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ON SOME CLASS OF NEARLY CONFORMALLY SYMMETRIC MANIFOLDS

BY

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1. Introduction. Let (M,g) and $(\overline{M},\overline{g})$ be two Riemannian or pseudo-Riemannian manifolds of class C^∞. A mapping $\gamma:(M,g)\to(\overline{M},\overline{g})$ is said to be *geodesic* if it preserves geodesics, i.e. maps geodesics of (M,g) onto geodesics of $(\overline{M},\overline{g})$. The metrics g and $\overline{g}$ are then said to be *geodesically corresponding*.

Suppose that both g and $\overline{g}$ are metrics on the same manifold M. Let $\mathfrak{F}(M)$ be the ring of differentiable functions and $\mathfrak{X}(M)$ the $\mathfrak{F}$-module of differentiable vector fields on M. Each of the conditions below is necessary and sufficient for the metrics g and $\overline{g}$ to be geodesically corresponding:

$$\overline{\nabla}_X Y = \nabla_X Y + (X\psi)Y + (Y\psi)X, \tag{1}$$

$$(\nabla_X \overline{g})(Y,Z) = 2(X\psi)\overline{g}(Y,Z) + (Y\psi)\overline{g}(X,Z) + (Z\psi)\overline{g}(X,Y), \tag{2}$$

for all $X,Y,Z\in\mathfrak{X}(M)$, where $\psi\in\mathfrak{F}(M)$, and ∇ and $\overline{\nabla}$ are the Levi-Civita connections with respect to g and $\overline{g}$.

A manifold (M,g) admits a geodesic mapping if and only if there exist a function $\varphi\in\mathfrak{F}(M)$ and a symmetric non-singular bilinear form a on M satisfying

$$(\nabla_X a)(Y,Z) = (Y\varphi)g(X,Z) + (Z\varphi)g(X,Y) \tag{3}$$

for all $X,Y,Z\in\mathfrak{X}(M)$ ([9]).

In a chart (U,x) on M the local components of $g,\overline{g},a,X\varphi$ and $X\psi$ given by

$$g_{ij}=g(X_i,X_j),\quad \overline{g}_{ij}=\overline{g}(X_i,X_j),\quad a_{ij}=a(X_i,X_j),$$
$$\varphi_i = X_i\varphi,\quad \psi_i = X_i\psi$$

satisfy

$$a_{ij} = \exp(2\psi)\overline{g}^{st}g_{si}g_{tj}, \tag{4}$$

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$$\varphi_i = -\exp(2\psi)\overline{g}^{st} g_{si} \psi_t, \tag{5}$$

where $X_i = \partial/\partial x^i \in \mathfrak{X}(U)$ and the $\overline{g}^{ij}$ are the components of $(\overline{g}_{ij})^{-1}$.

A geodesic mapping is said to be *non-trivial* if it is non-affine, which is equivalent to $\varphi \neq \text{const}$ on M.

By (1), the curvature tensors and Ricci tensors of (M, g) and $(M, \overline{g})$ are related by

$$\overline{R}(X,Y)Z = R(X,Y)Z + P(X,Z)Y - P(Y,Z)X, \tag{6}$$

$$\overline{S}(X,Y) = S(X,Y) + (n-1)P(X,Y), \tag{7}$$

where

$$P(X,Y) = H_\psi(X,Y) - (X\psi)(Y\psi) \tag{8}$$

and H_ψ is the Hessian of ψ.

Following W. Roter ([7]), the manifold (M, g) is said to be *nearly conformally symmetric* if the tensor

$$L(X,Y) = \frac{1}{n-2}\left[S(X,Y) - \frac{r}{2(n-1)} g(X,Y)\right]$$

is a Codazzi tensor, where S is the Ricci tensor and r denotes the scalar curvature.

N. S. Sinyukov ([9]) and E. N. Sinyukova ([10]) investigated manifolds whose Ricci tensor satisfies

$$(\nabla_X S)(Y,Z) = \sigma(X)g(Y,Z) + \nu(Y)g(X,Z) + \nu(Z)g(X,Y), \tag{9}$$

where σ and ν are some 1-forms. Such manifolds are known under different names (see [9], [10], [3]). In what follows a Riemannian (or pseudo-Riemannian) manifold satisfying (9) with non-constant scalar curvature will be called a *Sinyukov manifold*. Such manifolds always admit non-trivial geodesic mappings and every Sinyukov manifold is nearly conformally symmetric (see Lemma 1).

Let (M, g) admit a non-trivial geodesic mapping onto the manifold $(M, \overline{g})$ defined by the 1-form $d\psi$ (see (2)). In [5] it was proved that (M, g) ($\dim M \geq 3$) is a conformally flat Sinyukov manifold if and only if $(M, \widetilde{g} = \exp(2\psi)g)$ is of constant sectional curvature. In the present paper we prove that (M, g) with nowhere vanishing Weyl conformal curvature tensor is a Sinyukov manifold if and only if $(M, \widetilde{g} = \exp(2\psi)g)$ is either an Einstein manifold admitting non-trivial geodesic mappings or a Sinyukov manifold.

In [3] some properties of Sinyukov manifolds with non-null vector Φ defined by (24) below were investigated. In the present paper we deal with manifolds without any assumption on Φ. Finally, the local structure theorem for Sinyukov manifolds is given.

2. Preliminaries. If $\widetilde{g}$ is a metric on M and there exists $\lambda \in \mathfrak{F}(M)$ such that $\widetilde{g} = \exp(2\lambda)g$, then g and $\widetilde{g}$ are said to be *conformally related* or conformal to each other, and the transformation $g \to \widetilde{g}$ is called a *conformal change*. As is well-known, the Christoffel symbols, the curvature tensors and the Ricci tensors of the manifolds (M, g) and $(M, \widetilde{g})$ are then related by

$$\widetilde{\nabla}_X Y = \nabla_X Y + (X\lambda)Y + (Y\lambda)X - g(X, Y)\Lambda, \tag{10}$$

where the vector field Λ is defined by $g(X, \Lambda) = X\lambda$ for $X \in \mathfrak{X}(M)$;

$$\begin{aligned} g(\widetilde{R}(X, Y)Z, V) = g(R(X, Y)Z, V) &\\ + Q(X, Z)g(Y, V) - Q(Y, Z)g(X, V) &\\ + g(X, Z)Q(Y, V) - g(Y, Z)Q(X, V) & \end{aligned} \tag{11}$$

for arbitrary $X, Y, Z, V \in \mathfrak{X}(M)$;

$$Q(X, Y) + L(X, Y) = \widetilde{L}(X, Y), \tag{12}$$

where the tensor fields Q and L are given by

$$Q(X, Y) = H_\lambda(X, Y) - (X\lambda)(Y\lambda) + \tfrac{1}{2}g(\Lambda, \Lambda)g(X, Y), \tag{13}$$

$$L(X, Y) = \frac{1}{n-2}\left[S(X, Y) - \frac{r}{2(n-1)}g(X, Y)\right] \tag{14}$$

with H_λ being the Hessian of λ, and r standing for the scalar curvature of (M, g). The tensor field $\widetilde{L}$ on $(M, \widetilde{g})$ is defined analogously.

