ON SOME CLASS OF NEARLY
CONFORMALLY SYMMETRIC MANIFOLDS

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1. Introduction. Let \((M, g)\) and \((\overline{M}, \overline{g})\) be two Riemannian or pseudo-Riemannian manifolds of class \(C^\infty\). A mapping \(\gamma: (M, g) \to (\overline{M}, \overline{g})\) is said to be geodesic if it preserves geodesics, i.e. maps geodesics of \((M, g)\) onto geodesics of \((\overline{M}, \overline{g})\). The metrics \(g\) and \(\overline{g}\) are then said to be geodesically corresponding.

Suppose that both \(g\) and \(\overline{g}\) are metrics on the same manifold \(M\). Let \(\mathcal{F}(M)\) be the ring of differentiable functions and \(\mathfrak{X}(M)\) the \(\mathcal{F}\)-module of differentiable vector fields on \(M\). Each of the conditions below is necessary and sufficient for the metrics \(g\) and \(\overline{g}\) to be geodesically corresponding:

\[
\nabla_X Y = \nabla_X Y + (X\psi)Y + (Y\psi)X,
\]

\[
(\nabla_X \overline{g})(Y, Z) = 2(X\psi)\overline{g}(Y, Z) + (Y\psi)\overline{g}(X, Z) + (Z\psi)\overline{g}(X, Y),
\]

for all \(X, Y, Z \in \mathfrak{X}(M)\), where \(\psi \in \mathcal{F}(M)\), and \(\nabla\) and \(\overline{\nabla}\) are the Levi-Civita connections with respect to \(g\) and \(\overline{g}\).

A manifold \((M, g)\) admits a geodesic mapping if and only if there exist a function \(\varphi \in \mathcal{F}(M)\) and a symmetric non-singular bilinear form \(a\) on \(M\) satisfying

\[
(\nabla_X a)(Y, Z) = (Y\varphi)g(X, Z) + (Z\varphi)g(X, Y)
\]

for all \(X, Y, Z \in \mathfrak{X}(M)\) ([9]).

In a chart \((U, x)\) on \(M\) the local components of \(g, \overline{g}, a, X\varphi\) and \(X\psi\) given by

\[
g_{ij} = g(X_i, X_j), \quad \overline{g}_{ij} = \overline{g}(X_i, X_j), \quad a_{ij} = a(X_i, X_j), \quad \varphi_i = X_i\varphi, \quad \psi_i = X_i\psi
\]

satisfy

\[
a_{ij} = \exp(2\psi)\overline{g}^{st}g_{st}g_{ij},
\]

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\[ \varphi_i = -\exp(2\psi)g^{st_{i4}}g_{si}\psi_t, \]

where \(X_i = \partial/\partial x^i \in \mathfrak{X}(U)\) and the \(g^{ij}\) are the components of \((\tilde{g}_{ij})^{-1}\).

A geodesic mapping is said to be non-trivial if it is non-affine, which is equivalent to \(\varphi \neq \text{const} \) on \(M\).

By (1), the curvature tensors and Ricci tensors of \((M, g)\) and \((M, \tilde{g})\) are related by

\[
\begin{align*}
\tilde{R}(X,Y)Z &= R(X,Y)Z + P(X,Z)Y - P(Y,Z)X, \\
\tilde{S}(X,Y) &= S(X,Y) + (n-1)P(X,Y),
\end{align*}
\]

where

\[
P(X,Y) = H_\psi(X,Y) - (X\psi)(Y\psi)
\]

and \(H_\psi\) is the Hessian of \(\psi\).

Following W. Roter ([7]), the manifold \((M, g)\) is said to be nearly conformally symmetric if the tensor

\[
L(X,Y) = \frac{1}{n-2}\left[ S(X,Y) - \frac{r}{2(n-1)}g(X,Y) \right]
\]

is a Codazzi tensor, where \(S\) is the Ricci tensor and \(r\) denotes the scalar curvature.

N. S. Sinyukov ([9]) and E. N. Sinyukova ([10]) investigated manifolds whose Ricci tensor satisfies

\[
(\nabla_X S)(Y, Z) = \sigma(X)g(Y, Z) + \nu(Y)g(X, Z) + \nu(Z)g(X,Y),
\]

where \(\sigma\) and \(\nu\) are some 1-forms. Such manifolds are known under different names (see [9], [10], [3]). In what follows a Riemannian (or pseudo-Riemannian) manifold satisfying (9) with non-constant scalar curvature will be called a Sinyukov manifold. Such manifolds always admit non-trivial geodesic mappings and every Sinyukov manifold is nearly conformally symmetric (see Lemma 1).

Let \((M, g)\) admit a non-trivial geodesic mapping onto the manifold \((M, \tilde{g})\) defined by the 1-form \(d\psi\) (see (2)). In [5] it was proved that \((M, g)\) (dim \(M \geq 3\)) is a conformally flat Sinyukov manifold if and only if \((M, \tilde{g} = \exp(2\psi)g)\) is of constant sectional curvature. In the present paper we prove that \((M, g)\) with nowhere vanishing Weyl conformal curvature tensor is a Sinyukov manifold if and only if \((M, \tilde{g} = \exp(2\psi)g)\) is either an Einstein manifold admitting non-trivial geodesic mappings or a Sinyukov manifold.

