C O L L O Q U I U M   M A T H E M A T I C U M

VOL. LXVIII   1995   FASC. 1

A NILPOTENT LIE ALGEBRA AND EIGENVALUE ESTIMATES

BY

JACEK DZIUBAŃSKI (WROCŁAW), ANDRZEJ HULANICKI (WROCŁAW) AND JOE JENKINS (ALBANY, NEW YORK)

The aim of this paper is to demonstrate how a fairly simple nilpotent Lie algebra can be used as a tool to study differential operators on $\mathbb{R}^n$ with polynomial coefficients, especially when the property studied depends only on the degree of the polynomials involved and/or the number of variables.

The general idea of this algebra already appears in our paper [HJ] where we study operators of the form

$$-\frac{d^2}{dx^2}+|P|,$$

where P is a polynomial on the real line. In this case the algebra considered has a basis

$$\{X, Y_0, Y_1, \ldots, Y_d\}$$

and relations

$$[X, Y_k] = Y_{k+1}, \qquad Y_{d+1} = 0\ .$$

In the present paper we consider a Lie algebra $\mathcal{F}_\alpha$ (described in Section 1) which is generated by elements

$$X_1, \ldots, X_n, Y_1, \ldots, Y_m$$

and such that if $P_1, \ldots, P_m$ are polynomials on $\mathbb{R}^n$ of degree $\le d$, then

$$X_k \mapsto D_k, \quad Y_j \mapsto \text{ multiplication by } iP_j$$

extends to a representation π of $\mathcal{F}_\alpha$ by skew symmetric operators on $C_c^\infty(\mathbb{R}^n)$. Thus every element in the enveloping algebra of $\mathcal{F}_\alpha$ is mapped by the representation π onto a differential operator with polynomial coefficients on $\mathbb{R}^n$. In a similar fashion the infinitesimal generators of convolution semigroups on the Lie group $\exp \mathcal{F}_\alpha$ are mapped by π onto other operators of interest on $\mathbb{R}^n$.

1991 *Mathematics Subject Classification*: Primary 43A80; Secondary 22E30, 58G25.
This research was supported in part by the National Science Foundation.

This enables us to prove various generalizations of C. Fefferman's [Fe] estimates of the smallest eigenvalue of Schrödinger operators with nonnegative polynomial potentials.

The proofs obtained in this way may certainly not be the simplest possible because they make use of very nontrivial results by Helffer and Nourrigat [HN], and by Głowacki [Gł]. However, we do believe that the point of view presented here is illuminating and might be useful in other investigations.

1. The Lie algebra. Given $\alpha = (\alpha_j)$ where $\alpha_j = (\alpha_j^i) \in (\mathbb{Z}^+)^n$ for $j = 1, \ldots, m$ we define the Lie algebra $\mathcal{F}_\alpha$ as follows: As a vector space, $\mathcal{F}_\alpha$ has basis $\{X_1, \ldots, X_n, Y_j^{\beta_j} \mid 0 \le \beta_j \le \alpha_j\}$, where $\beta_j \le \alpha_j$ iff $\beta_j^i \le \alpha_j^i$ for $i = 1, \ldots, n$. (For later purposes we assume that $\mathcal{F}_\alpha$ is a Euclidean space for which this basis is orthonormal.) Let $\mathcal{X}, \mathcal{Y}$ denote the spans of the X_i's and the $Y_j^{\beta_j}$'s respectively. The nontrivial commutators are all determined by

$$[X_k, Y_j^{\beta_j}] = \begin{cases} Y_j^{\beta_j - e_k} & \text{if } \beta_j - e_k \ge 0, \\ 0 & \text{otherwise}, \end{cases} \tag{1.1}$$

where e_k is the n-tuple consisting of zeros except for a 1 in the kth position.

For $1 \le i \le n$, let $D_i = \partial_{x_i}$, and for $\beta_j \in (\mathbb{Z}^+ \cup \{0\})^n$ and $x \in \mathbb{R}^n$ set $D^\beta = D_1^{\beta_j^1} \ldots D_n^{\beta_j^n}$ and $D_x = \sum_i x_i D_i$. Let $\mathcal{P}(\mathbb{R}^n)$ denote the ring of real polynomials on $\mathbb{R}^n$, and given $\alpha = (\alpha_j^i)$ as above, define

$$\mathcal{P}_{\alpha_j} = \Big\{ P \in \mathcal{P}(\mathbb{R}^n) \,\Big|\, P(x) = \sum_{\beta \le \alpha_j} a_\beta x^\beta \Big\}.$$

