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NORMAL FORMS OF SYMPLECTIC STRUCTURES ON THE STRATIFIED SPACES

BY

W. DOMITRZ AND S. JANECZKO (WARSZAWA)

Introduction. In this paper we consider the singular symplectic spaces defined as follows. Let V be a stratified subspace of $\mathbb{R}^N$. We call it a *singular symplectic space* if there exists a differential 2-form ω on $\mathbb{R}^N$ such that the restriction of ω to each stratum is a symplectic form. Spaces of this type have been extensively studied by several authors [15, 10] in the context of Marsden–Weinstein singular reduction. An approach to the local classification of germs of such spaces was introduced in [8]. The germs of singular symplectic spaces are classified by the corresponding coisotropic varieties in the extended symplectic space.

One can get a representative example of a singular symplectic space obtained by symplectic reduction by taking the generating family in the following form (cf. [8]):

$$F:\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}^k\to\mathbb{R},\qquad F(q,\alpha,\lambda)=f(q,\lambda)+\alpha_1^2q_1+\alpha_2q_2,$$

where f is a smooth function such that $\partial^2f/\partial\lambda^2\neq 0$. The reduction map

$$\Phi_F|_{\Sigma_F}:(q,\alpha,\lambda)\to(\xi,\alpha)=\left(\frac{\partial F}{\partial\alpha}(q,\alpha,\lambda),\alpha\right)$$

of the corresponding coisotropic variety $C\subset T^*\mathbb{R}^m$ (cf. [8]) defines the symplectic space of bicharacteristics

$$W=\{(\xi,\alpha):\alpha_1\neq 0\}\cup\{(\xi_1,\alpha):\xi_1=0,\ \alpha_1=0\}$$

endowed with the symplectic form $\sum_{i=1}^n d\xi_i\wedge d\alpha_i$. In what follows, we shall take one component of W, i.e. $\{\alpha_1>0\}\cup\{\xi_1=0,\ \alpha_1=0\}$, as a local model of stratified symplectic space (we call it a symplectic flag). We obtain a generalization of this model by taking all pull-backs of this space by smooth maps which are diffeomorphisms on the maximal strata.

The main results of the paper are Darboux-type theorems for a certain class of stratified spaces and a general approach to the classification of such spaces by germs of generating families. The observation that in contrast to

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the singular symplectic reduction theory, symplectic structures on stratified spaces may be generated by singular symplectic two-forms in an ambient space, is also the starting point of our paper.

This article is organized in three sections. The first section classifies the symplectic structures on the flag model. An analogue of the Darboux theorem is proved and the prenormal form of a symplectic structure is derived. Finally, the stratified symplectic space with singular boundary of the maximal stratum is discussed. The second section gives a canonical construction of generating families for singular reduction together with the formulation of the problem of symplectic reduction from singular hypersurfaces of the symplectic space. The third section gives an example of a singular symplectic space encountered in classical physics of systems of rays.

1. Local models of singular symplectic structures. A stratified space each of whose strata is a symplectic manifold is called a *stratified symplectic space*. This notion was introduced in [15] (see also [8]) in the context of standard symplectic reduction. For our purposes we will use embedded symplectic spaces.

Definition 1.1. Let S be a stratified subset of $\mathbb{R}^N$ with each stratum S_i (even-dimensional) endowed with a symplectic structure ω_{S_i}. Assume that there exists a closed two-form ω on $\mathbb{R}^N$ such that $\omega|_{S_i} = \omega_{S_i}$. Then the pair (S,ω) is called a *singular symplectic space*.

A representative model of a singular symplectic space is a disjoint union of semialgebraic sets. We consider the elementary symplectic flag

$$S = S_{2n} \cup S_{2n-2} \subset \mathbb{R}^{2n};$$

$$S_{2n} = \{(x,y) \in \mathbb{R}^{2n} : x_1 > 0\}, \qquad S_{2n-2} = \{(x,y) \in \mathbb{R}^{2n} : x_1 = 0,\ y_1 = 0\},$$

endowed with a symplectic structure ω. By $\iota_k : S_k \to \mathbb{R}^N$ we denote the canonical inclusions of S_k, with $S_{2n-1} = \{x \in \mathbb{R}^{2n} : x_1 = 0\}$.

Now we have a natural extension problem: Let $\widetilde{\omega}$ be a symplectic form on S_{2n-2}; we ask about the existence of a closed smooth two-form on $\mathbb{R}^{2n}$ such that $\omega|_{S_{2n-2}} = \widetilde{\omega}$ and $\omega|_{S_{2n}}$ is symplectic.

The first step in approaching this problem is to classify singular symplectic spaces (S,ω), where ω provides a symplectic structure on $\mathbb{R}^{2n}$.

We denote by G_S the group of germs of diffeomorphisms $(\mathbb{R}^{2n},0) \to (\mathbb{R}^{2n},0)$ preserving S, i.e. if $\Phi \in G_S$ then $\Phi(S_{2n}) \subset S_{2n}$ and $\Phi(S_{2n-2}) \subset S_{2n-2}$. It is easy to see that

Proposition 1.1. *If* $\Phi \in G_S$, *then*

$$\begin{aligned}\Phi&(x_1,y_1,\dots,x_n,y_n)\\ &= (x_1\phi_1(x,y), x_1\phi_{12}(x,y) + y_1\phi_{22}(x,y), \phi_3(x,y),\dots,\phi_{2n}(x,y)),\end{aligned}$$

where $\phi_1, \phi_{12}, \phi_{22}, \phi_3, \ldots, \phi_{2n}$ *are smooth germs of functions on* $(\mathbb{R}^{2n}, 0)$ *and* $\phi_1(0) > 0$.

DEFINITION 1.2. Let ω_1, ω_2 be two symplectic structures on S (closed two-forms on $(\mathbb{R}^{2n}, 0)$). We say that ω_1 and ω_2 are *equivalent* ($\omega_1 \sim \omega_2$) if and only if there exists $\Phi \in G_S$ such that $\Phi^*\omega_1 = \omega_2$.

THEOREM 1.1 (Darboux form). *Let* ω *be a symplectic structure on* S. *Assume* ω *is a symplectic form on* $\mathbb{R}^{2n}$. *Then* ω *is equivalent to the Darboux form*:

$$\omega \sim \sum_{i=1}^{n} dx_i \wedge dy_i.$$

Proof. First we reduce ω to the Darboux form on S_{2n-2}. Let $\delta = \iota_{2n-1}^*\omega$. The rank of δ at 0 is $2n-2$, because

$$\iota_{2n-2}^*\omega = \sum_{i=2}^{n} dx_i \wedge dy_i. \tag{1}$$

We define the characteristic distribution on S_{2n-1} by

$$D = \bigcup_{p \in S_{2n-1}} D_p, \quad D_p = \{v \in T_pS_{2n-1} : \delta(v, w)=0 \text{ for every } w \in T_pS_{2n-1}\}.$$

Because δ is a closed two-form, D is involutive.