The Weyl conformal curvature tensor C satisfying

$$\begin{aligned} g(C(X, Y)Z, V) = g(R(X, Y)Z, V) + g(X, V)L(Y, Z) - g(Y, V)L(X, Z) &\\ + L(X, V)g(Y, Z) - L(Y, V)g(X, Z) & \end{aligned} \tag{15}$$

is invariant under conformal change, i.e. $\widetilde{C} = C$.

From (12) and (13) we get easily

$$g(C(X, Y)Z, \Lambda) = D(X, Y, Z) - \widetilde{D}(X, Y, Z), \tag{16}$$

where

$$D(X, Y, Z) = (\nabla_X L)(Y, Z) - (\nabla_Y L)(X, Z) \tag{17}$$

and the tensor field $\widetilde{D}$ on $(M, \widetilde{g})$ is defined in the same manner.

In the sequel we shall use the following theorem and lemmas.

THEOREM 1 ([9]). *If (M, g) admits a non-trivial geodesic mapping onto a manifold $(M, \overline{g})$ defined by a 1-form $d\psi$, then the manifold (M, a), where a satisfies* (3), *admits a geodesic mapping onto $(M, \widetilde{g} = \exp(2\psi)g)$ determined by the same 1-form $d\psi$.*

LEMMA 1 ([12], [1]). *On a Sinyukov manifold the tensor D given by* (17) *vanishes*, *i.e. L is a Codazzi tensor.*

Lemma 2 ([9]). *If on* (M,g) *relation* (9) *is satisfied at a point* p, *then*

$$\sigma(X)=\frac{n}{(n-1)(n+2)}(Xr),\qquad \nu(X)=\frac{n-2}{2(n-1)(n+2)}(Xr) \tag{18}$$

for any $X\in T_p(M)$. *Consequently*, (M,g) *is a Sinyukov manifold if and only if the scalar curvature* $r\neq$const *and the condition* (9) *holds everywhere on* M.

We define (1,1) tensor fields Ric and A as follows:

$$g(\mathrm{Ric}(X),Y)=S(X,Y), \tag{19}$$

$$g(A(X),Y)=a(X,Y) \tag{20}$$

for all $X,Y\in\mathfrak{X}(M)$.

Lemma 3 ([10], [3]). *If* (M,g) *is a Sinyukov manifold and* $d\varphi\neq 0$ *at a point* $p\in M$, *then*

$$a(\mathrm{Ric}(X),Y)=a(X,\mathrm{Ric}(Y)), \tag{21}$$

$$a(X,N)-\frac{\mathrm{tr}(A)}{n}\nu(X)=S(X,\Phi)-\frac{r}{n}(X\varphi), \tag{22}$$

$$\begin{aligned}(X\varphi)\Big[S(Y,Z)-\frac{r}{n}g(Y,Z)\Big]-(Y\varphi)\Big[S(X,Z)-\frac{r}{n}g(X,Z)\Big]\\ =\nu(X)\Big[a(Y,Z)-\frac{\mathrm{tr}(A)}{n}g(Y,Z)\Big]-\nu(Y)\Big[a(X,Z)-\frac{\mathrm{tr}(A)}{n}g(X,Z)\Big]\end{aligned} \tag{23}$$

at p *for all* $X,Y,Z\in T_p(M)$, *where* N *and* Φ *are given by*

$$g(X,N)=\nu(X),\qquad g(X,\Phi)=X\varphi. \tag{24}$$

3. Properties of conformal and geodesic mappings of Sinyukov manifolds. Let $p\in M$ be such that $d\varphi\neq 0$ and (3) hold at p. Choose a local coordinate system (U,x) so that $p\in U$. By $R^l{}_{ijk}, S_{ij}, \varphi_{ij}$ we denote the components of the tensors R, S and H_φ in this coordinate system. Differentiating covariantly (3) and applying the Ricci identity we get

$$a_{it}R^t{}_{jkl}+a_{tj}R^t{}_{ikl}=\varphi_{li}g_{jk}+\varphi_{lj}g_{ik}-\varphi_{ki}g_{jl}-\varphi_{kj}g_{il}. \tag{25}$$

Differentiating covariantly (25) with respect to x^m, contracting with g^{lm} and applying the Ricci identity, by (3) and (9), we obtain

$$\begin{aligned}4\varphi_tR^t{}_{jki}&=\varphi_kS_{ij}-\varphi_iS_{jk}\\ &\quad+\frac{n+2}{n-2}[\nu^ta_{tk}g_{ij}-\nu^ta_{ti}g_{kj}+\nu_ia_{kj}-\nu_ka_{ij}]+b_ig_{jk}-b_kg_{ij},\end{aligned} \tag{26}$$

where the ν_i are the components of the 1-form ν, the ν^i are the components of the field N (i.e. $\nu^i=g^{it}\nu_t$) whereas $b_i=\varphi_{it;s}g^{ts}$ and the semicolon denotes covariant differentiation on (M,g). Moreover, substituting (22) and

(23) into (26), we get

$$4\varphi_t R^t{}_{jki} = \frac{4}{n-2}(\varphi_i S_{jk} - \varphi_k S_{ij}) + \frac{n+2}{n-2}(\varphi^t S_{tk} g_{ij} - \varphi^t S_{ti} g_{kj}) + b_i g_{jk} - b_k g_{ij}, \tag{27}$$

where $\varphi^i = \varphi_t g^{ti}$ are the components of the field Φ.

