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ESTIMATES FOR THE BERGMAN AND SZEGÖ PROJECTIONS IN TWO SYMMETRIC DOMAINS OF $\mathbb{C}^n$

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1. Introduction. Let D denote each of the following domains in $\mathbb{C}^n$, $n \geq 3$:

(i) the tube $\Omega = \mathbb{R}^n + i\Gamma$ over the spherical cone

$$\Gamma = \{(y_1, \ldots, y_n) \in \mathbb{R}^n : y_1 > 0,\ y_1 y_2 - y_3^2 - \ldots - y_n^2 > 0\},$$

(ii) the Lie ball

$$\omega = \Big\{ z \in \mathbb{C}^n : \Big| \sum_{j=1}^n z_j^2 \Big| < 1,\ 1 - 2|z|^2 + \Big| \sum_{j=1}^n z_j^2 \Big|^2 > 0 \Big\}.$$

Obviously, the first domain is unbounded while the second one is bounded. It is well known that they are biholomorphically equivalent and, in Elie Cartan's classification of bounded symmetric domains [5], they are representatives of class IV (according to Hua's numbering [9]).

Let $H(D)$ denote the space of holomorphic functions in D and let dV be Lebesgue measure in $\mathbb{C}^n$. For every $p \geq 1$, the *Bergman space* $A^p(D)$ is defined by $A^p(D) = H(D) \cap L^p(D, dV)$. For every $f \in A^p(D)$, we set $\|f\|_{A^p(D)} = \|f\|_{L^p(D,dV)}$; for $p \geq 1$, this is a norm under which $A^p(D)$ is a Banach space. The *Bergman projection* P_D of D is the orthogonal projection of the Hilbert space $L^2(D, dV)$ onto its closed subspace $A^2(D)$. Moreover, P_D is the integral operator associated with a kernel $B_D(\cdot,\cdot)$ called the *Bergman kernel* of D. Finally, we shall let P_D^* denote the integral operator associated with the positive kernel $|B_D(\cdot,\cdot)|$.

Let us state our first results:

THEOREM 1. *For every* $p \in \big(1, \frac{3n-2}{2n}\big] \cup \big[\frac{3n-2}{n-2}, \infty\big)$, *the Bergman projection* P_D *is unbounded on* $L^p(D, dV)$.

THEOREM 2. *Let* $p \geq 1$. *The operator* P_D^* *is bounded on* $L^p(D, dV)$ *if and only if* $p \in \big(\frac{2n-2}{n}, \frac{2n-2}{n-2}\big)$. *Furthermore, the Bergman projection* P_D *is bounded from* $L^p(D, dV)$ *to* $A^p(D)$ *when* $p \in \big(\frac{2n-2}{n}, \frac{2n-2}{n-2}\big)$.

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For the tube domain Ω, some of these results were announced in [1]. The question whether P_D is bounded on $L^p(D, dV)$ when p belongs to $\left(\frac{3n-2}{2n}, \frac{2n-2}{n}\right] \cup \left[\frac{2n-2}{n-2}, \frac{3n-2}{n-2}\right)$ remains open. The case of all homogeneous Siegel domains of type II has recently been considered by D. Bekollé and A. Temgoua Kagou. They proved that there is a range of p, around 2, where the Bergman projection is bounded in L^p, while there is a range of p, around 1 and ∞, where it is unbounded (cf. [4]). In all cases the critical result is not known, except for the product of Cayley transforms of unit balls for which the Bergman projection is bounded in L^p for every $p > 1$.

For bounded domains, one can as well ask for (L^p, L^q) estimates with $q < p$. The case $p = \infty$ is of special interest because the Bergman projection of L^∞ can be described as the Bloch space of holomorphic functions. The Bloch space $\mathcal{B}$ is related to Hankel operators [16]. For a description in the case of the Lie ball, see [3] and [15]. The following two statements deal with this case:

THEOREM 3. *In the Lie ball ω of $\mathbb{C}^n$, the operator P_ω^* is bounded from $L^\infty(\omega)$ to $L^q(\omega, dV)$ if and only if $q < 2n/(n-2)$. Furthermore, the Bloch space $\mathcal{B}_\omega$ of ω is contained in $A^q(\omega)$ when $q < 2n/(n-2)$ and this inclusion is continuous.*