Making use of (1) we can write

$$\delta = \sum_{i=2}^{n} dx_i \wedge dy_i + dy_1 \wedge \sum_{i=2}^{n} (\alpha_i dx_i + \beta_i dy_i) + y_1\delta',$$

where α_i, β_i are smooth function-germs on S_{2n-1} and δ' is a closed two-form on S_{2n-1} depending on the differentials of $x_2, y_2, \ldots, x_n, y_n$. We have

$$\delta(0) = \sum_{i=2}^{n} dx_i \wedge dy_i + dy_1 \wedge \sum_{i=2}^{n} (\alpha_i(0) dx_i + \beta_i(0) dy_i).$$

Let $X_0 \in D_0$, i.e. $X_0 \rfloor \delta(0) = 0$. Then

$$X_0 = \lambda \frac{\partial}{\partial y_1}\Big|_0 + \sum_{i=2}^{n} \left(a_i \frac{\partial}{\partial x_i}\Big|_0 + b_i \frac{\partial}{\partial y_i}\Big|_0 \right),$$

where

$$a_i = -\lambda\beta_i(0), \quad b_i = \lambda\alpha_i(0), \quad i = 2, \ldots, n, \ \lambda \in \mathbb{R}.$$

So we can write

$$D_0 = \operatorname{span}\left\{ X_0 = \frac{\partial}{\partial y_1}\bigg|_0 + \sum_{i=2}^{n}\left(-\beta_i(0)\frac{\partial}{\partial x_i}\bigg|_0 + \alpha_i(0)\frac{\partial}{\partial y_i}\bigg|_0 \right)\right\},$$

$$T_0 S_{2n-1} = \operatorname{span}\left\{ X_0, \frac{\partial}{\partial x_i}\bigg|_0, \frac{\partial}{\partial y_i}\bigg|_0 \right\}_{i=2,\dots,n}.$$

We immediately deduce that D is transversal to S_{2n-2} around 0. We set $\overline{\delta} = \delta|_{S_{2n-2}} = \sum_{i=2}^{n} dx_i \wedge dy_i$. Then we extend $\overline{\delta}$ along the integral curves of D. This means that there is a natural diffeomorphism $\phi : (S_{2n-1}, 0) \to (S_{2n-1}, 0)$ preserving S_{2n-2} such that $\phi^*\delta = \sum_{i=2}^{n} dx_i \wedge dy_i$. Now we write

$$\Phi(x, y) = (x_1, \phi(y_1, x_2, y_2, \dots, x_n, y_n)), \qquad \Phi \in G_S,$$

and $\omega_1 = \Phi^*\omega$, $\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$. We use Moser's method [13] to show the equivalence of ω_1 and ω_0. We take the homotopy $\omega_t = t\omega_1 + (1-t)\omega_0$, $t \in [0,1]$. One can check that ω_t is a nondegenerate form for every $t \in [0,1]$. We seek for a smooth family $t \to \Phi_t$ such that

$$\Phi_t^*\omega_t = \omega_0, \qquad \Phi_0 = \mathrm{id}_{\mathbb{R}^{2n}}. \tag{2}$$

Differentiating (2) we get

$$L_{V_t}\omega_t + \omega_1 - \omega_0 = 0,$$

where L_{V_t} is the Lie derivative along the vector field V_t generated by the flow Φ_t. But

$$L_{V_t}\omega_t = d(V_t \rfloor \omega_t) + V_t \rfloor d\omega_t = d(V_t \rfloor \omega_t).$$

The differential $d(\omega_0 - \omega_1) = 0$ and $\iota_{2n-1}^*(\omega_0 - \omega_1) = 0$. So by the relative Poincaré Lemma (see e.g. [16]) there exists a one-form α such that $d\alpha = \omega_0 - \omega_1$ and α vanishes on S_{2n-1}. Thus

$$V_t \rfloor \omega_t = \alpha \quad \text{and} \quad \alpha_{(x,y)} = 0 \text{ for every } (x, y) \in S_{2n-1}. \tag{3}$$

Because ω_t is a nondegenerate form, (3) is always solvable with respect to V_t, and moreover, $V_t(x, y) = 0$ for every $(x, y) \in S_{2n-1}$. We deduce that Φ_t exists, preserves the submanifolds S_{2n-1} and S_{2n-2}, and by compactness of the interval [0, 1] we have $\Phi_1^*\omega_1 = \omega_0$. If Φ_1 preserves S_{2n} then it belongs to G_S, otherwise we take $\Gamma \circ \Phi_1 \in G_S$, where Γ is the symplectomorphism

$$\Gamma(x, y) = (-x_1, x_2, \dots, x_n, -y_1, y_2, \dots, y_n). \ \blacksquare$$

Before we pass to a more detailed classification we recall the basic results on the standard classification of singularities of differential forms [11].

Let ω be a germ of a closed two-form on $\mathbb{R}^{2n}$ at zero. We set

$$\Sigma_k = \{x \in \mathbb{R}^{2n} : \operatorname{rank}\omega(x) = 2n - k\} \quad \text{for } k \text{ even}.$$

Let $\omega^n = f\Omega$, where Ω is the volume form on $\mathbb{R}^{2n}$.

(i) If $f(0) \neq 0$ then ω is a symplectic form (Σ_0 in the standard notation) and by the Darboux theorem we obtain

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i \tag{4}$$

in local coordinates around zero.

(ii) Next we assume $f(0) = 0$ while $(df)(0) \neq 0$. We have $\Sigma_2 = \{f = 0\}$ and let $\iota : \Sigma_2 \to \mathbb{R}^{2n}$ be the inclusion. If $\iota^*\omega^{n-1}(0) \neq 0$ then in local coordinates

$$\omega = x_1 dx_1 \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_i \tag{5}$$

and this type of singular form ω is denoted by $\Sigma_{2,0}$ (and called *Martinet's singular symplectic form*).

Both of these types of forms Σ_0 and $\Sigma_{2,0}$ are locally stable (see [11]) and we use them in what follows.

PROPOSITION 1.2. *Let ω be a symplectic structure on S. Assume $f(0) = 0$ and $df(0) \neq 0$. Then ω is a singular form of type $\Sigma_{2,0}$ at zero, i.e. ω belongs to the standard orbit of* (5) *in* (ii).

Remark 1.1. A symplectic form ω on S may be quite singular in general. The singular set of ω is not visible from S (see Fig. 1). The above proposition says that typical symplectic forms on S can only have a $\Sigma_{2,0}$ type singularity in the ambient space, i.e. the next two types of stable singular 2-forms, $\Sigma_{2,2,0}$, do not appear in this approach.