Now, we shall prove

PROPOSITION 1. *If* (M, g) *is a Sinyukov manifold and the Weyl conformal curvature tensor* $C \neq 0$ *and* $d\varphi \neq 0$ *at a point* $p \in M$, *then*

$$g(\Phi, C(X, Y)Z) = 0, \tag{28}$$

$$g(N, C(X, Y)Z) = 0 \tag{29}$$

on some neighbourhood U_1 *of* p, *where* Φ *and* N *are as in* (24). *So, for the metrics* $\widetilde{g}_1 = \exp(2\varphi)g$ *and* $\widetilde{g}_2 = \exp(2\nu)g$, *where* $\nu \in \mathfrak{F}(U_1)$ *and* $X\nu = \nu(X)$, *the tensors* $\widetilde{L}_1, \widetilde{L}_2$ *defined by* (14) *are Codazzi tensors.*

Proof. Transvecting (27) with g^{jk} we get

$$b_i = \frac{n+6}{n-2}\varphi^t S_{ti} - \frac{4r}{(n-1)(n-2)}\varphi_i. \tag{30}$$

Substituting (30) into (27), in view of (15), we find (28). Beginning with (9) and following the above argument we obtain (29). Thus the proposition is proved.

PROPOSITION 2. *If* (M, g) *is a Sinyukov manifold*, *then on the set* $U_\varphi = \{p \in M : d\varphi \neq 0 \text{ at } p\}$ *the following identities hold*:

$$H_\varphi(X, Y) = \frac{1}{n-2}\Big[a(\mathrm{Ric}(X), Y) - \frac{r}{n(n-1)}a(X, Y) - \frac{\mathrm{tr}(A)}{n}S(X, Y) + (n-2)\varrho_1 g(X, Y)\Big] + F(X\varphi)(Y\varphi), \tag{31}$$

$$H_\psi(X, Y) - (X\psi)(Y\psi) = -\frac{1}{n-2}S(X, Y) + \frac{r}{n(n-1)(n-2)}g(X, Y) - \frac{\mathrm{tr}(A)}{n(n-2)}\exp(-2\psi)\overline{g}(\mathrm{Ric}(X), Y) + \overline{K}_1\overline{g}(X, Y) + F(X\psi)(Y\varphi), \tag{32}$$

$$(\nabla_X \nu)(Y) = \frac{1}{n-2}\Big[S(\mathrm{Ric}(X), Y) - \frac{r}{n-1}S(X, Y) + (n-2)\varrho_2 g(X, Y)\Big] + G\nu(X)\nu(Y), \tag{33}$$

where $F, G \in \mathfrak{F}(U_\varphi)$, $\overline{K}_1 = -\varrho_1 \exp(-2\psi) - \exp(-2\psi) g(\Phi, \Psi)$ *and* Ψ *is given by* $g(X, \Psi) = X\psi$, *and*

$$\varrho_1 = \frac{\Delta\varphi}{n} - \frac{1}{n(n-2)} g(\mathrm{Ric}(X), A(Y)) + \frac{r\,\mathrm{tr}(A)}{n(n-1)(n-2)}, \tag{34}$$

$$\varrho_2 = \frac{\Delta\nu}{n} - \frac{1}{n(n-2)} |S|^2 + \frac{r^2}{n(n-1)(n-2)} \tag{35}$$

with $\Delta\varphi$, $\Delta\nu$ *standing for the traces of the Hessian* H_φ *and* $\nabla\nu$ *with respect to* g. *If* Φ *or* N *is non-null*, *then* $F = 0$ *or* $G = 0$ *respectively.*

Proof. Transvecting (25) with φ^l and applying (27) and (30) we obtain

$$\begin{aligned} \frac{1}{n-2}\bigg[\Big(S_k^t a_{ti} - \frac{r}{n-1} a_{ik}\Big)\varphi_j - \Big(S_{ik} - \frac{r}{n-1} g_{ik}\Big) a_{jt}\varphi^t \\ + S_j^t \varphi_t a_{ik} - \varphi^t S_t^s a_{js} g_{ik} + \Big(S_k^t a_{tj} - \frac{r}{n-1} a_{jk}\Big)\varphi_i \\ - \Big(S_{jk} - \frac{r}{n-1} g_{jk}\Big) a_{it}\varphi^t + \varphi^t S_{ti} a_{jk} - \varphi^t S_t^s a_{is} g_{jk}\bigg] \\ = \varphi_{ki}\varphi_j + \varphi_{kj}\varphi_i - \varphi_{ti}\varphi^t g_{jk} - \varphi_{tj}\varphi^t g_{ik}, \end{aligned} \tag{36}$$

where $S_i^j = S_{it} g^{tj}$. Transvecting (36) with g^{jk} and making use of (21) we get

$$\begin{aligned} \varphi_{it}\varphi^t = \varrho_1 \varphi_i + \frac{1}{n-2} S_i^t a_{ts}\varphi^s - \frac{r}{n(n-1)(n-2)} a_{it}\varphi^t \\ - \frac{\mathrm{tr}(A)}{n(n-2)} S_{it}\varphi^t, \end{aligned} \tag{37}$$

where

$$\varrho_1 = \frac{\Delta\varphi}{n} - \frac{1}{n(n-2)} S^{ts} a_{ts} + \frac{r\,\mathrm{tr}(A)}{n(n-1)(n-2)}$$

and $S^{ij} = S_t^i g^{tj}$. Substituting (37) into (36), in view of (22) and (23), we easily obtain (31). Hence, by metric contraction and the use of (34), we have either $F = 0$, provided that Φ is non-null, or $F \neq 0$, provided that Φ is null. Moreover, (4) and (5) yield $\varphi_i = -a_{it}\psi^t$, whence, by covariant differentiation and the use of (3) and (31), we get (32). Finally, beginning with (9) relations (33) and (35) can be obtained in a similar way to (31) and (34). This completes the proof.

LEMMA 4. *If* (M, g) *is a Sinyukov manifold*, *then*

$$X\varphi = \omega\nu(X)$$

on U_φ, *where* $\omega \in \mathfrak{F}(U_\varphi)$ *and* $X \in \mathfrak{X}(U_\varphi)$.

P r o o f. Consider the following two cases.

