set of T. Moreover, S is asymptotically regular on C (see [7] and [5]), i.e.,

$$\lim_{n\to\infty} \|S^n x - S^{n+1}x\| = 0, \qquad x \in C. \tag{4}$$

The separability and weak sequential compactness of C make it possible to choose a subsequence $\{n_k\}$ of positive integers such that for every $x \in C$, $\{S^{n_k}x\}$ converges weakly. Now we define a sequence $\{x_n\}_{n=1}^{\infty}$ in C as follows:

$$x_0 \in C \text{ arbitrary}, \quad x_{m+1} = w\text{-}\lim_{k\to\infty} S^{n_k} x_m, \quad m \ge 0.$$

Note that since S is asymptotically regular on C, it follows that $x_{m+1} = w\text{-}\lim_{k\to\infty} S^{n_k+j} x_m$ for all integers $j \ge 0$. Write

$$R_m = \limsup_{k\to\infty} \|S^{n_k} x_m - x_{m+1}\|.$$

Then from Theorem 1, it follows that

$$R_m \le \mathrm{WCS}(X)^{-1} D[(S^{n_k} x_m)].$$

By the nonexpansiveness and asymptotic regularity of S and the weak lower semicontinuity of the norm of X, we get for all $k > i$,

$$\begin{aligned}\|S^{n_k}x_m - S^{n_i}x_m\| &\le \|S^{n_k-n_i}x_m - x_m\| \\ &\le \liminf_{j\to\infty} \|S^{n_k-n_i}x_m - S^{n_j+(n_k-n_i)}x_{m-1}\| \\ &\le \liminf_{j\to\infty} \|x_m - S^{n_j}x_{m-1}\| \le R_{m-1}.\end{aligned}$$

It follows that $R_m \le AR_{m-1} \le \ldots \le A^m R_0$, where $A = \mathrm{WCS}(X)^{-1} < 1$. Since

$$\begin{aligned}\|x_m - x_{m+1}\| &\le \limsup_{k\to\infty}(\|x_m - S^{n_k}x_m\| + \|x_{m+1} - S^{n_k}x_m\|) \\ &\le \limsup_{k\to\infty}(\liminf_{j\to\infty} \|S^{n_j+n_k}x_{m-1} - S^{n_k}x_m\|) + R_m \\ &\le \liminf_{j\to\infty} \|S^{n_j}x_{m-1} - x_m\| + R_m \\ &\le R_{m-1} + R_m \le (A^{m-1} + A^m)R_0,\end{aligned}$$

we see that $\{x_m\}$ is Cauchy. Let $x_\infty = \lim_{m\to\infty} x_m$. Then

$$\begin{aligned}\|S^{n_k}x_\infty - x_\infty\| &\le \|S^{n_k}x_\infty - S^{n_k}x_m\| + \|S^{n_k}x_m - x_m\| + \|x_m - x_\infty\| \\ &\le 2\|x_m - x_\infty\| + \liminf_{j\to\infty} \|S^{n_k}x_m - S^{n_j+n_k}x_{m-1}\| \\ &\le 2\|x_m - x_\infty\| + \liminf_{j\to\infty} \|x_m - S^{n_j}x_{m-1}\| \\ &\le 2\|x_m - x_\infty\| + R_{m-1} \to 0 \quad \text{as} \quad m \to \infty.\end{aligned}$$

Therefore, $S^{n_k}x_\infty = x_\infty$ for all k and the asymptotic regularity of S shows that x_∞ is a fixed point of S and hence of T. This completes the proof.

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