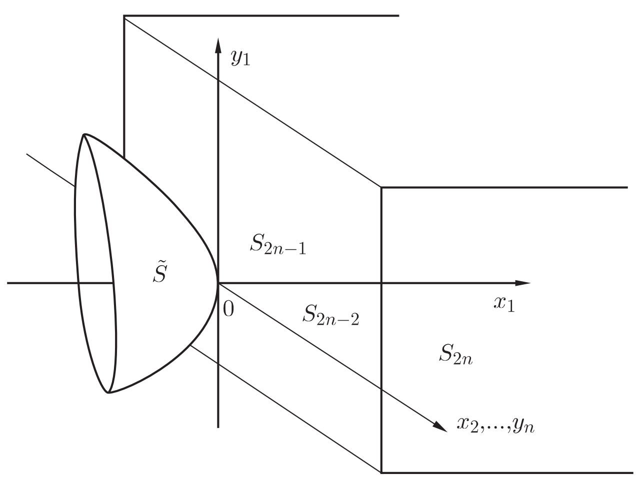

Fig. 1

Proof of Proposition 1.2. We see that ω is a symplectic form on S_{2n-2}. Let $\widetilde{S} = \{f = 0\}$, where $\omega^n = f\Omega$ and Ω is the standard volume form on $\mathbb{R}^{2n}$. We have $T_0\widetilde{S} = T_0 S_{2n-1}$, because ω is symplectic on S_{2n}. Further, $S_{2n-2} \subset S_{2n-1}$, so $T_0 S_{2n-2} \subset T_0 S_{2n-1}$ and $T_0 S_{2n-2} \subset T_0\widetilde{S}$. By assumption $\iota_{2n-2}^*\omega$ is symplectic. Thus $(\iota_{2n-2}^*\omega)^{n-1} \neq 0$ and this implies $(\iota^*\omega)^{n-1} \neq 0$, where $\iota : \widetilde{S} \to \mathbb{R}^{2n}$ is an embedding. ■

Let ω be a symplectic form on S, $\omega^n = f\Omega$, $f(0) = 0$ and $df(0) \neq 0$. Then $\frac{\partial f}{\partial x_i}(0) = 0$ and $\frac{\partial f}{\partial y_j}(0) = 0$ for $i = 2, \ldots, n$, $j = 1, \ldots, n$, so $\frac{\partial f}{\partial x_1}(0) \neq 0$. Thus

$$df \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge \ldots \wedge dx_n \wedge dy_n(0) \neq 0,$$

so $\{y_1, x_2, y_2, \ldots, x_n, y_n\}$ defines a coordinate system on $\widetilde{S} = \{f = 0\}$.

LEMMA 1.1. *By using elements of G_S one can reduce ω^n to the following normal form*:

$$(\pm x_1 + \psi(y_1, x_2, y_2, \ldots, x_n, y_n))\Omega.$$

Proof. Since the vector field $\partial/\partial x_1$ is transversal to the hypersurface $\widetilde{S}$ at 0, $\widetilde{S}$ is the graph of a smooth function ϕ on S_{2n-1}. Therefore one can write

$$f(x_1, y_1, \ldots, x_n, y_n) = r(x_1, y_1, \ldots, x_n, y_n)(x_1 - \phi(y_1, x_2, y_2, \ldots, x_n, y_n)).$$

The diffeomorphism

$$\Phi(x, y) = (x_1 - \phi(y_1, x_2, y_2, \ldots, x_n, y_n), y_1, x_2, y_2, \ldots, x_n, y_n)$$

reduces ω^n to the form $q(x, y)x_1\Omega$. Define a diffeomorphism

$$\Psi(x, y) = \left(\sqrt{2}x_1\sqrt{\frac{|g(0)|}{g(0)}g(x, y)}, y_1, x_2, y_2, \ldots, x_n, y_n\right),$$

where g is the smooth function

$$g(x, y) = \int_0^1 sq(sx_1, y_1, x_2, y_2, \ldots, x_n, y_n)\, ds.$$

Since

$$x_1^2 g(x, y) = \int_0^{x_1} tq(t, y_1, x_2, y_2, \ldots, x_n, y_n)\, dt,$$

we have $\Psi^*(\omega^n) = \pm x_1\Omega$. The diffeomorphism

$$\Upsilon(x, y) = \Bigg(x_1 + \sqrt{2}\phi(y_1, \ldots, x_n, y_n) \times\sqrt{\frac{|g(0)|}{g(0)}g(-\phi(y_1, \ldots, x_n, y_n), y_1, \ldots x_n, y_n)}, y_1, \ldots, x_n, y_n\Bigg)$$

reduces ω^n to the form

$$\omega^n = (\pm x_1 + h(y_1, \dots, x_n, y_n))\Omega.$$

The diffeomorphism $\Upsilon \circ \Psi \circ \Phi$ preserves the submanifolds S_{2n-2} and S_{2n-1}. If it preserves S_{2n} then it belongs to G_S, otherwise we take $\Gamma \circ \Upsilon \circ \Psi \circ \Phi \in G_S$, where Γ is the reflection of x_1. ■

DEFINITION 1.3. Let ψ_1, ψ_2 be function-germs on $(\mathbb{R}^{2n-1}, 0)$. We say that ψ_1, ψ_2 are *contact equivalent* if and only if there exists a diffeomorphism $\Phi : (\mathbb{R}^{2n-1}, 0) \to (\mathbb{R}^{2n-1}, 0)$ and a smooth function-germ $g : (\mathbb{R}^{2n-1}, 0) \to \mathbb{R}$, $g(0) \neq 0$, such that $\psi_1 = g \cdot (\psi_2 \circ \Phi)$.

Let ω_1, ω_2 be two symplectic forms on S. Let f_1, f_2 define their corresponding singular hypersurfaces, $\omega_1^n = f_1 \Omega$ and $\omega_2^n = f_2 \Omega$, and let ψ_1, ψ_2 be as in Lemma 1.1. By a simple check we obtain the following

PROPOSITION 1.3. *If ω_1 and ω_2 are equivalent, then ψ_1 and ψ_2 are contact equivalent.*

PROPOSITION 1.4. *Let ω be a symplectic form on S such that $f(0) = 0$, $df(0) \neq 0$. Then there exists a symplectic form ω_1 on S, equivalent to ω, such that*

$$\iota^* \omega_1 = \sum_{i=2}^{n} dx_i \wedge dy_i,$$

where $\iota : \widetilde{S} \to \mathbb{R}^{2n}$ is the embedding.