(i) The vector field Φ is null (see (24)). Since $\overline{g}(\mathrm{Ric}(X),Y)=\overline{g}(X,\mathrm{Ric}(Y))$ (cf. [8], p. 294), by (32), we get $X\varphi = -\tau(X\psi)$, $\tau \in \mathfrak{F}(U_\varphi)$. It follows that if Φ is null, then so is Ψ. From (4) and (5) we have

$$a_{it}\varphi^t = \tau\varphi_i. \tag{38}$$

Moreover, (37) yields

$$\left(\tau - \frac{\mathrm{tr}(A)}{n}\right)S_{it}\varphi^t = \left(\frac{r\tau}{n(n-1)} - (n-2)\varrho_1\right)\varphi_i. \tag{39}$$

In a local chart, (23) takes the form

$$\begin{aligned}\varphi_i\left(S_{jk} - \frac{r}{n}g_{jk}\right) - \varphi_j\left(S_{ik} - \frac{r}{n}g_{ik}\right)\\ = \nu_i\left(a_{jk} - \frac{\mathrm{tr}(A)}{n}g_{jk}\right) - \nu_j\left(a_{ik} - \frac{\mathrm{tr}(A)}{n}g_{ik}\right),\end{aligned} \tag{40}$$

whence, transvecting with φ^k and making use of (38) and (39), we get

$$S_{it}\varphi^t = \tau_1\varphi_i, \tag{41}$$

where $\tau_1 \in \mathfrak{F}(U_\varphi)$. Differentiating covariantly (41) and transvecting the resulting equation with φ^i we find $\nu_t\varphi^t = 0$. Finally, transvecting (40) with φ^j we obtain

$$\varphi_i\left(S_{tk}\varphi^t - \frac{r}{n}\varphi_k\right) = \nu_i\left(\tau - \frac{\mathrm{tr}(A)}{n}\right)\varphi_k.$$

Consider two cases.

1) $\tau = \mathrm{tr}(A)/n$. Then $S_{tk}\varphi^t = (r/n)\varphi_k$ at each point where $\varphi_i \neq 0$. Differentiating covariantly with respect to x^l and alternating the resulting equation in i, l, in view of (9) and (18), we have $\varphi_i\nu_l = \varphi_l\nu_i$, and the result follows.

2) $\tau \neq \mathrm{tr}(A)/n$. Alternating the above result in i, k and applying (41) we obtain the assertion.

(ii) The vector field Φ is non-null. Differentiating covariantly (22) and alternating the resulting equation, by (21), (18), (3), (31) and (33), we obtain

$$\begin{aligned}\frac{2(n+2)}{n}(\varphi_i\nu_j - \varphi_j\nu_i) + G(a_{it}\nu^t\nu_j - a_{jt}\nu^t\nu_i)\\ = F(S_{it}\varphi^t\varphi_j - S_{jt}\varphi^t\varphi_i).\end{aligned} \tag{42}$$

If Φ and N are non-null, then the result follows from (42) and Proposition 2. Finally, let N be a null vector field. Differentiating (33) covariantly, then applying the Ricci identity and comparing the resulting equation to (29), in view of (9) and (18), we have

$$\varrho_{2;i} = \frac{2}{n-2} S_{it}\nu^t + \varrho_3\nu_i, \tag{43}$$

where $\varrho_3 \in \mathfrak{F}(U_\varphi)$. On the other hand, (33) gives

$$S_i^t S_{tp}\nu^p - \frac{r}{n-1} S_{it}\nu^t + (n-2)\varrho_2\nu_i = 0.$$

Differentiating covariantly with respect to x^k, then transvecting with ν^k, by the use of (43), we obtain $S_{tp}\nu^t\nu^p = 0$. Hence, by transvection of (40) with $\nu^j\nu^k$, we have

$$-\varphi_t\nu^t \left(S_{it}\nu^t - \frac{r}{n}\nu_i \right) = \nu_i a_{tp}\nu^t\nu^p. \tag{44}$$

Now, transvecting (40) with $\varphi^i\nu^j$ and applying the last result, we get $S_{kt}\nu^t = (r/n)\nu_k$ at each point where $\varphi_t\varphi^t \neq 0$ and $\varphi_t\nu^t = 0$. Then transvection of (40) with ν^k results in $\nu_i a_{jt}\nu^t - \nu_j a_{it}\nu^t = 0$. Hence and from (42), by Proposition 2 we have $\varphi_i = \omega\nu_i$. On the other hand, if $\varphi_t\nu^t \neq 0$ in (44), then $S_{ti}\nu^t = \tau_2\nu_i$, $\tau_2 \in \mathfrak{F}(U_\varphi)$. Therefore, transvecting (40) with $\varphi^i\nu^j$ and using (22), we obtain $a_{it}\nu^t = \tau_3\nu_i$, $\tau_3 \in \mathfrak{F}(U_\varphi)$, whence, by (42), we have $\varphi_i = \omega\nu_i$ again. From the above considerations it follows that the case when Φ is non-null but N is null does not occur. This completes the proof.

4. Main results. From (23) and Lemma 4 it follows that

$$\omega\left(S_{ij} - \frac{r}{n} g_{ij} \right) = a_{ij} - \frac{\mathrm{tr}(A)}{n} g_{ij} + B\nu_i\nu_j, \tag{45}$$

in a local chart (U, x), where $B \in \mathfrak{F}(U_\varphi)$. Transvecting (45) with g^{ij} we get $B\nu_t\nu^t = 0$. Hence and from Lemma 4 it follows that if the vector field Φ is non-null, then $B = 0$. Now we shall prove

PROPOSITION 3. *Assuming that* (3) *and* (9) *are satisfied at a point* $p \in U_\varphi$ *and* Φ *is a null vector field*, *we have* $\nu(X) = 0$, $X \in T_p(M)$, *and* (U, g) *is an Einstein manifold.*