In [3] some properties of Sinyukov manifolds with non-null vector \(\Phi\) defined by (24) below were investigated. In the present paper we deal with manifolds without any assumption on \(\Phi\). Finally, the local structure theorem for Sinyukov manifolds is given.
2. Preliminaries. If $\tilde{g}$ is a metric on $M$ and there exists $\lambda \in \mathfrak{F}(M)$ such that $\tilde{g} = \exp(2\lambda)g$, then $g$ and $\tilde{g}$ are said to be conformally related or conformal to each other, and the transformation $g \to \tilde{g}$ is called a conformal change. As is well-known, the Christoffel symbols, the curvature tensors and the Ricci tensors of the manifolds $(M, g)$ and $(M, \tilde{g})$ are then related by

\begin{equation}
\tilde{\nabla}_X Y = \nabla_X Y + (X\lambda) Y + (Y\lambda) X - g(X, Y) \Lambda,
\end{equation}

where the vector field $\Lambda$ is defined by $g(X, \Lambda) = X\lambda$ for $X \in \mathfrak{X}(M)$;

\begin{equation}
g(\tilde{\mathcal{R}}(X, Y) Z, V) = g(R(X, Y) Z, V) + Q(X, Z) g(Y, V) - Q(Y, Z) g(X, V) + g(X, Z) Q(Y, V) - g(Y, Z) Q(X, V),
\end{equation}

for arbitrary $X, Y, Z, V \in \mathfrak{X}(M)$;

\begin{equation}
Q(X, Y) + L(X, Y) = \tilde{L}(X, Y),
\end{equation}

where the tensor fields $Q$ and $L$ are given by

\begin{equation}
Q(X, Y) = H_\lambda(X, Y) - (X\lambda)(Y\lambda) + \frac{1}{2} g(A, A) g(X, Y),
\end{equation}

\begin{equation}
L(X, Y) = \frac{1}{n-2} \left[ S(X, Y) - \frac{r}{2(n-1)} g(X, Y) \right]
\end{equation}

with $H_\lambda$ being the Hessian of $\lambda$, and $r$ standing for the scalar curvature of $(M, g)$. The tensor field $\tilde{L}$ on $(M, \tilde{g})$ is defined analogously.

The Weyl conformal curvature tensor $C$ satisfying

\begin{equation}
g(C(X, Y) Z, V) = g(R(X, Y) Z, V) + g(X, V) L(Y, Z) - g(Y, V) L(X, Z) + L(X, V) g(Y, Z) - L(Y, V) g(X, Z),
\end{equation}

is invariant under conformal change, i.e. $\tilde{C} = C$.

From (12) and (13) we get easily

\begin{equation}
g(C(X, Y) Z, \Lambda) = D(X, Y, Z) - \tilde{D}(X, Y, Z),
\end{equation}

where

\begin{equation}
D(X, Y, Z) = (\nabla_X L)(Y, Z) - (\nabla_Y L)(X, Z)
\end{equation}

and the tensor field $\tilde{D}$ on $(M, \tilde{g})$ is defined in the same manner.

In the sequel we shall use the following theorem and lemmas.

**Theorem** ([9]). If $(M, g)$ admits a non-trivial geodesic mapping onto a manifold $(M, \overline{g})$ defined by a 1-form $d\psi$, then the manifold $(M, a)$, where $a$ satisfies (3), admits a geodesic mapping onto $(M, \tilde{g} = \exp(2\psi)g)$ determined by the same 1-form $d\psi$.

**Lemma** ([12], [1]). On a Sinyukov manifold the tensor $D$ given by (17) vanishes, i.e. $\tilde{L}$ is a Codazzi tensor.
Lemma 2 ([9]). If on \((M, g)\) relation (9) is satisfied at a point \(p\), then
\[\sigma(X) = \frac{n}{(n-1)(n+2)}(Xr), \quad \nu(X) = \frac{n-2}{2(n-1)(n+2)}(Xr)\]
for any \(X \in T_p(M)\). Consequently, \((M, g)\) is a Sinyukov manifold if and only if the scalar curvature \(r \neq \text{const}\) and the condition (9) holds everywhere on \(M\).

We define \((1,1)\) tensor fields \(Ric\) and \(A\) as follows:
\[(19) \quad g(Ric(X), Y) = S(X, Y),\]
\[(20) \quad g(A(X), Y) = a(X, Y)\]
for all \(X, Y \in \mathfrak{X}(M)\).

Lemma 3 ([10], [3]). If \((M, g)\) is a Sinyukov manifold and \(d\phi \neq 0\) at a point \(p \in M\), then
\[(21) \quad a(Ric(X), Y) = a(X, Ric(Y)),\]
\[(22) \quad a(X, N) - \frac{\text{tr}(A)}{n} \nu(X) = S(X, \Phi) - \frac{r}{n}(X\phi),\]
\[(23) \quad (X\phi)
\begin{bmatrix}
    S(Y, Z) - \frac{r}{n}g(Y, Z) \\
    -(Y\phi)
\end{bmatrix}
\quad = \nu(X)
\begin{bmatrix}
    a(Y, Z) - \frac{\text{tr}(A)}{n}g(Y, Z) \\
    -\nu(Y)
\end{bmatrix} - \nu(Y)
\begin{bmatrix}
    a(X, Z) - \frac{\text{tr}(A)}{n}g(X, Z)
\end{bmatrix}\]

at \(p\) for all \(X, Y, Z \in T_p(M)\), where \(N\) and \(\Phi\) are given by
\[(24) \quad g(X, N) = \nu(X), \quad g(X, \Phi) = \Phi\phi.\]

3. Properties of conformal and geodesic mappings of Sinyukov manifolds. Let \(p \in M\) be such that \(d\phi \neq 0\) and (3) hold at \(p\). Choose a local coordinate system \((U, x)\) so that \(p \in U\). By \(R^t_{i,j,k}, S_{i,j}, \phi_{i,j}\) we denote the components of the tensors \(R, S\) and \(H_{\phi}\) in this coordinate system. Differentiating covariantly (3) and applying the Ricci identity we get
\[(25) \quad a_{it}R^t_{jkl} + a_{tj}R^t_{ikl} = \varphi_{it}g_{jk} + \varphi_{ij}g_{tk} - \varphi_{kj}g_{it} - \varphi_{kt}g_{jt}.\]

Differentiating covariantly (25) with respect to \(x^m\), contracting with \(g^{lm}\) and applying the Ricci identity, by (3) and (9), we obtain
\[(26) \quad 4\varphi_{i}R^i_{jkl} = \varphi_{k}S_{ij} - \varphi_{j}S_{ik}
\quad + \frac{n+2}{n-2}[\nu^i a_{ik}g_{ij} - \nu^i a_{ii}g_{kj} + \nu_i a_{k} - \nu_k a_{ij}] + b_{ig_{jk}} - b_{j}g_{ij},\]
where the \(\nu_i\) are the components of the 1-form \(\nu\), the \(\nu^i\) are the components of the field \(N\) (i.e. \(\nu^i = g^{im}\nu_m\)) whereas \(b_i = \varphi_{it}g_{it}\) and the semicolon denotes covariant differentiation on \((M, g)\). Moreover, substituting (22) and
(23) into (26), we get

\begin{equation}
4\varphi_t R^k_{jki} = \frac{4}{n-2} (\varphi_i S_{jk} - \varphi_k S_{ij}) + \frac{n+2}{n-2} (\varphi^i S_{tk} g_{ij} - \varphi^j S_{ti} g_{kj}) + b_i g_{jk} - b_j g_{ik},
\end{equation}

where \(\varphi^i = \varphi_t g^{ti}\) are the components of the field \(\Phi\).