Proof. First we reduce ω to a form such that $\iota_{2n-2}^* \omega$ is the Darboux form. Then making use of Lemma 1.1 we have

$$\widetilde{S} = \{\pm x_1 + \psi(y_1, x_2, y_2, \dots, x_n, y_n) = 0\}.$$

The last change of coordinates is the identity on S_{2n-1}, so $\iota_{2n-2}^* \omega$ is still in the Darboux form. So we can write

$$\begin{aligned}\omega = \sum_{i=2}^{n} dx_i \wedge dy_i + dx_1 \wedge \Big(\sum_{i=2}^{n} \alpha_i dx_i + \sum_{i=1}^{n} \beta_i dy_i\Big) \\ + dy_1 \wedge \Big(\sum_{i=2}^{n} \gamma_i dx_i + \sum_{i=2}^{n} \delta_i dy_i\Big) + x_1 \widehat{\omega} + y_1 \widetilde{\omega},\end{aligned}$$

where the 2-form $\widehat{\omega}$ depends on the differentials of $y_1, x_2, y_2, \dots, x_n, y_n$ and $\widetilde{\omega}$ depends on the differentials of $x_2, y_2, \dots, x_n, y_n$. So the pull-back of ω to $\widetilde{S}$ has the form

$$\begin{aligned}\sum_{i=2}^{n} dx_i \wedge dy_i \mp d\psi \wedge \Big(\sum_{i=2}^{n} \alpha_i dx_i + \sum_{i=1}^{n} \beta_i dy_i\Big) \\ + dy_1 \wedge \Big(\sum_{i=2}^{n} \gamma_i dx_i + \sum_{i=2}^{n} \delta_i dy_i\Big) \mp \psi \iota^* \widehat{\omega} + y_1 \iota^* \widetilde{\omega}\end{aligned}$$

and

$$\iota^*\omega(0) = \sum_{i=2}^{n} dx_i \wedge dy_i + dy_1 \wedge \Big(\sum_{i=2}^{n} \gamma_i(0) dx_i + \sum_{i=2}^{n} \delta_i(0) dy_i \Big),$$

because $\psi(0) = 0$ and $d\psi(0) = 0$.

Consider the involutive distribution

$$D = \bigcup_{p \in \widetilde{S}} D_p, \qquad D_p = \{v \in T_p\widetilde{S} : \forall w \in T_p\widetilde{S},\ \iota^*\omega(v,w) = 0\}.$$

One can easily check that

$$D_0 = \operatorname{span}\left\{ \frac{\partial}{\partial y_1}\Big|_0 + \sum_{i=2}^{n} \left(\gamma_i(0) \frac{\partial}{\partial y_i}\Big|_0 - \delta_i(0) \frac{\partial}{\partial x_i}\Big|_0 \right) \right\}. \tag{6}$$

We see that $\widetilde{S} \cap \{y_1 = 0\}$ is a $(2n-2)$-dimensional submanifold of $\widetilde{S}$ and $(x_2, y_2, \ldots, x_n, y_n)$ can be taken as coordinates on it. By (6), D is transversal to $\widetilde{S} \cap \{y_1 = 0\}$ at zero. Let τ denote a pull-back of $\iota^*\omega$ to $\widetilde{S} \cap \{y_1 = 0\}$. We can reduce τ to the Darboux form $\sum_{i=2}^{n} dx_i \wedge dy_i$. Then we extend τ, along the leaves of D, onto $\widetilde{S}$. Finally, we reduce $\iota^*\omega$ to the form $\sum_{i=2}^{n} dx_i \wedge dy_i$ preserving the set $\widetilde{S} \cap \{y_1 = 0\}$. The diffeomorphism of $\widetilde{S}$ constructed in this way has the form

$$\begin{aligned}&\Psi(y_1, x_2, y_2, \ldots, x_n, y_n)\\ &\quad = (y_1\psi_1(y_1, x_2, y_2, \ldots, x_n, y_n), \ldots, \psi_{2n-1}(y_1, x_2, y_2, \ldots, x_n, y_n)),\end{aligned}$$

because it preserves the hypersurface $\{y_1 = 0\}$ in $\widetilde{S}$. Thus

$$\Phi(x_1, y_1, x_2, y_2, \ldots, x_n, y_n) = (x_1, \Psi(y_1, x_2, y_2, \ldots, x_n, y_n))$$

is a diffeomorphism, $\Phi \in G_S$ and

$$\iota^*\Phi^*\omega = \sum_{i=2}^{n} dx_i \wedge dy_i. \ \blacksquare$$

Before we formulate the main theorem concerning the normal form of ω we prove some necessary facts ([11]).

Lemma 1.2. *Let τ be a k-form on $\mathbb{R}^n$ satisfying*

$$\frac{\partial}{\partial x_1} \rfloor \tau = 0, \tag{7}$$

$$\frac{\partial}{\partial x_1} \rfloor d\tau = 0. \tag{8}$$

Then $\tau = \pi^*\iota^*\tau$, where

$$\begin{aligned}\pi &: \mathbb{R}^n \to \{x_1 = 0\}, &\quad \pi(x_1, x_2, \ldots, x_n) &= (0, x_2, \ldots, x_n),\\ \iota &: \{x_1 = 0\} \to \mathbb{R}^n, &\quad \iota(x_2, \ldots, x_n) &= (0, x_2, \ldots, x_n).\end{aligned}$$

Proof. By (7), τ does not depend on dx_1. Equation (8) implies that the partial differentials of the coefficients of τ with respect to x_1 vanish. So the coefficients do not depend on x_1. ∎

LEMMA 1.3. *Let τ be a k-form on $\mathbb{R}^n$ satisfying*

$$\frac{\partial}{\partial x_1}\rfloor \tau = 0, \tag{9}$$

$$\frac{\partial}{\partial x_1}\rfloor d\tau = \varphi\tau, \tag{10}$$

where φ is a smooth function on $\mathbb{R}^n$. Then $\tau = \zeta\pi^*\iota^*\tau$, where ζ is a smooth function on $\mathbb{R}^n$ and $\zeta|_{\{x_1=0\}} = 1$.

Proof. We seek for a smooth function η on $\mathbb{R}^n$ such that $\eta|_{\{x_1=0\}} = 1$ and $\eta\tau$ satisfies the assumptions of Lemma 1.2. Then we have

$$\frac{\partial}{\partial x_1}\rfloor d(\eta\tau) = \frac{\partial\eta}{\partial x_1}\tau - d\eta \wedge \left(\frac{\partial}{\partial x_1}\rfloor \tau\right) + \eta\left(\frac{\partial}{\partial x_1}\rfloor d\tau\right).$$

By (9) and (10) we have

$$\frac{\partial\eta}{\partial x_1} + \eta\varphi = 0, \qquad \eta|_{\{x_1=0\}} = 1 \tag{11}$$

and solving this we get

$$\eta(x_1,\dots,x_n) = \exp\left(\int_0^{x_1} \varphi(s,x_2,\dots,x_n)\,ds\right).$$

We thus obtained η such that $\eta\tau = \pi^*\iota^*(\eta\tau)$ (by Lemma 1.2). However, $\iota^*(\eta\tau) = \iota^*\tau$, because $\eta|_{\{x_1=0\}} = 1$. So finally,

$$\tau = \frac{1}{\eta}\pi^*\iota^*\tau. \ ∎$$

Now we prove the main theorem concerning the normal form of the symplectic structure on S, the geometrical content of which is illustrated in Fig. 1.