Proof. Suppose that the vector field Φ is null. Differentiating covariantly (45), then making use of (3), (9) and (18) we get

$$\omega_k\left(S_{ij} - \frac{r}{n} g_{ij} \right) = B_k\nu_i\nu_j + B(\nu_{i;k}\nu_j + \nu_i\nu_{j;k}), \tag{46}$$

where $\omega_k = X_k\omega$, $B_k = X_k B$ and the semicolon stands for covariant differentiation on (M, g). If $\omega = \text{const}$, then $\nu_i = 0$ is a consequence of the results of [10]. If $\omega_k \neq 0$ at p, then, by covariant differentiation of $\varphi_i = \omega\nu_i$, we obtain $X_i\omega = \omega_1\nu_i$ and $X_i(\omega_1) = \omega_2\nu_i$, where $\omega_1, \omega_2 \in \mathfrak{F}(U)$. Moreover, differentiating covariantly (46) and applying the Ricci identity, by (29), we find

$$\nu_i T_{jkl} + \nu_j T_{ikl} = 0, \tag{47}$$

where

$$T_{jkl} = \left[\frac{B}{n-2}S_{jk} - \left(\frac{Br}{n(n-1)(n-2)} - \omega_1\right)g_{jk}\right]\nu_l - \left[\frac{B}{n-2}S_{jl} - \left(\frac{Br}{n(n-1)(n-2)} - \omega_1\right)g_{jl}\right]\nu_k. \tag{48}$$

If $\nu_i \neq 0$, then (47) results in $T_{jkl} = 0$. Thus, by (48),

$$\frac{B}{n-2}S_{ij} - \left(\frac{Br}{n(n-1)(n-2)} - \omega_1\right)g_{ij} = B_1\nu_i\nu_j, \qquad B_1 \in \mathfrak{F}(U).$$

Hence, metric contraction with respect to i, j gives

$$\omega_1 = -\frac{Br}{n(n-1)}. \tag{49}$$

Therefore

$$S_{ij} - \frac{r}{n}g_{ij} = B_2\nu_i\nu_j, \tag{50}$$

where $B_2 = (n-2)B_1/B$.

From (49) it follows that if $r \neq 0$, then $B_i = B_3\nu_i$, $B_3 \in \mathfrak{F}(U)$. Substituting (50) into (46) and taking into account the above considerations, we obtain $\nu_i\nu_{j;k} - \nu_k\nu_{j;i} = 0$ at each point where $B \neq 0$. Hence

$$\nu_{i;j} = G_1\nu_i\nu_j, \tag{51}$$

where $G_1 \in \mathfrak{F}(U)$. From (50) we obtain $(S_{it} - (r/n)g_{it})\nu^t = 0$, whence, by covariant differentiation and the use of (51) and (18), we have $\frac{n-2}{n}\nu_i\nu_j = 0$. So, $\nu_i = 0$ if Φ is a null vector field. Then (9) results in $(\nabla_X S)(Y, Z) = 0$, which implies $(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) = 0$ for $X, Y, Z \in T_p(M)$. Now the second part of our proposition is a consequence of the results of [11]. This completes the proof.

From Lemma 4, Proposition 3 and the results of [10] (cf. [3]) we obtain

THEOREM 2. *A manifold (M, g) admitting a non-trivial geodesic mapping onto a pseudo-Riemannian manifold is a Sinyukov manifold if and only if both $r \neq$ const and the condition*

$$a(X, Y) = \omega[S(X, Y) - (\sigma + c)g(X, Y)] \tag{52}$$

holds everywhere, where $\omega = \text{const} \neq 0$, $c = \text{const}$, $\sigma \in \mathfrak{F}(M)$ and $X\sigma = \sigma(X)$, $X, Y \in \mathfrak{X}(M)$.

From Theorem 2 we obtain

COROLLARY 1. *On a Sinyukov manifold* $X\varphi = \omega\nu(X)$, $\omega = \text{const} \neq 0$.

COROLLARY 2 ([9]). *A Sinyukov manifold* (M, g) *always admits a non-trivial geodesic mapping onto a pseudo-Riemannian manifold.*

Moreover, from Proposition 3 we have

COROLLARY 3. *On a Sinyukov manifold the vector field* Φ *is non-null.*

Now we shall prove

PROPOSITION 4. *Suppose that* (M, g) *is a Sinyukov manifold and let* $\overline{g}$ *be a metric satisfying* (2), *i.e.* $\overline{g}$ *is geodesically corresponding to* g. *If* $p \in U_\varphi$, *then*

$$H_\psi(X,Y) - (X\psi)(Y\psi) = -\frac{1}{(n-2)\omega}a(X,Y) + Kg(X,Y) + \overline{K}\overline{g}(X,Y) \tag{53}$$

at p, *where* $X, Y \in T_p(M)$,

$$K = -\frac{2}{n-2}(\sigma + c) + \frac{r}{(n-1)(n-2)}$$

and

$$\overline{K} = \left[\frac{\operatorname{tr}(A)}{n(n-2)}(\sigma + c) - \varrho_1 - g(\Psi, \Phi)\right]\exp(-2\psi) = \text{const}, \tag{54}$$

H_ψ *is the Hessian of the function* ψ, ϱ_1 *is given by* (34) *and* $X(\sigma + c) = \sigma(X)$.

Proof. Equation (53) results immediately from (32) and (52). Differentiating covariantly (31) and applying the Ricci identity, by (3), (9), (52) and (28), we obtain

$$X_i(\varrho_1) = \frac{2}{n-2}S_{it}\varphi^t + \left[\frac{2r}{(n-2)^2(n-1)} - \frac{2(n+2)}{n(n-2)^2}(\sigma + c)\right](X_i\varphi).$$

Then differentiating covariantly (54), by (3), (31), (32), (52) and the above identity, we easily find that $\overline{K}$ is constant on U_φ. Thus the proposition is proved.

Theorem 2 and Proposition 4 result in

COROLLARY 4. *On a Sinyukov manifold*,

$$g(\Psi, C(X,Y)Z) = 0. \tag{55}$$

Moreover, *on* $(M, \widetilde{g} = \exp(2\psi)g)$ *the tensor field* $\widetilde{L}$ *given by* (14) *is a Codazzi tensor.*

Proof. (52), (4) and (5) yield $-X\varphi = \omega[S(X, \Psi) - (\sigma + c)(X\psi)]$. Differentiating covariantly (53) and applying the Ricci identity, in view of

(3), (2) and the above equation, we have (55). Together with (16) and (17), this implies that $\widetilde{L}$ is a Codazzi tensor. This completes the proof.

Suppose that the 1-form $d\psi$ defines a geodesic mapping of a Sinyukov manifold (M,g) onto a pseudo-Riemannian manifold $(M,\overline{g})$. Theorem 1 states that the manifold (M,a), where a is given by (3), admits a geodesic mapping onto the manifold $(M,\widetilde{g}=\exp(2\psi)g)$ determined by the same 1-form $d\psi$.