Let $\Omega = \{(P_1, \ldots, P_m) \mid P_j \in \mathcal{P}_{\alpha_j} \text{ for each } j\}$. For $\omega = (P_1, \ldots, P_m) \in \Omega$, set $V_\omega = \{x \in \mathbb{R}^n \mid D_x P_j \equiv 0 \text{ for all } 1 \le j \le m\}$. Let $C_c^\infty(\mathbb{R}^n/V_\omega)$ denote the smooth functions on $\mathbb{R}^n$ that are invariant under translation by elements of V_ω and compactly supported on any subspace complementary to V_ω. Denote by $\mathcal{F}_\omega(\mathbb{R}^n/V_\omega)$ (respectively $\mathcal{F}_\omega(\mathbb{R}^n)$) the Lie algebra of operators on $C_c^\infty(\mathbb{R}^n/V_\omega)$ (respectively $C_c^\infty(\mathbb{R}^n)$) generated by the D_i's and multiplication by iP, denoted M_{iP}, as P ranges over ω. Define the mappings $\pi^\omega : \mathcal{F}_\alpha \to \mathcal{F}_\omega(\mathbb{R}^n/V_\omega)$ and $\Pi^\omega : \mathcal{F}_\alpha \to \mathcal{F}_\omega(\mathbb{R}^n)$ by

$$\pi^\omega, \Pi^\omega : \begin{cases} X_i \mapsto D_i, \\ Y_j^{\alpha_j} \mapsto M_{iP_j}, \\ \text{and, inductively, if } Y_j^\beta \mapsto M_{iP(Y_j^\beta)}, \text{ then} \\ [X_k, Y_j^\beta] \mapsto M_{iD_k P(Y_j^\beta)}, \end{cases} \tag{1.2}$$

and extend linearly to $\mathcal{F}_\alpha$.

LEMMA 1.3. *π^ω and Π^ω are Lie algebra homomorphisms.*

Proof. The only point to check is that the maps are well defined. For this, it suffices to note that if $[X_k, \sum_{j,\beta} a_{j,\beta} Y_j^\beta] = 0$ then the images are zero, but this is clear from (1.1) and (1.2).

Given ω, define the linear functional ξ_ω on $\mathcal{F}_\alpha$ by setting $\langle \xi_\omega, X_i\rangle = 0$ for each $1 \le i \le n$, and $\langle \xi_\omega, Y_j^\beta\rangle = Q(0)$, where $\pi^\omega(Y_j^\beta) = M_{iQ}$. Set $\mathcal{X}_\omega = \{X \in \mathcal{X} \mid \pi^\omega([X, Y_j^{\alpha_j}]) = 0 \text{ for each } 1 \le j \le m\}$.

LEMMA 1.4. *$\mathcal{X}_\omega + \mathcal{Y}$ is the maximal subalgebra subordinate to ξ_ω, and π^ω is the (infinitesimal) representation associated with ξ_ω via the Kirillov correspondence. In particular, if $V_\omega \ne \{0\}$, then Π^ω is reducible.*

Proof. It is clear that $\mathcal{X}_\omega + \mathcal{Y}$ is a subalgebra. Since for $X \in \mathcal{X}_\omega$ and $Y \in \mathcal{Y}$, $\langle \xi_\omega, [X,Y]\rangle = P([X,Y])(0) = 0$, where $\pi^\omega([X,Y]) = M_{iP([X,Y])}$, the subalgebra $\mathcal{X}_\omega + \mathcal{Y}$ is also subordinate. To see that it is maximal, suppose that $X \in \mathcal{X} \setminus \mathcal{X}_\omega$. Then there is a $k \in \mathbb{Z}^+$ and a $1 \le j \le m$ such that $D_x^{k+1} P_j \equiv 0$ but $D_x^k P_j \not\equiv 0$. Thus, $D_x^k P_j = Q$, where Q is a nonzero polynomial which depends only on coordinates orthogonal to x in $\mathbb{R}^n$. If $Q(0) \ne 0$, then

$$\langle \xi_\omega, [\overbrace{X, [X, \ldots, [X}^{k \text{ times}}, Y_j^{\alpha_j}]] \ldots]\rangle = Q(0) \ne 0,$$

where $X = \sum_i x_i X_i$. If $Q(0) = 0$, then there are an integer l and a vector $x' = (x'_1, \ldots, x'_n) \in \mathbb{R}^n$ such that $D_{x'}^l Q(0) \ne 0$. Then

$$\langle \xi_\omega, [\overbrace{X, [X, \ldots, [X,}^{k \text{ times}}\overbrace{[X', \ldots, [X'}^{l \text{ times}}, Y_j^{\alpha_j}]] \ldots]\rangle = D_{x'}^l Q(0) \ne 0,$$

where $X' = \sum_i x'_i X_i$. Thus, $\mathcal{X}_\omega + \mathcal{Y}$ is a maximal subalgebra subordinate to ξ_ω.