THEOREM 1.2. *Let ω be a symplectic structure on S. Assume $f(0) = 0$ and $df(0) \neq 0$. Then ω is equivalent to the form*

$$dh \wedge \left(dy_1 + dg + \sum_{i=2}^{n} x_i dy_i\right) + h\sum_{i=2}^{n} dx_i \wedge dy_i,$$

where g is a smooth function-germ

$$(x_2,y_2,\dots,x_n,y_n) \to g(x_2,y_2,\dots,x_n,y_n),$$

and h is a smooth function-germ such that

$$h(x,y) = \sqrt[n]{\frac{\pm 1}{2(n-1)!}(\pm x_1 + \psi(y_1, x_2, y_2, \ldots, x_n, y_n))^2 + 1},$$

where ψ is a germ at zero of a smooth function, $\psi(0) = 0$, $\frac{\partial \psi}{\partial x_i}(0) = 0$, $i = 2, \ldots, n$, $\frac{\partial \psi}{\partial y_i}(0) = 0$, $i = 1, \ldots, n$, and $\widetilde{S} = \{(x,y) : \pm x_1 + \psi = 0\}$.

Proof. By Proposition 1.4 we have $\widetilde{S} = \{(x,y) : \pm x_1 + \psi = 0\}$, where ψ does not depend on x_1, and $\iota^*\omega = \sum_{i=2}^n dx_i \wedge dy_i$. There exists a 1-form α such that $\omega = d\alpha$. Then

$$d(\iota^*\alpha) = \iota^*\omega = \sum_{i=2}^n dx_i \wedge dy_i$$

and so

$$\iota^*\alpha = d\phi + \sum_{i=2}^n x_i dy_i.$$

The form α can be chosen in such a way that $\iota^*\alpha \wedge (d(\iota^*\alpha))^{n-1} \neq 0$ ([11]). Thus

$$d\phi \wedge dx_2 \wedge dy_2 \wedge \ldots \wedge dx_n \wedge dy_n(0) \neq 0.$$

In consequence, $\frac{\partial \phi}{\partial y_1}(0) \neq 0$. Therefore,

$$\phi(y_1, x_2, y_2, \ldots, x_n, y_n) = y_1 \zeta(y_1, x_2, y_2, \ldots, x_n, y_n) + g(x_2, y_2, \ldots, x_n, y_n),$$

where ζ and g are smooth on $\widetilde{S}$. Now, by means of $\Phi \in G_S$, where

$$\begin{aligned} &\Phi(x_1, y_1, x_2, y_2, \ldots, x_n, y_n) \\ &\qquad = (x_1, y_1\zeta(y_1, x_2, y_2, \ldots, x_n, y_n), x_2, y_2, \ldots, x_n, y_n), \end{aligned}$$

we reduce $\iota^*\alpha$ to the form

$$\iota^*\alpha = dy_1 + dg + \sum_{i=2}^n x_i dy_i$$

preserving the form of $\iota^*\omega$ and $\widetilde{S}$.

LEMMA 1.4. *There exists a vector field X transversal to $\widetilde{S}$ at 0 and such that*

$$X \rfloor \alpha = 0, \tag{12}$$

$$X \rfloor d\alpha = \varphi\alpha, \tag{13}$$

where $\varphi : \mathbb{R}^{2n} \to \mathbb{R}$ is a smooth function.

P r o o f o f L e m m a 1.4. From the standard classification of 1-forms we know ([11]) that α may be reduced to the form

$$\alpha = \left(1 \pm \frac{x_1^2}{2}\right) dy_1 + \sum_{i=2}^{n} x_i dy_i.$$

So $d(\alpha) = \pm x_1 dx_1 \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_i$ and

$$\begin{aligned}\alpha \wedge (d\alpha)^{n-1} = (n-1)((1 \pm x_1^2/2) & dy_1 \wedge dx_2 \wedge dy_2 \wedge \ldots \wedge dx_n \wedge dy_n \\ & \pm x_1 x_2 dy_2 \wedge dx_1 \wedge dy_1 \wedge dx_3 \wedge dy_3 \wedge \ldots \wedge dx_n \wedge dy_n + \ldots \\ & \pm x_1 x_n dy_n \wedge dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge \ldots \wedge dx_{n-1} \wedge dy_{n-1}).\end{aligned}$$

Consider the vector field X defined by $X \rfloor \Omega = \alpha \wedge (d\alpha)^{n-1}$. We have

$$X = (n-1)\left(\left(1 \pm \frac{x_1^2}{2}\right) \frac{\partial}{\partial x_1} \pm x_1 \sum_{i=2}^{n} x_i \frac{\partial}{\partial x_i}\right).$$

We also check that

$$X \rfloor \alpha = 0, \qquad X \rfloor d\alpha = \pm (n-1) x_1 \alpha,$$

and we see that X is transversal to $\widetilde{S}$ at 0, which finishes the proof of Lemma 1.4.

LEMMA 1.5. *By using an element of* G_S *one can reduce* α *to the form*

$$\alpha = h\left(dy_1 + dg + \sum_{i=2}^{n} x_i dy_i\right),$$

where $h : \mathbb{R}^n \to \mathbb{R}$ *is a smooth function such that* $h|_{\tilde{S}} = 1$.

P r o o f o f L e m m a 1.5. The diffeomorphism

$$\Phi(x_1, y_1, \ldots, x_n, y_n) = (x_1 \pm \psi(y_1, \ldots, x_n, y_n), y_1, \ldots, x_n, y_n),$$

transforms $\widetilde{S} = \{x_1 = \mp\psi\}$ onto $\{x_1 = 0\}$. Let X be a vector field introduced in Lemma 1.4. Thus $\Phi_* X$ is transversal to $\{x_1 = 0\}$ at 0 and we can transform $\Phi_* X$ into $\partial/\partial x_1$ (cf. [2]) by a diffeomorphism Ψ such that $\Psi|_{\{x_1=0\}} = \mathrm{id}$. However, $\partial/\partial x_1$ satisfies the assumptions of Lemma 1.3 ((9), (10)). So we immediately obtain

$$((\Psi \circ \Phi)^{-1})^* \alpha = \zeta \pi_{2n-1}^* \iota_{2n-1}^* ((\Psi \circ \Phi)^{-1})^* \alpha. \tag{14}$$

One can write Ψ in the form

$$\begin{aligned}&\Psi(x_1, y_1, \ldots, x_n, y_n) \\ &\quad = (x_1 \psi_1(x,y), x_1 \psi_{12}(x,y) + y_1 \psi_{22}(x,y), \psi_3(x,y), \ldots, \psi_{2n}(x,y)).\end{aligned}$$