THEOREM 3 ([5]). *A manifold (M,g) $(\dim M\geq 3)$ is a conformally flat Sinyukov manifold if and only if $(M,\widetilde{g}=\exp(2\psi)g)$ is of constant sectional curvature.*

From (2) and (10) we obtain

LEMMA 5. *On a manifold $(M,\widetilde{g}=\exp(2\psi)g)$,*

$$(\widetilde{\nabla}_X\overline{g})(Y,Z)=\widetilde{\varphi}(Y)\widetilde{g}(X,Z)+\widetilde{\varphi}(Z)\widetilde{g}(X,Y), \tag{56}$$

where $\widetilde{\varphi}(X)=\overline{g}(X,\Psi)\exp(-2\psi)$. Thus, on $(M,\widetilde{g})$ the tensor $\overline{g}$ satisfies the same condition as does the tensor a on (M,g).

THEOREM 4. *Suppose that a manifold (M,g) admits a non-trivial geodesic mapping onto a manifold $(M,\overline{g})$ defined by a 1-form $d\psi$. Let $U_C=\{p\in M : C\neq 0 \text{ at } p\}$, where C is the Weyl conformal curvature tensor. Then (U_C,g) is a Sinyukov manifold if and only if either*

(i) $(U_C,\widetilde{g}=\exp(2\psi)g)$ *is an Einstein manifold which admits a geodesic mapping determined by the* 1*-form* $-d\psi$, *or*

(ii) $(U_C,\widetilde{g}=\exp(2\psi)g)$ *is a Sinyukov manifold which admits a geodesic mapping determined by the* 1*-form* $-d\psi$.

Proof. On $(M,\widetilde{g}=\exp(2\psi)g)$, by (12)–(14), (52) and Proposition 4, we get

$$\widetilde{S}(X,Y)=(n-2)\overline{K}\overline{g}(X,Y)+\widetilde{K}\widetilde{g}(X,Y), \tag{57}$$

where

$$\widetilde{K}=\frac{\widetilde{r}}{2(n-1)}+\frac{r}{2(n-1)}\exp(-2\psi)$$
$$-(\sigma+c)\exp(-2\psi)+\frac{n-2}{n}g(\Psi,\Psi)\exp(-2\psi).$$

Differentiating covariantly (57) and making use of (56) we have

$$(\widetilde{\nabla}_Z\widetilde{S})(X,Y)=\widetilde{\nu}(X)\widetilde{g}(Y,Z)+\widetilde{\nu}(Y)\widetilde{g}(X,Z)+\widetilde{\sigma}(Z)\widetilde{g}(X,Y), \tag{58}$$

where $\widetilde{\nu}(X)=(n-2)\overline{K}\widetilde{\varphi}(X)$, $\widetilde{\sigma}(X)=X(\widetilde{K})$. As in [9], p. 131 (see also Lemma 2), one can prove that

$$\widetilde{\nu}(X)=\frac{n-2}{2(n-1)(n-2)}(X\widetilde{r}),\qquad \widetilde{\sigma}(X)=\frac{n}{(n-1)(n+2)}(X\widetilde{r}). \tag{59}$$

Consider the following two cases.

(i) The scalar curvature $\widetilde{r}$ of $(M,\widetilde{g})$ is constant. Since $(M,\widetilde{g})$ admits a non-trivial geodesic mapping onto (M,a), we see, by the above considerations, that $\widetilde{r}=\mathrm{const}$ if and only if $\overline{K}=0$. Then (57) implies that $(M,\widetilde{g})$ is an Einstein manifold. Conversely, if $(M,\widetilde{g})$ is an Einstein manifold which admits a geodesic mapping corresponding to $-d\psi$, then, as in [9], p. 130 (see also [12]), we easily conclude that (M,g) is a Sinyukov manifold.

(ii) If $\widetilde{r}$ is not constant, then from (58) and (59) it follows that $(M,\widetilde{g})$ is a Sinyukov manifold. This completes the proof.

Notice that if $(M,\widetilde{g})$ is an Einstein manifold, then, by the results of [6], so is (M,a). Hence and from Theorem 4 we have

COROLLARY 5. *If* $\widetilde{g}=\exp(2\psi)g$ *is an Einstein metric, then* $\widetilde{a}=\exp(2\psi)a$ *is a Sinyukov metric.*

5. Local structure theorem. The local structure theorem for conformally flat Sinyukov manifolds is given in [5]. Let a be a differentiable symmetric bilinear form on $U_a\subseteq M$ satisfying (3) and having t different eigenvalues $\overset{1}{\lambda},\ldots,\overset{t}{\lambda}$. From the very definition, at each point $p\in U_a$ they coincide with the eigenvalues of the endomorphism A_p of the tangent space $T_p(M)$ corresponding to a, i.e. $g(AX,Y)=a(X,Y)$ for all $X,Y\in\mathfrak{X}(U_a)$. Let (U,x) be a chart on M such that $U\subseteq U_a$. Suppose that $\overset{\alpha}{v}$ is an eigenvector of the matrix a_{ij} corresponding to the eigenvalue $\overset{\alpha}{\lambda}$, i.e. satisfying the condition

$$(a_{ij}-\overset{\alpha}{\lambda}g_{ij})\overset{\alpha}{v}{}^j=0. \tag{60}$$

Following [4], one can prove that $\varphi_t\overset{\alpha}{v}{}^t=0$ and $\overset{\alpha}{v}_i=\overset{\alpha}{B}(X_i\overset{\alpha}{\lambda})$, where $\overset{\alpha}{B}\in\mathfrak{F}(U)$. Transvecting (60) with ψ^i and making use of (4) and (5) we have $\psi_i\overset{\alpha}{v}{}^i=0$. From [2], it follows that if (M,g) admits a geodesic mapping then $\exp(-2\psi)=\prod_{\alpha=1}^t(f_\alpha)^{\tau_\alpha}$, where τ_α denotes the algebraic multiplicity of $\overset{\alpha}{\lambda}=f_\alpha(x^{n_\alpha+\tau_\alpha})$, $n_1=0$, $n_\beta=\tau_1+\ldots+\tau_{\beta-1}$, $\beta=2,\ldots,t$. Hence

$$\overset{\alpha}{v}_{i_\alpha}=\overset{\alpha}{F}\psi_{i_\alpha}\quad\text{and}\quad\overset{\alpha}{v}_j=0\quad\text{for } j\neq i_\alpha, \tag{61}$$

where $i_\alpha=n_\alpha+1,\ldots,n_\alpha+\tau_\alpha$, $\overset{\alpha}{F}\in\mathfrak{F}(U)$, $\alpha=1,\ldots,t$.