THEOREM 4. *The Bergman projection P_ω is unbounded from $L^\infty(\omega)$ to $L^q(\omega, dV)$ when $q \geq 4n/(n-2)$. Furthermore, there is no continuous inclusion of the Bloch space $\mathcal{B}_\omega$ into $A^q(\omega)$ when $q \geq 4n/(n-2)$.*

The case of the Lie ball in Theorems 1 and 2, as well as Theorems 3 and 4, will be deduced from the case of the unbounded domain Ω via a transfer principle based on two tools:

(i) an explicit linear fractional mapping Φ of ω into Ω given in [5],

(ii) the following well-known change of variable formula for the Bergman kernel:

$$B_\omega(\zeta, z') = B_\Omega(\Phi(\zeta), \Phi(z'))J\Phi(\zeta)\overline{J\Phi(z')}.$$

In the Hardy space setting, we shall also apply our transfer principle to the Szegö projection. More precisely, the Shilov boundary of the tube $\mathcal{T} = \mathbb{R}^n + i\mathcal{C}$ over a self-dual cone $\mathcal{C}$ is $\mathbb{R}^n$. The *Hardy space* $H^p(\mathcal{T})$, $0 < p < \infty$, consists of those holomorphic functions $f(x+iy)$ on $\mathcal{T}$ which satisfy

$$\|f\|_{H^p(\mathcal{T})} = \sup_{y \in \mathcal{C}} \Big(\int_{\mathbb{R}^n} |f(x+iy)|^p \, dx \Big)^{1/p} < \infty.$$

For $p \geq 1$, such functions have boundary values, namely

$$\lim_{y \to 0,\, y \in \mathcal{C}} f(x+iy) = f(x)$$

exists in the L^p norm (cf. [13], p. 119).

In particular, for $p = 2$, the integral representation of each H^2 function f in terms of its boundary values is

$$f(s+it) = \int_{\mathbb{R}^n} S_{\mathcal{T}}(s+it,x)f(x)\,dx,$$

where $S_{\mathcal{T}}$ is the *Szegö kernel* of Ω given by (cf. [8])

$$S_{\mathcal{T}}(s+it,x) = \tau_n \int_{\mathcal{C}} e^{i\lambda\cdot(s-x+it)}\,d\lambda.$$

Moreover, the boundary value functions form a closed subspace of $L^2(\mathbb{R}^n)$. The *Szegö projection* $\mathbb{S}_{\mathcal{T}}$ of $\mathcal{T}$ is the orthogonal projection of $L^2(\mathbb{R}^n)$ onto this subspace and it is given by

$$\mathbb{S}_{\mathcal{T}}f(s) = \lim_{t\to 0,\, t\in\mathcal{C}} \int_{\mathbb{R}^n} S_{\mathcal{T}}(s+it,x)f(x)\,dx \qquad (s\in\mathbb{R}^n),$$

where the limit is taken in the L^2 norm. The analytic continuation of $\mathbb{S}_{\mathcal{T}}f, f\in L^2(\mathbb{R}^n)$, to $\mathcal{T}$ is the H^2 function also denoted by $\mathbb{S}_{\mathcal{T}}f$ and defined by

$$\mathbb{S}_{\mathcal{T}}f(s+it) = \int_{\mathbb{R}^n} S_{\mathcal{T}}(s+it,x)f(x)\,dx \quad (s+it\in\Omega).$$

The following theorem has been known for 20 years (cf. [12] and [6], [14]):

THEOREM 5. *The Szegö projection is unbounded on* L^p, $p\neq 2$, $p\in(1,\infty)$, *in the tube over an irreducible self-dual cone of rank greater than* 1.

Now, let $\mathcal{D}$ be a standard bounded realization of the tube $\mathcal{T}$. Such a domain $\mathcal{D}$ is called a *standard bounded symmetric domain of tube type*. The definition of Hardy spaces on $\mathcal{D}$ is as follows. We denote by $\partial_0\mathcal{D}$ the Shilov boundary of $\mathcal{D}$ and by $d\sigma$ a measure on $\partial_0\mathcal{D}$ which is invariant under the stability group of the origin. The Hardy space $\mathcal{H}^p(\mathcal{D})$, $p\geq 1$, consists of those holomorphic functions f on $\mathcal{D}$ which satisfy

$$\|f\|_{H^p(\mathcal{D})} = \sup_{0<r<1}\Big(\int_{\partial_0\mathcal{D}} |f(r\xi)|^p\,d\sigma(\xi)\Big)^{1/p} < \infty.$$