Define

$$\Upsilon(x_1, y_1, \ldots, x_n, y_n) = (x_1 \mp \psi_1(x,y)\psi(y_1, \ldots, x_n, y_n), y_1, \ldots, x_n, y_n).$$

Then

$$(\Upsilon \circ \Psi \circ \Phi)(x_1, y_1, \ldots, x_n, y_n)$$
$$= (x_1\psi_1(x,y), x_1\psi_{12}(x,y) + y_1\psi_{22}(x,y), \psi_3(x,y), \ldots, \psi_{2n}(x,y)).$$

If $\psi_1 > 0$ then $\Lambda = \Upsilon \circ \Psi \circ \Phi \in G_S$, otherwise we take $\Lambda = \Gamma \circ \Upsilon \circ \Psi \circ \Phi \in G_S$ instead, where Γ is the reflection of x_1. It is easy to check that

$$\iota_{2n-1}^*((\Psi \circ \Phi)^{-1})^*\alpha = dy_1 + dg + \sum_{i=2}^n x_i dy_i.$$

Now using Υ in (14) we obtain the assertion of Lemma 1.5.

By Lemma 1.5 we get

$$\alpha = h\Big(dy_1 + dg + \sum_{i=2}^n x_i dy_i\Big),$$

where $h|_{\tilde{S}} = 1$. This implies

$$\omega = d\alpha = dh \wedge \Big(dy_1 + dg + \sum_{i=2}^n x_i dy_i\Big) + h\sum_{i=2}^n dx_i \wedge dy_i.$$

We also have

$$(d\alpha)^n = n!h^{n-1}\frac{\partial h}{\partial x_1}\Omega.$$

On the other hand, by Lemma 1.1 we have $\omega^n = (\pm x_1 + \psi)\Omega$ preserving the form of α. Hence

$$n!h^{n-1}\frac{\partial h}{\partial x_1} = \pm x_1 + \psi$$

and

$$\frac{\partial h^n}{\partial x_1} = \frac{1}{(n-1)!}(\pm x_1 + \psi)$$

with an extra condition $h|_{\{x_1 = \mp\psi\}} = 1$. Solving this equation we get

$$h^n = \frac{1}{(n-1)!}\left(\frac{\pm x_1^2}{2} + \psi x_1\right) + \eta,$$

where η does not depend on x_1. Inserting $x_1 = \mp\psi$ to this equation we obtain

$$\eta = 1 \pm \frac{1}{(n-1)!}\frac{\psi^2}{2}$$

and finally

$$h = \sqrt[n]{\frac{\pm 1}{2(n-1)!}(\pm x_1 + \psi)^2 + 1}. \ \blacksquare$$

Remark 1.2. Consider the semialgebraic set $S = S_{2n} \cup S_{2n-2} \subset \mathbb{R}^{2n}$, where

$$S_{2n} = \{(x,y) \in \mathbb{R}^{2n} : x_1^3 > y_1^2\}, \quad S_{2n-2} = \{(x,y) \in \mathbb{R}^{2n} : x_1 = 0,\ y_1 = 0\}.$$

The difference between this space and the previous space S_n is that in the above model ∂S_{2n} is a singular set (see Fig. 2).

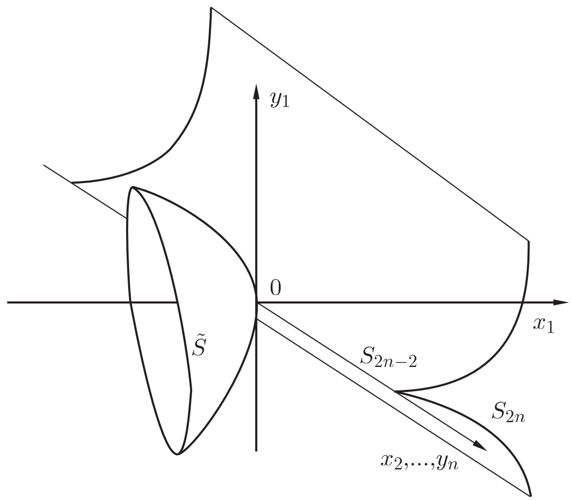

Fig. 2

We endow S with a symplectic structure ω. As before, G_S denotes the group of diffeomorphisms $(\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ preserving S. Let ω_1, ω_2 be two symplectic structures on S. We say that ω_1 and ω_2 are *G_S-equivalent* if and only if $\Phi^*\omega_1 = \omega_2$ for some $\Phi \in G_S$. Now we can show the following

PROPOSITION 1.5. *Let ω be a symplectic structure on S. Assume $f(0) = 0$ and $df(0) \neq 0$. Then ω is a singular form of type $\Sigma_{2,0}$ at zero.*

Proof. By straightforward use of the proof of Proposition 1.2.

An analogous Darboux theorem for the space S is proved by Arnold ([3]). Namely, if ω is a symplectic structure on $\mathbb{R}^{2n}$ then ω is G_S-equivalent to the Darboux form, i.e.

$$\omega \sim \sum_{i=1}^{n} dx_i \wedge dy_i.$$

2. Generating families for singular reduction. So far we discussed a concrete model of a singular symplectic space. We now consider the more general framework enabling us to classify the local types of singular reduced symplectic spaces.

Let (M,ω) be a symplectic manifold. Let $C\subset M$ be an embedded, connected, coisotropic submanifold of M, i.e. for every $x\in C$ the orthogonal space $C_x^{\perp}=(T_xC)^{\perp}$, with respect to ω, is contained in T_xC (see [16]). Then $\dim C_x^{\perp}=\operatorname{codim} C$ and $D=\bigcup_{x\in C}C_x^{\perp}$ is the characteristic distribution of $\omega|_C$. This distribution is integrable and its maximal connected integral manifolds are called *characteristics*. They form the characteristic foliation of C. Let $\varrho : C\to Y$ be the canonical projection along characteristics. If Y admits a differentiable structure and ϱ is a submersion, then there is a unique symplectic structure β on Y such that

$$\varrho^*\beta=\omega|_C.$$

In analogy to the theory of Morse families, which generate Lagrangian submanifolds, the notion of generating families for coisotropic submanifolds was introduced in [8]. We recall this construction.

We assume $M=T^*X$, $Y=T^*N$, $X\cong\mathbb{R}^m$, $N\cong\mathbb{R}^n$, and M and Y are endowed with the Liouville symplectic structures ω_X and ω_N respectively.