LEMMA 6. *On a Sinyukov manifold the eigenvectors of the matrix* $a_{ij}(p)$, $p\in U_a$, *are non-null.*

Proof. Suppose, to the contrary, that the eigenvector $\overset{\alpha}{v}$ corresponding to the eigenvalue $\overset{\alpha}{\lambda}$ is a null vector. Differentiating covariantly (61) with respect to x^k, then transvecting the resulting equation with $\overset{\alpha}{v}{}^i$ and applying the relation $\psi_{ji_\alpha} = 0$ for $j \neq j_\alpha$, we obtain $\psi_{kt} \overset{\alpha}{v}{}^t = 0$. Therefore, from (53) and (60), we have

$$-\frac{1}{(n-2)\omega} \overset{\alpha}{\lambda} + K + \overline{K}(f_\alpha)^{-1} \prod_{\beta=1}^{t} (f_\beta)^{-\tau_\beta} = 0.$$

Since

$$X_i K = -\frac{2}{(n-2)\omega} \varphi_i \quad \text{and} \quad \varphi = \frac{1}{2} \sum_{\beta=1}^{t} \tau_\beta f_\beta$$

(see [2]), it is easily seen that the above relation is false if the manifold admits a non-trivial geodesic mapping. This completes the proof.

Assume that a manifold (M, g) admits a geodesic mapping onto a manifold $(M, \overline{g})$. If at $p \in M$ the eigenvectors of the matrix $a_{ij}(p)$ are non-null, then in some neighbourhood of p there exists a coordinate system such that the components of the metric tensors g and $\overline{g}$ take the form ([2])

$$(62) \qquad \begin{aligned} g_{\mu\mu} &= e_\mu \prod_{\substack{\beta=1 \\ \beta \neq \mu}}^{t} (f_\beta - f_\mu)^{\tau_\beta}, \qquad \overline{g}_{\mu\mu} = \prod_{\beta=1}^{t} (f_\beta)^{-\tau_\beta} (f_\mu)^{-1} g_{\mu\mu}, \\ g_{i_\varrho j_\varrho} &= \prod_{\substack{\beta=1 \\ \beta \neq \varrho}}^{t} (f_\varrho - f_\beta)^{\tau_\beta} \overset{\varrho}{g}_{i_\varrho j_\varrho}, \qquad \overline{g}_{i_\varrho j_\varrho} = \prod_{\beta=1}^{t} (f_\beta)^{-\tau_\beta} (f_\varrho)^{-1} g_{i_\varrho j_\varrho}, \end{aligned}$$

where $f_\mu = f_\mu(x^\mu)$, $f_\varrho = \text{const} \neq 0$, $e_\mu = \pm 1$, $\mu = 1, \ldots, k$, $\varrho = k+1, \ldots, t$, $t \leq 2k+1$, $\tau_1 = \ldots = \tau_k = 1$, $\tau_\varrho > 1$, $i_\varrho, j_\varrho = n_\varrho + 1, n_\varrho + 2, \ldots, n_\varrho + \tau_\varrho$, $n_1 = 0$, $n_\gamma = \tau_1 + \tau_2 + \ldots + \tau_{\gamma-1}$, $\gamma = 2, \ldots, t$ and $\overset{\varrho}{g}_{i_\varrho j_\varrho}(x^{n_\varrho+1}, \ldots, x^{n_\varrho+\tau_\varrho})$ are metric tensors on τ_ϱ-dimensional submanifolds $\overset{\varrho}{M}$, $\exp(-2\psi) = \prod_{\alpha=1}^{t} (f_\alpha)^{\tau_\alpha}$.

The following lemma is a consequence of (25), (31), (15) and (52).

LEMMA 7. *If (M, g) is a Sinyukov manifold and the Weyl conformal curvature tensor $C \neq 0$ at a point p then, at p,*

$$(63) \qquad a(X, C(Y, Z)V) + a(V, C(Y, Z)X) = 0.$$

Taking into account (63) in the coordinate system in which the metric has the form (62) and applying the equality $a_{i_\alpha j_\alpha} = f_\alpha g_{i_\alpha j_\alpha}$ we find

LEMMA 8. *If $\overset{\varrho}{g}$ are metrics of one-dimensional manifolds, then the adjoint metric*

$$\overset{*}{g} = \sum_{\mu=1}^{k} \prod_{\substack{\beta=1 \\ \beta \neq \mu}}^{t} (f_\beta - f_\mu)^{\tau_\beta} (dx^\mu)^2 + \sum_{\varrho=k+1}^{t} \prod_{\substack{\beta=1 \\ \beta \neq \varrho}}^{t} (f_\varrho - f_\beta)^{\tau_\beta} (dy^\varrho)^2$$

is a metric of a conformally flat manifold. In particular, if $a_{ij}(p)$, $p \in U_a$, has n distinct eigenvalues, then (U_a, g) is a conformally flat Sinyukov manifold.

THEOREM 5. *Suppose that a 1-form $d\psi$ defines a geodesic mapping of a Sinyukov manifold (M,g) with $C \neq 0$ everywhere on M. If $\widetilde{g} = \exp(2\psi)g$ is a Sinyukov metric, then on a neighbourhood of each point $p \in M$ there exists a coordinate system such that the metrics g and $\widetilde{g}$ take one of the following forms*:

(i) *if $k = 1$ and $t = 2$, then*

$$g = \frac{1}{4(c_1 - x^1)W_1(x^1)}\,(dx^1)^2 + (c_1 - x^1)\overset{2}{h}_{\alpha\beta}\,dx^\alpha dx^\beta, \qquad \widetilde{g} = (x^1)^{-1}g, \tag{64}$$

where

$$W_1(z) = A_2 z^2 + A_1 z + A_0,$$

$A_0, A_2, c_1 = \text{const} \neq 0$, $A_1 = \text{const}$, $\overset{2}{h} = \overset{2}{h}(x^2,\ldots,x^n)$ *is an $(n-1)$-dimensional Einstein metric with the Ricci tensor*

$$\overset{2}{S} = -(n-2)W_1(c_1)\,\overset{2}{h},$$

$\alpha, \beta = 2,\ldots,n$;