Let ϱ_ω denote the representation of $\mathcal{F}_\omega(\mathbb{R}^n/V_\omega)$ corresponding to ξ_ω. Then ϱ_ω is obtained by composing the representation of $\mathcal{F}_\omega(\mathbb{R}^n/V_\omega)$ corresponding to "evaluation at 0", ϱ'_ω, with π^ω, i.e. $\varrho_\omega = \varrho'_\omega \circ \pi^\omega$. The simply connected group corresponding to $\mathcal{F}_\omega(\mathbb{R}^n/V_\omega)$ is of the form $T \cdot M$, where $T \simeq V_\omega^\perp$ consists of translations on $C_c^\infty(\mathbb{R}^n/V_\omega)$, and M consists of multiplications by e^{iQ}, as Q ranges over the polynomials in $\mathcal{F}_\omega(\mathbb{R}^n/V_\omega)$. We know that $\varrho'_\omega(tm)f(u) = a(m,u)f(u+t)$, where $|a(m,u)| = 1$. Thus, if $m = e^{iQ}$, then $\varrho'_\omega(m)f(u) = a(m,u)f(u)$ and since $\pi^\omega(\mathcal{Y})$ is subordinate to the functional, $\varrho'_\omega(m)f(u) = m(u)f(u)$, i.e. $a(m,u) = m(u)$, which shows the equivalence.

Set $\mathcal{X}^\perp = \{\xi \in \mathcal{F}_\alpha^* \mid \xi_{|\mathcal{X}} = 0\}$, and define $\mathcal{Y}^\perp$ similarly. Then clearly $\mathcal{X}^\perp = \{\xi_\omega \mid \omega \in \Omega\}$. Thus, given $\lambda \in \mathcal{F}_\alpha^*$ there is a unique $\omega \in \Omega$ such that λ agrees with ξ_ω on $\mathcal{Y}$. Let λ_ω denote the restriction of λ to $\mathcal{X}_\omega$. Then $\mathcal{X}_\omega + \mathcal{Y}$ is a maximal subalgebra subordinate to $\lambda_\omega + \xi_\omega$. We denote by $\pi^{\lambda,\omega}$

the associated irreducible representation. Then $\pi^{\lambda,\omega}(Z) = i\lambda_\omega(Z)I + \pi^\omega(Z)$ for all $Z \in \mathcal{F}_\alpha$.

For $x \in \mathbb{R}^n$ and $Q \in \mathcal{P}_{\alpha_j}$, set $Q_x(y) = Q(x+y)$ for all $y \in \mathbb{R}^n$, and set $\omega_x = ((P_1)_x, \ldots, (P_m)_x)$.

LEMMA 1.5. *Each co-adjoint orbit in $\mathcal{F}_\alpha^*$ contains some $\lambda_\omega + \xi_\omega$, and the co-adjoint orbit of $\lambda_\omega + \xi_\omega$, $\mathcal{O}(\lambda_\omega + \xi_\omega)$, is given by*

$$\mathcal{O}(\lambda_\omega + \xi_\omega) = \lambda_\omega + \{\xi \in \mathcal{Y}^\perp \mid \xi_{|\mathcal{X}_\omega} = 0\} + \{\xi_{\omega_x} \mid x \in \mathbb{R}^n\}. \tag{1.6}$$

Proof. Let $x = (x_i) \in \mathbb{R}^n$ and set $X = \sum_i x_i X_i$. If $\pi^\omega(Y) = M_{iQ}$, then

$$\begin{aligned}\langle \mathrm{Ad}^*(\exp X)\xi_\omega, Y\rangle &= \langle \xi_\omega, \mathrm{Ad}(\exp X)Y\rangle \\ &= \sum_k \frac{1}{k!}\langle \xi_\omega, [X,[X,\ldots,[X,Y]]\ldots]\rangle \\ &= \sum_k \frac{1}{k!} D_x^k Q(0) = Q(x).\end{aligned}$$

Since $\langle \xi_{\omega_x}, Y_j^{\alpha_j}\rangle = (P_j)_x(0) = P_j(x)$, one sees that $\mathrm{Ad}^*(\exp X)\xi_\omega = \xi_{\omega_x}$.