Definition 2.1. A smooth function (germ) $F: X\times N\times\mathbb{R}^K\to\mathbb{R}$ is called a C-*generating family* if the smooth map $(q,\alpha,\lambda)\to\frac{\partial F}{\partial\lambda}(q,\alpha,\lambda)\in\mathbb{R}^K$ is nonsingular on the stationary set

$$\Sigma_F=\left\{(q,\alpha,\lambda):\frac{\partial F}{\partial\lambda}(q,\alpha,\lambda)=0\right\}$$

and the smooth map $(q,\alpha,\lambda)|_{\Sigma_F}\to\left(\frac{\partial F}{\partial\alpha}(q,\alpha,\lambda),\alpha\right)$ is surjective.

In what follows we assume that Σ_F is a smooth component of the stationary set of dimension $m+n$. We easily see that the image of $\Phi_F:\Sigma_F\to T^*X$, $\Phi_F(q,\alpha,\lambda)=\left(\frac{\partial F}{\partial q}(q,\alpha,\lambda),q\right)$, provides a coisotropic variety C generated by F. We have

$$C=\Big\{(p,q)\in T^*X:\exists_{(\alpha,\lambda)\in N\times\mathbb{R}^K}\text{ such that } p_i=\frac{\partial F}{\partial q_i}(q,\alpha,\lambda),\quad \frac{\partial F}{\partial\lambda_k}(q,\alpha,\lambda)=0,\ 1\le i\le m,\ 1\le k\le K\Big\}. \tag{15}$$

The variety C obtained in this way is not necessarily smooth. There is a class of families F which provides smooth coisotropic submanifolds.

Definition 2.2. Let $F: X\times N\times\mathbb{R}^K\to R$ be a C-generating family. F is called a C-*Morse family* if the smooth map $\Psi_F:(q,\alpha,\lambda)\to\left(\frac{\partial F}{\partial\alpha}(q,\alpha,\lambda),\alpha\right)\in T^*N$ is regular along the stationary set Σ_F.

We can easily see that the variety C, generated by a C-Morse family via (15), is an immersed submanifold of T^*X. The immersion is given by $\Phi_F:\Sigma_F\to T^*X$.

PROPOSITION 2.1 ([8]). *For each coisotropic germ* $(C,0) \subset T^*X$, *there exists a* C-*Morse family germ* $F : (X \times N \times \mathbb{R}^K, (0,0,0)) \to \mathbb{R}$ *such that* $(C,0)$ *is defined by* (15).

If $(C,0)$ is the germ of a coisotropic submanifold then the corresponding C-Morse family generating $(C,0)$ is not uniquely defined. However, we can easily show that all C-Morse families generating $(C,0)$ are equivalent, i.e. if F_1, F_2 generate $(C,0)$ then there exists a diffeomorphism $\Phi(q,\alpha,\lambda) = (q,\alpha,\Lambda(q,\alpha,\lambda))$ such that $F_1 = F_2 \circ \Phi$.

Let $C \subset T^*X$ be a coisotropic variety defined by a C-generating family F. Then C is a stratifiable variety, $C = \bigcup_i C^{d_i}$, $d_1 > \ldots > d_k$, $d_i = \dim C^{d_i}$. We write $\widetilde{C} = \left\{(p,q,\alpha,\lambda) : p = \frac{\partial F}{\partial q}(q,\alpha,\lambda),\ \frac{\partial F}{\partial \lambda}(q,\alpha,\lambda) = 0\right\} \subset T^*X \times N \times \Lambda$. So we have a corresponding stratification of $\widetilde{C}$: $\widetilde{C} = \bigcup_i \widetilde{C}^{n_i}$, $C = \pi_{T^*X}(\widetilde{C})$, and of Σ_F: $\Sigma_F = \bigcup_i K^{s_i}$, $\Sigma_F = \pi_{X\times N\times\Lambda}(\widetilde{C})$.

COROLLARY 2.1. *The* $\{\Psi_F(K^{s_i})\}$ *form a collection of coisotropic varieties of* T^*N *if* $\dim \Psi_F(K^{s_i}) > \dim N$. *Their corresponding* C-*generating families are defined by the diagram*:

$$\begin{array}{ccc} & \Sigma_F & \\ {\scriptstyle\Psi_F}\swarrow & & \searrow{\scriptstyle\Phi_F} \\ T^*N & ---- & T^*X \end{array}$$

i.e. the q *variable in* F *is replaced by* α.

Let V be a hypersurface of $X \times N \times \Lambda$, $\dim V = 2n + K$, transversal to Σ_F. The manifold $\Xi = V \cap \Sigma_F$, $\dim \Xi = 2n$, is a parameterizing space of the reduced symplectic manifold of bicharacteristics in T^*N. The space of bicharacteristics intersecting Ξ, say $\widetilde{\Xi}$, is endowed with the symplectic structure $\Omega = \Psi_F|_{\Xi}^{*}\omega_N$ provided $\Psi_F|_{\Xi}$ is a regular map.

EXAMPLE 2.1. We consider the stable germ of the C-generating family (see [8], p. 440)

$$F : (\mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}, 0) \to \mathbb{R}, \qquad F(q,\alpha,\lambda) = \lambda^3 + \alpha\lambda + (q_1^2 - q_2^2 - q_3^2)\lambda.$$

This family generates the coisotropic variety

$$\{(p,q) : q_2p_1 + q_1p_2 = 0,\ q_3p_1 + q_1p_3 = 0\},$$
$$\Psi_F|_{\Sigma_F}(q,\lambda) = (\lambda, q_2^2 + q_3^2 - q_1^2 - 3\lambda^2) \in T^*\mathbb{R},$$

where

$$\Sigma_F = \{(q,\alpha,\lambda) : 3\lambda^2 + \alpha + q_1^2 - q_2^2 - q_3^2 = 0\}.$$

Let $V = \{(q,\alpha,\lambda) : q_1 = 0,\ q_2 = 0\}$. Then the corresponding variety Ξ of bicharacteristics is endowed with the singular symplectic structure $\Omega = 2q_3 d\lambda \wedge dq_3$.

DEFINITION 2.3. We say V is a *coisotropic hypersurface* if and only if each stratum of $\mathcal{X}_V$ is a coisotropic or isotropic submanifold of (M,ω).

Remark 2.1. We easily see that a typical hypersurface V defined by a polynomial equation $F(x,y)=0$ is not coisotropic. As an example, consider the cusp-edge surface V in $\mathbb{R}^4$ endowed with a symplectic form ω in general position with respect to V. In this case $\omega|_{\operatorname{Sing} V}$ is a symplectic form. It is shown in [3] that (V,ω) is diffeomorphic to $(\{x_1^3-y_1^2=0\},\ dx_1\wedge dy_1+dx_2\wedge dy_2)$ and the reduced symplectic space of $V-\operatorname{Sing} V$ is isomorphic to the singular edge of V.