Now, we shall prove

**Proposition 1.** If \((M, g)\) is a Sinyukov manifold and the Weyl conformal curvature tensor \(C \neq 0\) and \(d\varphi \neq 0\) at a point \(p \in M\), then

\begin{equation}
g(\Phi, C(X, Y) Z) = 0,
\end{equation}

\begin{equation}
g(N, C(X, Y) Z) = 0
\end{equation}
on some neighbourhood \(U_1\) of \(p\), where \(\Phi\) and \(N\) are as in (24). So, for the metrics \(\tilde{g}_1 = \exp(2\varphi) g\) and \(\tilde{g}_2 = \exp(2\nu) g\), where \(\nu \in \mathcal{F}(U_1)\) and \(X\nu = \nu(X)\), the tensors \(L_1, L_2\) defined by (14) are Codazzi tensors.

**Proof.** Transvecting (27) with \(g_{jk}\) we get

\begin{equation}
b_i = \frac{n+6}{n-2} \varphi^i S_{ti} - \frac{4r}{(n-1)(n-2)} \varphi_i.
\end{equation}

Substituting (30) into (27), in view of (15), we find (28). Beginning with (9) and following the above argument we obtain (29). Thus the proposition is proved.

**Proposition 2.** If \((M, g)\) is a Sinyukov manifold, then on the set \(U_\varphi = \{p \in M : \varphi \neq 0 \text{ at } p\}\) the following identities hold:

\begin{equation}
H_\varphi(X, Y) = \frac{1}{n-2} \left[ a(Ric(X), Y) - \frac{r}{n(n-1)} g(X, Y) 
- \frac{\text{tr}(A)}{n} S(X, Y) + (n-2) \varphi_1 g(X, Y) \right] + F(X\varphi)(Y\varphi),
\end{equation}

\begin{equation}
H_\psi(X, Y) - (X\psi)(Y\psi) = - \frac{1}{n-2} S(X, Y) + \frac{r}{n(n-1)(n-2)} g(X, Y)
- \frac{\text{tr}(A)}{n(n-2)} \exp(-2\psi) \bar{g}(Ric(X), Y)
+ K_1 \bar{g}(X, Y) + F(X\psi)(Y\psi),
\end{equation}

\begin{equation}(\nabla_X\nu)(Y) = \frac{1}{n-2} \left[ S(Ric(X), Y) - \frac{r}{n-1} S(X, Y) 
+ (n-2) \varphi_2 g(X, Y) \right] + G\nu(X)\nu(Y),
\end{equation}
where $F, G \in \mathcal{F}(U, \phi)$, $K_1 = -\varrho_1 \exp(-2\psi) - \exp(-2\psi) g(\Phi, \Psi)$ and $\Psi$ is given by $g(X, \Psi) = X \psi$, and

\begin{align*}
(34) \quad & \varrho_1 = \frac{\Delta \varphi}{n} - \frac{1}{n(n - 2)} g(\text{Ric}(X), A(Y)) + \frac{r \text{tr}(A)}{n(n - 1)(n - 2)}, \\
(35) \quad & \varrho_2 = \frac{\Delta \nu}{n} - \frac{1}{n(n - 2)} |S|^2 + \frac{r^2}{n(n - 1)(n - 2)}
\end{align*}

with $\Delta \varphi, \Delta \nu$ standing for the traces of the Hessian $H \varphi$ and $\nabla \nu$ with respect to $g$. If $\Phi$ or $N$ is non-null, then $F = 0$ or $G = 0$ respectively.

**Proof.** Transvecting (25) with $\varphi^t$ and applying (27) and (30) we obtain

\begin{align*}
(36) \quad & \frac{1}{n - 2} \left[ (S^t_i a_{ti} - \frac{r}{n - 1} a_{ik}) \varphi_j - (S^t_i - \frac{r}{n - 1} g_{ik}) a_{jt} \varphi^t \\
& + S^t_j a_{it} - \varphi^t S^t_i a_{js} g_{ik} + (S^t_i a_{tj} - \frac{r}{n - 1} a_{jk}) \varphi_i \\
& - (S^t_j - \frac{r}{n - 1} g_{jk}) a_{it} \varphi^t + \varphi^t S^t_i a_{ij} - \varphi^t S^t_i a_{is} g_{jk} \right] \\
& = \varphi_i \varphi_j + \varphi_{ij} \varphi_i - \varphi_{ij} \varphi^t g_{jk} - \varphi_{ij} \varphi^t g_{ik},
\end{align*}

where $S^t_i = S^t_{ij} g^{ij}$. Transvecting (36) with $g^{ik}$ and making use of (21) we get

\begin{align*}
(37) \quad & \varphi_i \varphi^t = \varrho_1 \varphi_i + \frac{1}{n - 2} S^t_i a_{ts} \varphi^s - \frac{r}{n(n - 1)(n - 2)} a_{it} \varphi^t \\
& - \frac{\text{tr}(A)}{n(n - 2)} S^t_i \varphi^t,
\end{align*}

where

\begin{align*}
& \varrho_1 = \frac{\Delta \varphi}{n} - \frac{1}{n(n - 2)} S^t_{st} a_{ts} + \frac{r \text{tr}(A)}{n(n - 1)(n - 2)}
\end{align*}

and $S^t_i = S^t_{ij} g^{ij}$. Substituting (37) into (36), in view of (22) and (23), we easily obtain (31). Hence, by metric contraction and the use of (34), we have either $F = 0$, provided that $\Phi$ is non-null, or $F \neq 0$, provided that $\Phi$ is null. Moreover, (4) and (5) yield $\varphi_i = -a_{it} \varphi^t$, whence, by covariant differentiation and the use of (3) and (31), we get (32). Finally, beginning with (9) relations (33) and (35) can be obtained in a similar way to (31) and (34). This completes the proof.