Let $Z_1, \ldots, Z_n$ be an orthonormal basis for $\mathcal{X}$ such that $Z_1, \ldots, Z_k$ is the orthogonal complement to $\mathcal{X}_\omega$ in $\mathcal{X}$. Then there exist $Y_1, \ldots, Y_k \in \mathcal{Y}$ such that $\langle \xi_\omega, [Y_j, Z_i]\rangle = \delta_{ij}$. Let Z_i^* denote the element of $\mathcal{Y}^\perp$ dual to Z_i. Then

$$\begin{aligned}\langle \mathrm{Ad}^*(\exp Y_i)\xi_\omega, Z_j\rangle &= \langle \xi_\omega, Z_j + [Y_i, Z_j]\rangle \\ &= \langle \xi_\omega, Z_j\rangle + \delta_{ij} = \langle \xi_\omega + Z_i^*, Z_j\rangle,\end{aligned}$$

which completes the proof, since clearly $\mathrm{Ad}^*(\exp \mathcal{F}_\alpha)\lambda_\omega = \lambda_\omega$.

The space of co-adjoint orbits in the dual of a nilpotent Lie algebra can be topologized using the quotient topology, or it can be given the Fell topology via the Kirillov correspondence. These topologies are equivalent according to Brown [Br] and generally not Hausdorff. Let F_α be the simply connected nilpotent Lie group with Lie algebra $\mathcal{F}_\alpha$. For $\lambda \in \mathcal{F}_\alpha^*$ let π^λ be the irreducible unitary representation of F_α corresponding to λ. The following lemma of Riemann–Lebesgue type comes from Fell (cf. [Fell, Corollary 1]); however, for the group F_α it has a direct simple proof.

LEMMA 1.7. *For every $K \in L^1(F_\alpha)$,*

$$\lim_{\mathcal{O}(\lambda)\to\infty} \|\pi_K^\lambda\|_{\mathrm{op}} = 0. \tag{1.8}$$

Proof. It is enough to prove (1.8) for K in the Schwartz space of functions on F_α. Since $\mathcal{O}(\lambda) = \mathcal{O}(\lambda_\omega + \xi_\omega)$ (cf. Lemma 1.5), we can certainly assume that $\lambda = \lambda_\omega + \xi_\omega$. We already know that $W = \mathcal{X}_\omega + \mathcal{Y}$ is a maximal subalgebra subordinate to λ. Obviously W is an ideal and $\mathcal{F}_\alpha/\mathcal{W}$ is abelian. Let S_λ be the orthogonal complement to $\mathcal{X}_\omega$ in $\mathbb{R}^n$. The representation π^λ

acts on $L^2(S_\lambda)$. The kernel $K_\lambda(x,s)$ of the operator π_K^λ has the form

$$K_\lambda(x,s) = \int\limits_W K(w, s-x)\exp(i\langle \mathrm{Ad}_x^*\lambda, w\rangle)\,dw \tag{1.9}$$
$$= K(\widehat{\mathrm{Ad}_x^*}\lambda, s-x).$$

Since $\|\pi_K^\lambda\|_{\mathrm{op}} \le \max\{\sup_{x\in S_\lambda}\int_{S_\lambda}|K_\lambda(x,s)|\,ds,\ \sup_{s\in S_\lambda}\int_{S_\lambda}|K_\lambda(x,s)|\,dx\}$ and K is from the Schwartz class, (1.8) follows.