We *conjecture* that if $\operatorname{Sing} V$ is a coisotropic (or Lagrangian) submanifold then (V,ω) is locally diffeomorphic to $(\{x_1^3-x_2^2=0\},\ \sum_{i=1}^n dx_i\wedge dy_i)$. Let $\Phi:\mathbb{R}^{2n-1}\to\mathbb{R}^{2n}$ be the following parameterization of $\{x_1^3-x_2^2=0\}$:

$$\Phi(s,y_1,y_2,x_3,y_3,\ldots,x_n,y_n)=(s^2,y_1,s^3,y_2,x_3,y_3,\ldots,x_n,y_n).$$

Then

$$\Phi^*\omega=ds\wedge d(3s^2y_2+2sy_1)+\sum_{i=2}^n dx_i\wedge dy_i.$$

Let $\pi:\mathbb{R}^{2n-1}\to\mathbb{R}^{2n-2}$ be the mapping

$$\pi(s,y_1,y_2,x_3,y_3,\ldots,x_n,y_n)=(s,3s^2y_2+2sy_1,x_3,y_3,\ldots,x_n,y_n).$$

Let S be the image of π. Then

$$S=\{(x,y)\in\mathbb{R}^{2n-2}:x_1\neq 0\}\cup\{(x,y)\in\mathbb{R}^{2n-2}:x_1=0,\ x_2=0\}$$

and

$$\pi^*\Big(\sum_{i=1}^{n-1}dx_i\wedge dy_i\Big)=\Phi^*\omega.$$

The reduced space S endowed with the Darboux form on $\mathbb{R}^{2n-2}$ is a singular symplectic space in the sense of Section 1.

3. Examples of singular symplectic spaces. Let (M,ω) be the symplectic space of oriented lines (optical rays) in Euclidean space. This space is obtained by reduction from the level set of the free particle Hamiltonian (see [1]),

$$C=\{(p,q)\in T^*\mathbb{R}^n:\|p\|=1\}. \tag{16}$$

We restrict ourselves to geometrical optics on the plane. We have a standard optical system illustrated in Fig. 3 with interface refractive index $n(q)<1$.

As in the standard construction of the phase space for optical systems [9], we take the following three copies of the space (M,ω):

(M_1,ω_1) — space of incoming rays,
(M_2,ω_2) — space of outgoing refracted rays,
(M_3,ω_3) — space of outgoing reflected rays.

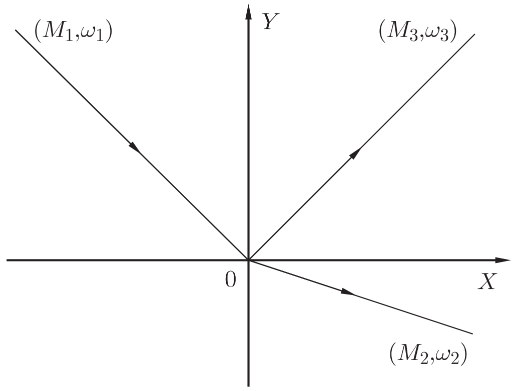

Fig. 3

We identify M_i with the chart on M defined by p_1, i.e. the rays directed transversally to the OY-line. The Darboux parameterization of this set of lines is given by (y, p) where y is the y-coordinate of an intersection point of the ray with the OY-axis and p is the y-component of the direction of the ray.

This phase space of rays is a stratified set,

$$S = S_6 \cup S_4, \tag{17}$$

where

$$S_6 = \{((y_1, p_1), (y_2, p_2), (y_3, p_3)) \in M_1 \times M_2 \times M_3 : p_1 < 0,\ p_2 < 0,\ p_3 > 0\},$$
$$S_4 = \{((y_1, p_1), (y_2, p_2), (y_3, p_3)) \in M_1 \times M_2 \times M_3 : \\ p_1 < 0,\ p_3 > 0,\ p_2 = 0,\ y_2 = 0\},$$

endowed with the canonical symplectic structure

$$\Omega = \omega_2 \oplus \omega_3 \ominus \omega_1 = \pi_2^*\omega_2 + \pi_3^*\omega_3 - \pi_1^*\omega_1.$$

In this structure reflection and refraction are each given by graphs of symplectomorphisms [8], in the partial symplectic spaces $(M_1 \times M_3, \omega_3 \ominus \omega_1)$ and $(M_1 \times M_2, \omega_2 \ominus \omega_1)$ respectively.

A suitable phase space for these collected phenomena, including internal reflection, is the singular symplectic space (S, Ω).

Let (M_i, ω_i), $i = 1, 2, 3$, be three symplectic manifolds, $\dim M_i = 2n$. We consider the product symplectic manifold

$$(\mathcal{M}, \Omega) = (M_1 \times M_2 \times M_3, \omega_3 \ominus (\omega_1 \ominus \omega_2)).$$

The wave front evolution or transformation of systems of rays is constructed as a symplectic image [7] by a class of generalized symplectic relations corresponding to concrete optical systems.

DEFINITION 3.1. A *cyclic symplectic relation* in $(\mathcal{M}, \Omega)$ is a smooth submanifold $\mathcal{L}$ of $\mathcal{M}$, $\dim \mathcal{L} = 2n$, such that the canonical projections $\pi_{12}(\mathcal{L})$, $\pi_{13}(\mathcal{L})$, $\pi_{23}(\mathcal{L})$ are Lagrangian submanifolds in the corresponding symplectic structures $(M_1 \times M_2, \omega_{12})$, $(M_1 \times M_3, \omega_{13})$, $(M_2 \times M_3, \omega_{23})$, where

$$\omega_{12} = \omega_2 \ominus \omega_1, \quad \omega_{13} = \omega_3 \ominus \omega_1, \quad \omega_{23} = \omega_3 \ominus \omega_2.$$

A straightforward generalization of our 6-dimensional optical example gives the following physically relevant

CONJECTURE 3.1. *An interface optical system is defined by a cyclic symplectic relation in the singular symplectic space* $(S_6 \cup S_4, \Omega)$.

EXAMPLE 3.1. We consider refraction and reflection on the plane with refraction index $n > 1$ illustrated in Fig. 3. Here we have

$$\omega_1 = dy_1 \wedge dp_1, \quad \omega_2 = n dy_2 \wedge dp_2, \quad \omega_3 = dy_3 \wedge dp_3$$

and a cyclic symplectic relation $\mathcal{L}$ defining this concrete simple optical system is

$$\mathcal{L} = \big\{((p_1, y_1), (p_2, y_2), (p_3, y_3)) : p_1 = -p_3,\ y_1 = -y_3,$$
$$\sqrt{1 - p_1^2} = \sqrt{1 - p_2^2},\ y_2 p_1 = n p_2 y_1\big\}.$$

It is easy to check that $\pi_{ij}(\mathcal{L})$ are Lagrangian submanifolds in the respective symplectic spaces.

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INSTITUTE OF MATHEMATICS
WARSAW UNIVERSITY OF TECHNOLOGY
PL. POLITECHNIKI 1
00-661 WARSZAWA, POLAND

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