**Lemma 4.** If $(M, g)$ is a Sinyukov manifold, then

\[ X \varphi = \omega \nu(X) \]

on $U$, where $\omega \in \mathcal{F}(U, \phi)$ and $X \in \mathcal{X}(U)$. 

Proof. Consider the following two cases.

(i) The vector field $\Phi$ is null (see (24)). Since $\varphi(\text{Ric}(X), Y) = \varphi(X, \text{Ric}(Y))$ (cf. [8], p. 294), by (32), we get $X\varphi = -\tau(X\psi)$, $\tau \in \mathfrak{F}(U_\varphi)$. It follows that if $\Phi$ is null, then so is $\Psi$. From (4) and (5) we have

$$a_{it}\varphi^t = \tau\varphi_i.$$  

Moreover, (37) yields

$$S_{it}\varphi^t = \left(\tau - \frac{\text{tr}(A)}{n}\right)\varphi_i = \left(\frac{rr}{n(n-1)} - (n-2)g_{1}\right)\varphi_i.  \tag{38}$$

In a local chart, (23) takes the form

$$\varphi_i(S_{jk} - \frac{r}{n}g_{jk}) = \varphi_j(S_{ik} - \frac{r}{n}g_{ik}) = \nu_i \left(a_{jk} - \frac{\text{tr}(A)}{n}g_{jk}\right) - \nu_j \left(a_{ik} - \frac{\text{tr}(A)}{n}g_{ik}\right), \tag{39}$$

whence, transvecting with $\varphi^k$ and making use of (38) and (39), we get

$$S_{it}\varphi^t = \tau_1\varphi_i, \tag{40}$$

where $\tau_1 \in \mathfrak{F}(U_\varphi)$. Differentiating covariantly (41) and transvecting the resulting equation with $\varphi^t$ we find $\nu_t\varphi^t = 0$. Finally, transvecting (40) with $\varphi^t$ we obtain

$$\varphi_i(S_{tk}\varphi^t - \frac{r}{n}\varphi_k) = \nu_i \left(\tau - \frac{\text{tr}(A)}{n}\right)\varphi_k.  \tag{41}$$

Consider two cases.

1) $\tau = \text{tr}(A)/n$. Then $S_{tk}\varphi^t = (r/n)\varphi_k$ at each point where $\varphi_i \neq 0$. Differentiating covariantly with respect to $x^l$ and alternating the resulting equation in $i, l$, in view of (9) and (18), we have $\varphi_i\nu_l = \varphi_l\nu_i$, and the result follows.

2) $\tau \neq \text{tr}(A)/n$. Alternating the above result in $i, k$ and applying (41) we obtain the assertion.

(ii) The vector field $\Phi$ is non-null. Differentiating covariantly (22) and alternating the resulting equation, by (21), (18), (3), (31) and (33), we obtain

$$\frac{2(n+2)}{n} \left(\varphi_i\nu_j - \varphi_j\nu_i\right) + G(a_{it}\nu^t\nu_j - a_{jt}\nu^t\nu_i) \tag{42}$$

$$= F(S_{it}\varphi^t\varphi_j - S_{jt}\varphi^t\varphi_i).$$

If $\Phi$ and $N$ are non-null, then the result follows from (42) and Proposition 2. Finally, let $N$ be a null vector field. Differentiating (33) covariantly, then applying the Ricci identity and comparing the resulting equation to (29), in
of (9) and (18), we have

\begin{equation}
\rho_{2,i} = \frac{2}{n-2} S_{it} \nu^t + \omega_{3,i},
\end{equation}

where \(\omega_{3} \in \mathcal{F}(U_\varphi)\). On the other hand, (33) gives

\[
S_t^i S_{ip} \nu^p - \frac{r}{n-1} S_{it} \nu^t + (n-2) \rho_2 \nu_i = 0.
\]

Differentiating covariantly with respect to \(x^k\), then transvecting with \(\nu^k\), by the use of (43), we obtain

\[
S_{tp} \nu^t \nu^p = 0.\]

Hence, by transvection of (40) with \(\nu^i \nu^j\), we have

\begin{equation}
-\varphi_i \nu^t \left( S_{it} \nu^t - \frac{r}{n} \nu_i \right) = \nu_t a_{ip} \nu^p \nu^p.
\end{equation}

Now, transvecting (40) with \(\varphi^i \nu^j\) and applying the last result, we get

\[
S_{it} \nu^t = \tau_{2,t} \nu_i, \quad \tau_{2} \in \mathcal{F}(U_\varphi).
\]

Therefore, transvecting (40) with \(\nu^i \nu^j\) and using (22), we obtain

\[
a_{it} \nu^t = \tau_{3,t} \nu_i, \quad \tau_{3} \in \mathcal{F}(U_\varphi),
\]

whence, by (42), we have \(\varphi_i = \omega \nu_i\) again. From the above considerations it follows that the case when \(\Phi\) is null-buth \(N\) is null does not occur. This completes the proof.

4. Main results. From (23) and Lemma 4 it follows that

\begin{equation}
\omega \left( S_{ij} - \frac{r}{n} g_{ij} \right) = a_{ij} - \frac{\text{tr}(A)}{n} g_{ij} + B \nu_i \nu_j,
\end{equation}

in a local chart \((U, x)\), where \(B \in \mathcal{F}(U_\varphi)\). Transvecting (45) with \(g^{ij}\) we get

\[
B \nu_i \nu^p = 0.
\]

Hence and from Lemma 4 it follows that if the vector field \(\Phi\) is non-null, then \(B = 0\). Now we shall prove

**Proposition 3.** Assuming that (3) and (9) are satisfied at a point \(p \in U_\varphi\) and \(\Phi\) is a null vector field, we have \(\nu(X) = 0\), \(X \in T_p(M)\), and \((U, g)\) is an Einstein manifold.