Let $\{a_1,\dots,a_n,b_1,\dots,b_m\}\subset\mathbb{R}^+$, the positive reals. Define a one-parameter group of automorphisms on $\mathcal{F}_\alpha, \{\delta_t\}_{t>0}$, by setting $\delta_t X_i = t^{1/a_i}X_i$, $\delta_t Y_j^{\alpha_j} = t^{1/b_j}Y_j^{\alpha_j}$, and inductively by $\delta_t[X, Y_j^\beta] = [\delta_t X, \delta_t Y_j^\beta]$, where $0\le\beta\le\alpha_j$. We define δ_t on F_α, the simply connected group with Lie algebra $\mathcal{F}_\alpha$, by setting $\delta_t(\exp Z) = \exp(\delta_t Z)$, and we define δ_t^* on $\mathcal{F}_\alpha^*$ by duality, i.e. $\langle\delta_t^*\xi, Z\rangle = \langle\xi,\delta_t Z\rangle$. Since δ_t is an automorphism, the representation associated with $\delta_t^*\xi_\omega$ is $\pi^\omega\circ\delta_t$. Thus it is easy to check that

$$\delta_t^*\,\xi_\omega = \xi_{\delta_t\omega},$$

where

$$\delta_t\omega = (\delta_t P_1,\dots,\delta_t P_m),$$

and

$$\delta_t P_j(x_1,\dots,x_n) = t^{1/b_j}P_j(t^{1/a_1}x_1,\dots,t^{1/a_n}x_n).$$

A function P defined on $\mathcal{F}_\alpha$ (or $\mathcal{F}_\alpha^*$) is said to be *homogeneous of degree r* if $P(\delta_t Z) = t^r P(Z)$ for $Z\in\mathcal{F}_\alpha$ (or $Z\in\mathcal{F}_\alpha^*$). A *homogeneous gauge* on $\mathcal{F}_\alpha$ or $\mathcal{F}_\alpha^*$ is a continuous function, homogeneous of degree one, that is positive on nonzero elements. Examples that are of particular interest here are given (on $\mathcal{X}^\perp$) by

$$\varrho^B(\xi_\omega) = \inf_{\gamma_1,\dots,\gamma_n>0}\Big\{\sum_{k=1}^n \gamma_k^{-a_k} + \max_{|x_k|\le B\gamma_k}\sum_{j=1}^m |P_j(x)|^{b_j}\Big\}, \tag{1.10}$$

where B is any positive constant.

2. The estimates. Recall that F_α is the simply connected nilpotent Lie group with Lie algebra $\mathcal{F}_\alpha$. Let $\mathcal{L}$ be a homogeneous distribution of degree one on F_α such that $\mathcal{L} = \mathcal{L}_0 + \mu$, where μ is a bounded measure and $\mathcal{L}_0$ is compactly supported. We define a left-invariant convolution operator L on F_α,

$$L : C_c^\infty(F_\alpha)\to L^1(F_\alpha),\qquad f\mapsto f*\mathcal{L}.$$

We assume that

$$(\mathcal{L}, f^* * f) = (f*\mathcal{L}, f)\ge 0\quad\text{for } f\in C_c^\infty(F_\alpha),\ \text{where } f^*(x) = \overline{f(x^{-1})},$$

and that $-L$ is the infinitesimal generator of a convolution semigroup $f \mapsto f * p_t$ such that $p_t \in L^1(F_\alpha)$. We denote by $\pi^\xi_{p_t}$ the operator $\int_{F_\alpha} p_t(x)\pi^\xi_x\,dx$, and by $-\pi^\xi_L$ the generator of the semigroup $\{\pi^\xi_{p_t}\}_{t>0}$.

Let ϱ be a homogeneous gauge on $\mathcal{F}^*_\alpha$ with respect to the dilations $\{\delta_t\}_{t>0}$. We define

$$\varrho(\mathcal{O}(\xi)) = \inf\{\varrho(\eta) \mid \eta \in \mathcal{O}(\xi)\}.$$

Let

$$\lambda(\pi^\xi_L) = \inf\{(\pi^\xi_L f, f) \mid \|f\|_{L^2} = 1\}.$$

THEOREM 2.1. *There is a constant $c > 0$ such that*

$$\lambda(\pi^\xi_L) \geq c\varrho(\mathcal{O}(\xi)). \tag{2.2}$$

P r o o f. Obviously both sides of (2.2) are constant on orbits. Also, since L is homogeneous of degree one,

$$\lambda(\pi^{\delta^*_r\xi}_L) = r\lambda(\pi^\xi_L) \quad \text{and} \quad \varrho(\mathcal{O}(\delta^*_r\xi)) = r\varrho(\mathcal{O}(\xi)).$$

Therefore to prove (2.2) it suffices to show that there exists an $R > 0$ such that if $\varrho(\mathcal{O}(\xi)) = R$, then $\lambda(\pi^\xi_L) \geq 1$.