(ii) *if $k = 1$ and $t = 3$, then we have*

$$g = \frac{(n-2)\omega}{W_2(x^1)}\,(dx^1)^2 + \sum_{\varrho=2}^{3}(c_\varrho - x^1)\overset{\varrho}{h}_{i_\varrho j_\varrho}\,dx^{i_\varrho}\,dx^{j_\varrho}, \qquad \widetilde{g} = (x^1)^{-1}g, \tag{65}$$

where

$$W_2(z) = 4(c_2 - z)(c_3 - z)(c_1 + z),$$

$c_\varrho, c_1, \omega = \text{const} \neq 0$, $\varrho = 2,3$, $\overset{2}{h} = \overset{2}{h}(x^2,\ldots,x^{\tau_2+1})$ *is a τ_2-dimensional Einstein metric with the Ricci tensor*

$$\overset{2}{S} = (\tau_2 - 1)(c_2 - c_3)(c_1 + c_2)K\overset{2}{h},$$

$\overset{3}{h} = \overset{3}{h}(x^{\tau_2+2},\ldots,n)$ *is a τ_3-dimensional Einstein metric with te Ricci tensor*

$$\overset{3}{S} = (\tau_3 - 1)(c_3 - c_2)(c_1 + c_3)K\overset{3}{h},$$

$K = \frac{1}{(n-2)\omega}$, $i_2, j_2 = 2,\ldots,\tau_2+1$, $i_3, j_3 = \tau_2+2,\ldots,n$, $1 + \tau_2 + \tau_3 = n$;

(iii) *if $k > 1$, then*

$$g = \sum_{\mu=1}^{k}\prod_{\substack{\eta=1\\ \eta\neq\mu}}^{k}\frac{x^\eta - x^\mu}{W_3(x^\mu)}(dx^\mu)^2 + \sum_{\varrho=k+1}^{t}\prod_{\mu=1}^{k}(f_\varrho - x^\mu)\overset{\varrho}{g}_{i_\varrho j_\varrho}\,dx^{i_\varrho}\,dx^{j_\varrho}, \tag{66}$$

$$\widetilde{g} = (x^1 \ldots x^k)^{-1}g,$$

where

$$W_3(z) = (-1)^{k+1} 4A_{k+2} z^{k+2} + A_{k+1} z^{k+1} + \ldots + A_1 z + 4A_0,$$

$A_0, A_1, \ldots, A_{k+2} = \text{const}$, $A_0, A_{k+2} \neq 0$, $f_\varrho = \text{const} \neq 0$, *and the* f_ϱ *are roots of the polynomial* W_3, $\overset{\varrho}{g}$ *are* τ_ϱ*-dimensional Einstein metrics with the Ricci tensors*

$$\overset{\varrho}{S} = (\tau_\varrho - 1) K_\varrho \overset{\varrho}{g}$$

and $K_\varrho = (-1)^{k+1} \frac{1}{4} W_3'(f_\varrho)$, $k > 1$, $t \leq 2k+1$, $\varrho = k+1, \ldots, t$, $i_\varrho, j_\varrho = n_\varrho + 1, \ldots, n_\varrho + \tau_\varrho$, $n_1 = 0$, $n_\gamma = \tau_1 + \tau_2 + \ldots + \tau_{\gamma-1}$, $\gamma = 2, \ldots, t$, $\tau_\varrho > 1$.

Proof. Solving (53) in the local coordinate system in which g and $\overline{g}$ are of the form (62) and using the equality $a_{i_\alpha j_\alpha} = f_\alpha g_{i_\alpha j_\alpha}$, in the same way as in the proof of Theorem 3 of [1], we obtain our assertion.

THEOREM 6. *Let* $\mathbb{R}^n$ *be endowed with a metric of the form either* (64) *or* (65) *or* (66), *where* h, $\overset{2}{h}$ *or* $\overset{3}{h}$ *and at least one of the forms* $\overset{\varrho}{g}$ *are non-conformally flat Einstein metrics. Then* $(\mathbb{R}^n, g)$ *(and* $(\mathbb{R}^n, \widetilde{g})$*) is a non-conformally flat Sinyukov manifold.*

Proof. By elementary computation one can easily verify that (9) holds on $(\mathbb{R}^n, g)$ (and the analogous condition is satisfied on $(\mathbb{R}^n, \widetilde{g})$). The components of the 1-form σ ($\nu = \frac{n-2}{2n}\sigma$) are respectively:

1) for the metric (64):

$$\sigma_1 = -nA_2,\ \sigma_\alpha = 0 \quad \left(\widetilde{\sigma}_1 = \frac{-nA_0 c_1}{(x^1)^2},\ \widetilde{\sigma}_\alpha = 0\right), \quad \alpha = 2, \ldots, n,$$

2) for the metric (65):

$$\sigma_1 = \frac{n}{(n-2)\omega},\ \sigma_\alpha = 0 \quad \left(\widetilde{\sigma}_1 = \frac{-nc_1 c_2 c_3}{(n-2)\omega (x^1)^2},\ \widetilde{\sigma}_\alpha = 0\right), \quad \alpha = 2, \ldots, n,$$

3) for the metric (66):

$$\sigma_\mu = nA_{k+2},\ \sigma_{i_\varrho} = 0 \quad \left(\widetilde{\sigma}_\mu = \frac{-nA_0}{(x^\mu)^2},\ \widetilde{\sigma}_{i_\varrho} = 0\right), \quad \mu = 1, \ldots, k.$$

Moreover, in the metrics (64), (65) and (66), the conformal curvature tensor $C \neq 0$ if and only if $\overset{2}{h}$ (resp. $\overset{2}{h}$ or $\overset{3}{h}$, resp. at least one of $\overset{\varrho}{g}$) is a non-conformally flat metric. This completes the proof.

Remark. In [4] the local structure theorem for Einstein manifolds admitting geodesic mappings is proved. If $\widetilde{g} = \exp(2\psi) g$ is an Einstein manifold, then, by Theorem 4(i), Corollary 5 and the results of [4], the local structure of Sinyukov manifolds can be easily obtained. This, together with Theorem 5, provides a complete description of the local structure of Sinyukov manifolds.

From Theorems 5, 6 and the results of [4] we have the following

COROLLARY 6. *If M is a Sinyukov manifold and* $\dim M \leq 4$, *then M is conformally flat.*

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