**Proof.** Suppose that the vector field \(\Phi\) is null. Differentiating covariantly (45), then making use of (3), (9) and (18) we get

\begin{equation}
\omega_k \left( S_{ij} - \frac{r}{n} g_{ij} \right) = B_k \nu_i \nu_j + B(\nu_i ; k \nu_j + \nu_i \nu_{j,k}),
\end{equation}

where \(\omega_k = X_k \omega, B_k = X_k B\) and the semicolon stands for covariant differentiation on \((M, g)\). If \(\omega = \text{const}\), then \(\nu_i = 0\) is a consequence of the results of [10]. If \(\omega_k \neq 0\) at \(p\), then, by covariant differentiation of \(\varphi_i = \omega \nu_i\), we obtain \(X_i \omega = \omega_1 \nu_i\) and \(X_i (\omega_1) = \omega_2 \nu_i\), where \(\omega_1, \omega_2 \in \mathcal{F}(U)\). Moreover,
differentiating covariantly (46) and applying the Ricci identity, by (29), we find
\[(47)\]
\[\nu_i T_{jkl} + \nu_j T_{ikl} = 0,\]
where
\[(48)\]
\[T_{jkl} = \left[ \frac{B}{n - 2} S_{jk} - \left( \frac{Br}{n(n - 1)(n - 2)} - \omega_1 \right) g_{jk} \right] \nu_l - \left[ \frac{B}{n - 2} S_{jl} - \left( \frac{Br}{n(n - 1)(n - 2)} - \omega_1 \right) g_{jl} \right] \nu_k.\]

If \(\nu_i \neq 0\), then (47) results in \(T_{jkl} = 0\). Thus, by (48),
\[\frac{B}{n - 2} S_{ij} - \left( \frac{Br}{n(n - 1)(n - 2)} - \omega_1 \right) g_{ij} = B_1 \nu_i \nu_j, \quad B_1 \in \mathfrak{F}(U).\]

Hence, metric contraction with respect to \(i, j\) gives
\[(49)\]
\[\omega_1 = -\frac{Br}{n(n - 1)}.\]

Therefore
\[(50)\]
\[S_{ij} - \frac{r}{n} g_{ij} = B_2 \nu_i \nu_j,\]
where \(B_2 = (n - 2)B_1/B\).

From (49) it follows that if \(r \neq 0\), then \(B_i = B_3 \nu_i, B_3 \in \mathfrak{F}(U)\). Substituting (50) into (46) and taking into account the above considerations, we obtain \(\nu_i \nu_{j,k} - \nu_k \nu_{j,i} = 0\) at each point where \(B \neq 0\). Hence
\[(51)\]
\[\nu_{i,j} = G_1 \nu_i \nu_j,\]
where \(G_1 \in \mathfrak{F}(U)\). From (50) we obtain \((S_{ij} - (r/n) g_{ij}) \nu_i^l = 0\), whence, by covariant differentiation and the use of (51) and (18), we have \(\frac{n - 2}{n} \nu_i \nu_j = 0\).

From (49) it follows that if \(r \neq 0\), then \(B_i = B_3 \nu_i, B_3 \in \mathfrak{F}(U)\). Substituting (50) into (46) and taking into account the above considerations, we obtain \(\nu_i \nu_{j,k} - \nu_k \nu_{j,i} = 0\) at each point where \(B \neq 0\). Hence
\[(51)\]
\[\nu_{i,j} = G_1 \nu_i \nu_j,\]
where \(G_1 \in \mathfrak{F}(U)\). From (50) we obtain \((S_{ij} - (r/n) g_{ij}) \nu_i^l = 0\), whence, by covariant differentiation and the use of (51) and (18), we have \(\frac{n - 2}{n} \nu_i \nu_j = 0\).

From Theorem 2 we obtain
Corollary 1. On a Sinyukov manifold $X\varphi = \omega \nu(X)$, $\omega = \text{const} \neq 0$.

Corollary 2 ([9]). A Sinyukov manifold $(M, g)$ always admits a non-trivial geodesic mapping onto a pseudo-Riemannian manifold.

Moreover, from Proposition 3 we have

Corollary 3. On a Sinyukov manifold the vector field $\Phi$ is non-null.

Now we shall prove

Proposition 4. Suppose that $(M, g)$ is a Sinyukov manifold and let $\bar{g}$ be a metric satisfying (2), i.e. $\bar{g}$ is geodesically corresponding to $g$. If $p \in U_\varphi$, then

$$H_\varphi(X, Y) - (X\varphi)(Y\varphi) = -\frac{1}{(n-2)}a(X, Y) + Kg(X, Y) + K\bar{g}(X, Y)$$

at $p$, where $X, Y \in T_p(M)$,

$$K = -\frac{2}{n-2}(\sigma + c) + \frac{r}{(n-2)(n-1)}$$

and

$$K = \left[ \frac{\text{tr}(A)}{n(n-2)}(\sigma + c) - \varphi_1 - g(\varphi, \Phi) \right] \exp(-2\psi) = \text{const},$$

$H_\varphi$ is the Hessian of the function $\psi$, $\varphi_1$ is given by (34) and $X(\sigma + c) = \sigma(X)$.

Proof. Equation (53) results immediately from (32) and (52). Differentiating covariantly (31) and applying the Ricci identity, by (3), (9), (52) and (28), we obtain

$$X_i(\varphi_1) = \frac{2}{n-2}S_{\varphi \varphi}^j + \left[ \frac{2r}{(n-2)^2(n-1)} - \frac{2(n+2)}{n(n-2)^2}(\sigma + c) \right] (X_i \varphi).$$

Then differentiating covariantly (54), by (3), (31), (32), (52) and the above identity, we easily find that $K$ is constant on $U_\varphi$. Thus the proposition is proved.

Theorem 2 and Proposition 4 result in

Corollary 4. On a Sinyukov manifold,

$$g(\varsigma, C(X, Y)Z) = 0.$$

Moreover, on $(M, \tilde{g} = \exp(2\psi)g)$ the tensor field $\tilde{L}$ given by (14) is a Codazzi tensor.