Note that

$$\lambda(\pi^\xi_L) = -\log(\|\pi^\xi_{p_1}\|_{\mathrm{op}}). \tag{2.3}$$

By Lemma 1.7 there is an R such that $\|\pi^\xi_{p_1}\|_{\mathrm{op}} \leq 1/e$ for $\varrho(\mathcal{O}(\xi)) \geq R$. This combined with (2.3) ends the proof.

3. Applications. For an element X in the Lie algebra of a Lie group G and $0 < a < 2$, the operator $|X|^a$ is defined on $C^\infty_{\mathrm{c}}(G)$ into $L^1(G)$ by

$$-|X|^a f = c\int_0^\infty t^{-1-a/2} f * \mu_t\,dt,$$

where $\{\mu_t\}_{t>0}$ is the semigroup generated by X^2. For a unitary representation π of G we have

$$\pi_{|X|^a} = |\pi_X|^a\ .$$

If the operator $\pi_X f(x) = iP(x)f(x)$, then $\pi_{|X|^a} f(x) = |P(x)|f(x)$. The following is an example of possible application of the theorem of the preceding section.

THEOREM 3.1. *Let $\omega = (P_1, \ldots, P_m)$ be polynomials on $\mathbb{R}^n$ such that $P_j \in \mathcal{P}_{\alpha_j}$, and let*

$$H = H(\omega) = \sum_{k=1}^n |D_k|^{a_k} + \sum_{j=1}^m |P_j|^{b_j},$$

where $a_k \le 2$ *and* $b_j \le 1$, *be an operator defined on* $C_c^\infty(\mathbb{R}^n)$. *Let*

$$\lambda(\omega) = \inf_{\substack{y \in \mathbb{R}^n \\ \gamma_1, \ldots, \gamma_n > 0}} \Big\{ \sum_{k=1}^n \gamma^{-a_k} + \max_{|y_k - x_k| \le \gamma_k} \sum_{j=1}^m |P_j(x)|^{b_j} \Big\}.$$

Then there is a constant $c = c(b_1, \ldots, b_m, a_1, \ldots, a_n)$, *otherwise independent of* $P_1, \ldots, P_m$, *such that*

$$\lambda(H(\omega)) = \inf \operatorname{Spectrum} H(\omega) \ge c\lambda(\omega). \tag{3.2}$$

Proof. We proceed by induction with respect to n. Assume first that the family ω is irreducible. We consider the algebra $\mathcal{F}_\alpha$ and we note that if

$$L = \sum_{k=1}^n |X_k|^{a_k} + \sum_{j=1}^m |Y_j^{\alpha_j}|^{b_j},$$

then $\pi_L^\omega = H$ and π^ω is irreducible. If we put $\delta_r X_k = r^{1/a_k} X_k$, $\delta_r Y_j^{\alpha_j} = r^{1/b_j} Y_j^{\alpha_j}$, we see that L is homogeneous of degree one. To see that L satisfies the conditions of Theorem 2.1 we use a theorem by P. Głowacki [Gł], which states that if $-L$ is the infinitesimal generator of a semigroup of probability measures, $\{\mu_t\}_{t>0}$, which satisfies the Rockland condition, as $-L$ clearly does, then μ_t is absolutely continuous. So (3.2) in that case is a consequence of (1.10) and Theorem 1.2.

Now assume that ω is reducible, that is, $V_\omega \ne \{0\}$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the canonical basis of $\mathbb{R}^n$. For $\gamma > 0$ and $k = 1, \ldots, n$ define $[-\gamma, \gamma]_k = \{t\mathbf{e}_k : |t| \le \gamma\} \subset \mathbb{R}^n$.

The following lemma is very easy to prove.

LEMMA 3.3. *There exists a constant* $B > 1$, $B = B(n)$, *such that for every linear subspace* $W \subset \mathbb{R}^n$ *and for every rectangle* $\mathcal{Q} = [-\gamma_1, \gamma_1]_1 + \ldots + [-\gamma_n, \gamma_n]_n \subset \mathbb{R}^n$ *there are vectors* $\mathbf{e}_{n_1}, \ldots, \mathbf{e}_{n_l}$ *from the canonical basis such that* $\operatorname{lin}\{\mathbf{e}_{n_1}, \ldots, \mathbf{e}_{n_l}\} \oplus W = \mathbb{R}^n$, $l + \dim W = n$, *and* $\mathcal{Q} \subset [-B\gamma_{n_1}, B\gamma_{n_1}]_{n_1} + \ldots + [-B\gamma_{n_l}, B\gamma_{n_l}]_{n_l} + W$.

Let $\mathcal{S}_\omega$ be the set of all linear subspaces S such that $S \oplus V_\omega = \mathbb{R}^n$ and S has a basis $\mathbf{e}_{n_1}, \ldots, \mathbf{e}_{n_l}$ consisting of vectors from the canonical basis. For $S \in \mathcal{S}_\omega$ with the basis $\{\mathbf{e}_{n_1}, \ldots, \mathbf{e}_{n_l}\}$ denote by P_j^S the polynomial P_j considered on S. Clearly the family $\{P_j^S\}_{j=1}^n$ is irreducible. By the induction hypothesis,

$$\lambda(H(\omega)) \ge \lambda_S \Big(\sum_{i=1}^l |D_{n_i}|^{a_{n_i}} + \sum_{j=1}^m |P_j^S|^{b_j} \Big) \ge c_S \varrho_S(\mathcal{O}(P_1^S, \ldots, P_m^S)),$$

where

$$\varrho_S((P_1^S,\dots,P_m^S)) = \inf_{\gamma_1,\dots,\gamma_l>0}\Big\{\sum_{i=1}^{l}\gamma_i^{-a_{n_i}} + \max_{|x_{n_i}|\le\gamma_i}\sum_{j=1}^{m}|P_j^S|^{b_j}\Big\},$$

and $c_S>0$ does not depend on $\omega\in\Omega$. Since the number of elements of $\mathcal{S}_\omega$ is bounded by 2^n, it suffices to show that there is a constant $c>0$ such that

$$\sum_{S\in\mathcal{S}_\omega}\varrho_S(\mathcal{O}(P_1^S,\dots,P_m^S)) \ge c\varrho(\mathcal{O}(P_1,\dots,P_m)).$$

Fix $\varepsilon>0$ and set $\varepsilon_0=\varepsilon/2^n$. For S as above let $x^S\in S\subset\mathbb{R}^n$ and $\gamma_{n_1}^S,\dots,\gamma_{n_l}^S>0$ be such that

$$\varepsilon_0+\varrho_S^B(\mathcal{O}(P_1^S,\dots,P_m^S)) \ge \sum_{i=1}^{l}(\gamma_{n_i}^S)^{-a_{n_i}} + \max_{|x_{n_i}|\le B\gamma_{n_i}^S}\sum_{j=1}^{m}|(P_j^S)_{x^S}|^{b_j}.$$

Let $\gamma_k=\min_S\{\gamma_{n_i}^S : n_i=k\}$. If $\mathbf{e}_k\in V_\omega$ for some k, we put $\gamma_k=\infty$. Let $\mathcal{Q}=[-\gamma_1,\gamma_1]_1+\dots+[-\gamma_n,\gamma_n]_n\subset\mathbb{R}^n$. By Lemma 3.3 there is $S_0\in\mathcal{S}_\omega$ such that S_0 has a basis $\mathbf{e}_{n_1}^{S_0},\dots,\mathbf{e}_{n_l}^{S_0}$ consisting of vectors from the canonical basis, and $\mathcal{Q}\subset[-B\gamma_{n_1},B\gamma_{n_1}]_{n_1}+\dots+[-B\gamma_{n_l},B\gamma_{n_l}]_{n_l}+V_\omega$. Then

$$\begin{aligned}\max_{|x_{n_i}|\le B\gamma_{n_i}^{S_0}}&\sum_{j=1}^{m}|(P_j^{S_0})_{x^{S_0}}|^{b_j}\\ &\ge \max_{|x_{n_i}|\le B\gamma_{n_i}}\sum_{j=1}^{m}|(P_j^{S_0})_{x^{S_0}}|^{b_j} \ge \max_{x\in\mathcal{Q}}\sum_{j=1}^{m}|(P_j)_{x^{S_0}}|^{b_j}.\end{aligned}$$

This gives

$$\begin{aligned}\varepsilon+&\sum_{S\in\mathcal{S}_\omega}\varrho_S^B(\mathcal{O}(P_1^S,\dots,P_m^S))\\ &\ge \sum_{S\in\mathcal{S}_\omega}\Big(\sum_{i=1}^{l}(\gamma_{n_i}^S)^{-a_{n_i}} + \max_{|x_{n_i}|\le B\gamma_{n_i}^S}\sum_{j=1}^{m}|(P_j)_{x^S}|^{b_j}\Big)\\ &\ge \gamma_1^{-a_1}+\dots+\gamma_n^{-a_n}+\max_{|x_k|\le\gamma_k}\sum_{j=1}^{m}|(P_j)_{x^{S_0}}|^{b_j} \ge \varrho(\mathcal{O}(P_1,\dots,P_m)).\end{aligned}$$