Proof. (52), (4) and (5) yield $-X\varphi = \omega[S(X, \varsigma) - (\sigma + c)(X\varphi)]$. Differentiating covariantly (53) and applying the Ricci identity, in view of
Suppose that the 1-form $d\psi$ defines a geodesic mapping of a Sinyukov manifold $(M, g)$ onto a pseudo-Riemannian manifold $(M, \bar{g})$. Theorem 1 states that the manifold $(M, a)$, where $a$ is given by (3), admits a geodesic mapping onto the manifold $(M, \tilde{g} = \exp(2\psi)g)$ determined by the same 1-form $d\psi$.

Theorem 3 ([5]). A manifold $(M, g)$ (dim $M \geq 3$) is a conformally flat Sinyukov manifold if and only if $(M, \tilde{g} = \exp(2\psi)g)$ is of constant sectional curvature.

From (2) and (10) we obtain Lemma 5. On a manifold $(M, \tilde{g} = \exp(2\psi)g)$,
\begin{equation}
(\tilde{\nabla}_Y g)(X,Z) = \tilde{\varphi}(Y)\tilde{g}(X,Z) + \tilde{\varphi}(Z)\tilde{g}(X,Y),
\end{equation}
where $\tilde{\varphi}(X) = \tilde{\vartheta}(X,\Psi)\exp(-2\psi)$. Thus, on $(M, \tilde{g})$ the tensor $\tilde{\vartheta}$ satisfies the same condition as does the tensor $a$ on $(M, g)$.

Theorem 4. Suppose that a manifold $(M, g)$ admits a non-trivial geodesic mapping onto a manifold $(M, \bar{g})$ defined by a 1-form $d\psi$. Let $U_C = \{p \in M : C \neq 0 \text{ at } p\}$, where $C$ is the Weyl conformal curvature tensor. Then $(U_C, g)$ is a Sinyukov manifold if and only if either
\begin{enumerate}
  \item $(U_C, \tilde{g} = \exp(2\psi)g)$ is an Einstein manifold which admits a geodesic mapping determined by the 1-form $-d\psi$, or
  \item $(U_C, \tilde{g} = \exp(2\psi)g)$ is a Sinyukov manifold which admits a geodesic mapping determined by the 1-form $-d\psi$.
\end{enumerate}

Proof. On $(M, \tilde{g} = \exp(2\psi)g)$, by (12)–(14), (52) and Proposition 4, we get
\begin{equation}
\tilde{S}(X,Y) = (n-2)\tilde{K}\tilde{g}(X,Y) + \tilde{K}\tilde{g}(X,Y),
\end{equation}
where
\begin{equation*}
\tilde{K} = \frac{\tilde{r}}{2(n-1)} + \frac{r}{2(n-1)} \exp(-2\psi) - (\sigma + c) \exp(-2\psi) + \frac{n-2}{n} g(\Psi, \Psi) \exp(-2\psi).
\end{equation*}
Differentiating covariantly (57) and making use of (56) we have
\begin{equation}
(\tilde{\nabla}_Z \tilde{S})(X,Y) = \tilde{\nu}(X)\tilde{g}(Y,Z) + \tilde{\nu}(Y)\tilde{g}(X,Z) + \tilde{\sigma}(Z)\tilde{g}(X,Y),
\end{equation}
where $\tilde{\nu}(X) = (n-2)\tilde{K}\tilde{\varphi}(X)$, $\tilde{\sigma}(X) = X(\tilde{K})$. As in [9], p. 131 (see also Lemma 2), one can prove that
\begin{equation}
\tilde{\nu}(X) = \frac{n-2}{2(n-1)(n-2)}(X\tilde{r}), \quad \tilde{\sigma}(X) = \frac{n}{(n-1)(n+2)}(X\tilde{r}).
\end{equation}
Consider the following two cases.

(i) The scalar curvature $\tilde{r}$ of $(M, \tilde{g})$ is constant. Since $(M, \tilde{g})$ admits a non-trivial geodesic mapping onto $(M, a)$, we see, by the above considerations, that $\tilde{r} = \text{const}$ if and only if $K = 0$. Then (57) implies that $(M, \tilde{g})$ is an Einstein manifold. Conversely, if $(M, \tilde{g})$ is an Einstein manifold which admits a geodesic mapping corresponding to $-d\psi$, then, as in [9], p. 130 (see also [12]), we easily conclude that $(M, g)$ is a Sinyukov manifold.

(ii) If $\tilde{r}$ is not constant, then from (58) and (59) it follows that $(M, \tilde{g})$ is a Sinyukov manifold. This completes the proof.

Notice that if $(M, \tilde{g})$ is an Einstein manifold, then, by the results of [6], so is $(M, a)$. Hence and from Theorem 4 we have

**Corollary 5.** If $\tilde{g} = \exp(2\psi)g$ is an Einstein metric, then $\tilde{a} = \exp(2\psi)a$ is a Sinyukov metric.

5. Local structure theorem. The local structure theorem for conformally flat Sinyukov manifolds is given in [5]. Let $a$ be a differentiable symmetric bilinear form on $U_a \subseteq M$ satisfying (3) and having $t$ different eigenvalues $\lambda_1, \ldots, \lambda_t$. From the very definition, at each point $p \in U_a$ they coincide with the eigenvalues of the endomorphism $A_p$ of the tangent space $T_p(M)$ corresponding to $a$, i.e. $g(AX, Y) = a(X, Y)$ for all $X, Y \in X(U_a)$.