Finally, since the homogeneous norm ϱ_S^B is comparable with ϱ_S, there is a constant $c>0$ such that

$$\begin{aligned}\sum_{S\in\mathcal{S}_\omega}&\varrho_S(\mathcal{O}(P_1^S,\dots,P_m^S))\\ &\ge c\sum_{S\in\mathcal{S}_\omega}\varrho_S^B(\mathcal{O}(P_1^S,\dots,P_m^S)) \ge c\varrho(\mathcal{O}(P_1,\dots,P_m)).\end{aligned}$$

This establishes the theorem.

Remarks. If we put $a_k = 2, m = 1$ and $b_1 = 1$, then $H = -\Delta + |V|$, where V is a polynomial. Inequality (3.2) then becomes the content of Fefferman's Theorem 2, p. 144 of [Fe].

Another type of operator whose smallest eigenvalue can be estimated in a similar manner is

$$H = \sum_{k=1}^{n} (-1)^{a_k} D_k^{2a_k} + \sum_{j=1}^{m} P_j^2,$$

where $P_1, \ldots, P_m$ are polynomials such that $P_j \in \mathcal{P}_{\alpha_j}$, and a_k are positive integers. Then there is a constant $c = c\,(a_1, \ldots, a_n, \alpha_1, \ldots, \alpha_m)$, independent of $\omega = (P_1, \ldots, P_m)$, such that

$$\inf \text{Spectrum}\, H(\omega) \ge c \inf_{\substack{y \in \mathbb{R}^n \\ \gamma_1, \ldots, \gamma_n > 0}} \Big\{ \sum_{k=1}^{n} \gamma_k^{-2a_k} + \max_{|x_k - y_k| \le \gamma_k} \sum_{j=1}^{m} P_j(x)^2 \Big\}.$$

This follows in the same way from a theorem of Folland–Stein [FS] that the operator

$$-L = \sum_{k=1}^{n} (-1)^{a_k} X_k^{2a_k} + \sum_{j=1}^{m} Y_j^2$$

is the infinitesimal generator of a semigroup of convolution operators $f \mapsto f * p_t$, where $p_t \in L^1(F_\alpha)$.

REFERENCES

[Br] I. D. Brown, *Dual topology of a nilpotent Lie group*, Ann. Sci. École Normale Sup. (4) 6 (1973), 407–411.

[Fe] C. L. Fefferman, *The uncertainty principle*, Bull. Amer. Math. Soc. 9 (1983), 129–206.

[Fell] J. M. G. Fell, *The dual spaces of C^*-algebras*, Trans. Amer. Math. Soc. 94 (1960), 365–403.

[FS] G. B. Folland and E. M. Stein, *Hardy Spaces on Homogeneous Groups*, Princeton Univ. Press, Princeton, N.J., 1982.

[Gł] P. Głowacki, *The Rockland condition for non-differential convolution operators*, Duke Math. J. 58 (1989), 371–395.

[HN] B. Helffer et J. Nourrigat, *Caractérisation des opérateurs hypoelliptiques homogènes invariants à gauche sur un groupe gradué*, Comm. Partial Differential Equations 4 (1978), 899–958.

[HJ] A. Hulanicki and J. W. Jenkins, *Nilpotent Lie groups and eigenfunction expansions of Schrödinger operators II*, Studia Math. 87 (1987), 239–252.

[HJL] A. Hulanicki, J. W. Jenkins and J. Ludwig, *Minimum eigenvalues for positive Rockland operators*, Proc. Amer. Math. Soc. 94 (1985), 718–720.

INSTITUTE OF MATHEMATICS
WROCŁAW UNIVERSITY
PL. GRUNWALDZKI 2/4
50-384 WROCŁAW, POLAND
E-mail: JDZIUBAN@MATH.UNI.WROC.PL

DEPARTMENT OF MATHEMATICS AND STATISTICS
THE UNIVERSITY AT ALBANY/SUNY
ALBANY, NEW YORK 12222
U.S.A.
E-mail: JWJ71@ALBNY1VX.BITNET

Reçu par la Rédaction le 8.10.1990;
en version modifiée le 10.12.1993