Let $(U, x)$ be a chart on $M$ such that $U \subseteq U_a$. Suppose that $\alpha v$ is an eigenvector of the matrix $a_{ij}$ corresponding to the eigenvalue $\lambda$, i.e. satisfying the condition

\[(a_{ij} - \lambda g_{ij}) \alpha v^j = 0.\]

Following [4], one can prove that $\psi \alpha v^i = 0$ and $\alpha v_i = \bar{B}(X_i, \alpha)$, where $\bar{B} \in \mathfrak{g}(U)$. Transvecting (60) covariantly (61) with $\psi^i$ and making use of (4) and (5) we have

\[\psi^i \alpha v^i = 0.\]

From [2], it follows that if $(M, g)$ admits a geodesic mapping then $\exp(-2\psi) = \prod_{\alpha=1}^t (f_\alpha)^{\tau_\alpha}$, where $\tau_\alpha$ denotes the algebraic multiplicity of $\lambda = f_\alpha(x^{n_\alpha} + \tau_\alpha)$, $n_1 = 0$, $n_\beta = \tau_1 + \ldots + \tau_{\beta-1}$, $\beta = 2, \ldots, t$. Hence

\[\alpha v_i = \bar{F}\psi_i \quad \text{and} \quad \alpha v_j = 0 \quad \text{for} \quad j \neq i,\]

where $i_\alpha = n_\alpha + 1, \ldots, n_\alpha + \tau_\alpha$, $\bar{F} \in \mathfrak{g}(U)$, $\alpha = 1, \ldots, t$.

**Lemma 6.** On a Sinyukov manifold the eigenvectors of the matrix $a_{ij}(p)$, $p \in U_a$, are non-null.

**Proof.** Suppose, to the contrary, that the eigenvector $\alpha v$ corresponding to the eigenvalue $\lambda$ is a null vector. Differentiating covariantly (61) with
respect to $x^k$, then transvecting the resulting equation with $\bar{v}^i$ and applying the relation $\psi_{ij\alpha} = 0$ for $j \neq j_i$, we obtain $\psi_{k\ell} \bar{v}^\ell = 0$. Therefore, from (53) and (60), we have

$$-\frac{1}{(n-2)\omega} \lambda + K + R(a)_{\beta}^{-1} \prod_{\beta=1}^t (f_{\beta})^{-\tau_{\beta}} = 0.$$  

Since

$$X_i K = -\frac{2}{(n-2)\omega} \varphi_i$$  

and $\varphi = \frac{1}{2} \sum_{\beta=1}^t \tau_{\beta} f_{\beta}$

(see [2]), it is easily seen that the above relation is false if the manifold admits a non-trivial geodesic mapping. This completes the proof.

Assume that a manifold $(M, g)$ admits a geodesic mapping onto a manifold $(\overline{M}, \overline{g})$. If at $p \in M$ the eigenvectors of the matrix $a_{ij}(p)$ are non-null, then in some neighbourhood of $p$ there exists a coordinate system such that the components of the metric tensors $g$ and $\overline{g}$ take the form ([2])

$$g_{\mu\nu} = e_\mu \prod_{\beta=1}^t (f_{\beta} - f_{\mu})^{\tau_{\nu}}; \quad \overline{g}_{\mu\nu} = \prod_{\beta=1}^t (f_{\beta})^{-\tau_{\mu}} (f_{\beta})^{-1} g_{\mu\nu},$$

(62)

$$g_{\nu j\alpha} = \prod_{\beta=1}^t (f_{\beta} - f_{\nu})^{\gamma_{\beta}} g_{\nu j\alpha}; \quad \overline{g}_{\nu j\alpha} = \prod_{\beta=1}^t (f_{\beta})^{-\tau_{\nu}} (f_{\beta})^{-1} g_{\nu j\alpha},$$

where $f_{\mu} = f_{\mu}(x^\mu), \overline{g}_{\nu j\alpha}$ is const $\neq 0$, $e_\mu = \pm 1, \mu = 1, \ldots, k, \nu = k + 1, \ldots, t, t \leq 2k + 1, \tau_1 = \ldots = \tau_k = 1, \tau_\nu > 1, i_\nu, j_\nu = n_\nu + 1, n_\nu + 2, \ldots, n_\nu + \tau_\nu, n_1 = 0, n_\nu = \tau_1 + \tau_2 + \ldots + \tau_{\nu-1}, \gamma = 2, \ldots, t$ and $\overline{g}_{\nu j\alpha}(x^{n_\nu+1}, \ldots, x^{n_\nu+\tau_\nu})$ are metric tensors on $\tau_\nu$-dimensional submanifolds $\overline{M}, \exp(-2\psi) = \prod_{\alpha=1}^k (f_{\alpha})^{\tau_{\alpha}}$.

The following lemma is a consequence of (25), (31), (15) and (52).

**Lemma 7.** If $(M, g)$ is a Sinyukov manifold and the Weyl conformal curvature tensor $C \neq 0$ at a point $p$ then, at $p$,

$$a(X, C(Y, Z)V) + a(V, C(Y, Z)X) = 0.$$  

(63)

Taking into account (63) in the coordinate system in which the metric has the form (62) and applying the equality $a_{\nu j\alpha} = f_{\nu} g_{\nu j\alpha}$, we find

**Lemma 8.** If $\overline{g}$ are metrics of one-dimensional manifolds, then the adjoint metric

$$\overline{g} = \sum_{\mu=1}^k \prod_{\beta=1}^t (f_{\beta} - f_{\mu})^{\tau_{\beta}} (dx^\mu)^2 + \sum_{\nu=k+1}^t \prod_{\beta=1}^t (f_{\beta} - f_{\nu})^{\tau_{\beta}} (dy^\nu)^2$$

...
is a metric of a conformally flat manifold. In particular, if $a_{ij}(p), p \in U_a$, has $n$ distinct eigenvalues, then $(U_a, g)$ is a conformally flat Sinyukov manifold.

**Theorem 5.** Suppose that a 1-form $d\psi$ defines a geodesic mapping of a Sinyukov manifold $(M, g)$ with $C \neq 0$ everywhere on $M$. If $\tilde{g} = \exp(2\psi)g$ is a Sinyukov metric, then on a neighbourhood of each point $p \in M$ there exists a coordinate system such that the metrics $g$ and $\tilde{g}$ take one of the following forms:

(i) if $k = 1$ and $t = 2$, then

$$g = \frac{1}{4(c_4 - x^4)W_4(x^4)}(dx^4)^2 + (c_5 - x^4)h_{\alpha\beta}dx^\alpha dx^\beta, \quad \tilde{g} = (x^4)^{-1}g,$$

where

$$W_1(z) = A_2z^2 + A_1z + A_0,$$

$$A_0, A_1, c_4 = \text{const} \neq 0, A_1 = \text{const}, \quad h = \frac{2}{c_5}(x^2, \ldots, x^n) \text{ is an } (n - 1)-\text{dimensional Einstein metric with the Ricci tensor }$$

$$\mathcal{S} = -(n - 2)W_4(c_4)^2h,$$

$\alpha, \beta = 2, \ldots, n$;

(ii) if $k = 1$ and $t = 3$, then we have

$$g = \frac{(n - 2)\omega}{W_2(x^4)}(dx^4)^2 + \sum_{\varphi = 2}^{4}(c_6 - x^4)\eta_{\varphi j, \rho}dx^\rho dx^{\varphi j}, \quad \tilde{g} = (x^4)^{-1}g,$$

where

$$W_2(z) = 4(c_2 - c_3)(c_3 - z)(c_1 + z),$$

$c_4, c_1, \omega = \text{const} \neq 0, \quad g = 2, 3, \quad h = \frac{2}{c_7}(x^2, \ldots, x^{\tau_1 + 1}) \text{ is a } \tau_2-\text{dimensional Einstein metric with the Ricci tensor }$$

$$\mathcal{S} = (\tau_2 - 1)(c_2 - c_3)(c_1 + c_2)^2h,$$

$$3h = 3h(x^{\tau_2 + 2}, \ldots, n) \text{ is a } \tau_3-\text{dimensional Einstein metric with the Ricci tensor }$$

$$\mathcal{S} = (\tau_3 - 1)(c_3 - c_2)(c_1 + c_3)^3h,$$

$K = \frac{1}{(n - 2)\omega}, \quad i_2, i_3, j_2 = 2, \ldots, \tau_2 + 1, \quad i_3, j_3 = \tau_2 + 2, \ldots, n, \quad 1 + \tau_2 + \tau_3 = n$;

(iii) if $k > 1$, then

$$g = \sum_{\mu = 1}^{k} \prod_{\eta = 1}^{k} x^{\eta} - x^{\mu} \frac{1}{W_4(x^{4})}(dx^4)^2 + \sum_{\varphi = k + 1}^{t} \prod_{\eta = 1}^{k} (f_\varphi - x^{\mu})\eta_{\varphi j, \rho}dx^\rho dx^{\varphi j},$$

$$\tilde{g} = (x^4 \ldots x^k)^{-1}g,$$
where
\[ W_3(z) = (-1)^{k+1} 4 A_{k+2} z^{k+2} + A_{k+1} z^{k+1} + A_1 z + 4 A_0, \]
where \( A_0, A_1, \ldots, A_{k+2} = \text{const} \), \( A_0, A_{k+2} \neq 0 \), \( f_0 = \text{const} \neq 0 \), and the \( f_0 \) are roots of the polynomial \( W_3 \).

\[ \hat{g} = (\tau_0 - 1) K_0 \hat{g} \]
and \( K_0 = (-1)^{k+1} \frac{1}{4} W_3'(f_0) \), \( k > 1 \), \( t \leq 2k + 1 \), \( \phi = k + 1, \ldots, t \), \( i, j_0 = n_0 + 1, \ldots, n_\phi + \tau_0, n_1 = 0, n_\gamma = \tau_1 + \tau_2 + \ldots + \tau_{\gamma-1}, \gamma = 2, \ldots, t, \tau_0 > 1. \)

**Proof.** Solving (53) in the local coordinate system in which \( g \) and \( \hat{g} \) are of the form (62) and using the equality \( a_{i_0 j_0} = f_\alpha g_{i_0 j_0} \), in the same way as in the proof of Theorem 3 of [1], we obtain our assertion.

**Theorem 6.** Let \( \mathbb{R}^n \) be endowed with a metric of the form either (64) or (65) or (66), where \( h, \hat{h} \) or \( \overline{h} \) and at least one of the forms \( \hat{g} \) are non-conformally flat Einstein metrics. Then \((\mathbb{R}^n, g)\) (and \((\mathbb{R}^n, \overline{g})\)) is a non-conformally flat Sinyukov manifold.

**Proof.** By elementary computation one can easily verify that (9) holds on \((\mathbb{R}^n, g)\) (and the analogous condition is satisfied on \((\mathbb{R}^n, \overline{g})\)). The components of the 1-form \( \sigma \) \((\nu = \frac{n-2}{2n} \sigma)\) are respectively:

1) for the metric (64):
\[ \sigma_1 = -n A_2, \sigma_\alpha = 0 \quad \left( \tilde{\sigma}_1 = \frac{-n A_0 c_1}{(x^1)^2}, \tilde{\sigma}_\alpha = 0 \right), \quad \alpha = 2, \ldots, n, \]

2) for the metric (65):
\[ \sigma_1 = \frac{n}{(n-2)\omega}, \sigma_\alpha = 0 \quad \left( \tilde{\sigma}_1 = \frac{-n c_1 c_2 c_3}{(n-2)\omega(x^1)^2}, \tilde{\sigma}_\alpha = 0 \right), \quad \alpha = 2, \ldots, n, \]

3) for the metric (66):
\[ \sigma_\mu = n A_{k+2}, \sigma_{i_\nu} = 0 \quad \left( \tilde{\sigma}_\mu = \frac{-n A_0}{(2\nu)^2}, \tilde{\sigma}_{i_\nu} = 0 \right), \quad \mu = 1, \ldots, k. \]

Moreover, in the metrics (64), (65) and (66), the conformal curvature tensor \( C \neq 0 \) if and only if \( h \) (resp. \( \hat{h} \) or \( \overline{h} \), resp. at least one of \( \hat{g} \)) is a non-conformally flat metric. This completes the proof.

**Remark.** In [4] the local structure theorem for Einstein manifolds admitting geodesic mappings is proved. If \( \overline{g} = \exp(2\psi)g \) is an Einstein manifold, then, by Theorem 4(i), Corollary 5 and the results of [4], the local structure of Sinyukov manifolds can be easily obtained. This, together
with Theorem 5, provides a complete description of the local structure of
Sinyukov manifolds.

From Theorems 5, 6 and the results of [4] we have the following

**Corollary 6.** If \( M \) is a Sinyukov manifold and \( \dim M \leq 4 \), then \( M \) is
conformally flat.

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