Those functions have radial boundary values, i.e.

$$\lim_{r\to 1} f(r\xi) = f(\xi), \qquad \xi\in\partial_0\mathcal{D},$$

exists in the L^p norm. Moreover, for $p = 2$, the integral representation of every H^2 function in terms of its boundary values is

$$f(z') = \int_{\partial_0\mathcal{D}} S_{\mathcal{D}}(z',\xi)f(\xi)\,d\sigma(\xi) \qquad (z'\in\mathcal{D}),$$

where $S_{\mathcal{D}}$ is the Szegö kernel of $\mathcal{D}$. Furthermore, the boundary value functions form a closed subspace of $L^2(\partial_0\mathcal{D}, d\sigma)$; the Szegö projection $\mathbb{S}_{\mathcal{D}}$ of $\mathcal{D}$ is the orthogonal projection of $L^2(\partial_0\mathcal{D}, d\sigma)$ onto this subspace and it is given by

$$\mathbb{S}_{\mathcal{D}}f(\xi) = \lim_{r\to 1,\, 0<r<1} \int_{\partial_0\mathcal{D}} S_{\mathcal{D}}(r\xi, \eta) f(\eta) d\sigma(\eta) \qquad (\xi \in \partial_0\mathcal{D}),$$

where the limit is taken in the L^2 norm. The analytic continuation of $\mathbb{S}_{\mathcal{D}}f$, $f \in L^2(\partial_0\mathcal{D}, d\sigma)$, to $\mathcal{D}$ is the H^2 function defined by

$$\mathbb{S}_{\mathcal{D}}f(z') = \int_{\partial_0\mathcal{D}} S_{\mathcal{D}}(z', \eta) f(\eta)\, d\sigma(\eta) \qquad (z' \in \mathcal{D}).$$

From our transfer principle, using restricted nontangential convergence, we shall deduce from Theorem 5 the following result:

THEOREM 6. *For every standard irreducible bounded symmetric domain $\mathcal{D}$ of tube type whose rank is greater than* 1, *the Szegö projection is unbounded on $L^p(\partial_0\mathcal{D}, d\sigma)$, $p \in (1, \infty)$, $p \neq 2$.*

On the other hand, we recall that for the unit ball and its Cayley transform, the Szegö projection is bounded on L^p for all $p > 1$ ([11]), but the general case of nontubular domains of rank greater than 1 remains open.

The proof of Theorem 6 is given below for the particular case of the Lie ball ω of $\mathbb{C}^n$, $n \geq 3$. In this case, the following analogue of Theorem 3 for the Szegö projection is due to B. Jöricke [10] (we shall, however, give a different proof based on our transfer principle):

THEOREM 7. *For the Lie ball ω of $\mathbb{C}^n$, $n \geq 3$, the Szegö projection is unbounded from $L^\infty(\partial_0\omega)$ to $L^q(\partial_0\omega, d\sigma)$ when $q \geq 2n/(n-2)$.*

We shall proceed as follows. In the second section, we prove Theorems 1 and 2 in the tube Ω. In fact, Theorem 1 in Ω is a straightforward consequence of the characterization obtained in [2] of those $p \in [1, \infty)$ such that for each $\zeta \in \Omega$, the Bergman kernel $B_\Omega(\zeta, z)$ belongs to $L^p(\Omega, dV(z))$ (Section 2.1). Section 2.2 is devoted to the proof of Theorem 2 in Ω; for the sufficiency, we apply Schur's lemma (cf. [7]) with the same test functions as in [1].

In the third section, we prove Theorems 1 and 2 in the Lie ball and, in the fourth section, we prove Theorems 3 and 4. By means of our transfer principle, we carry over the estimates to the tube Ω where more is known and our computation techniques are more powerful.

Finally, we prove Theorem 6 and we give another proof of B. Jöricke's result (Theorem 7).

2. Proofs of Theorems 1 and 2 in the tube Ω

2.1. *Proof of Theorem* 1 *in* Ω. The tube Ω is a symmetric Siegel domain of type I (hence of type II). Thus the general theorems proved by S. G. Gindikin [8] can be applied to Ω. In particular, the Bergman kernel of Ω has the following expression:

PROPOSITION 2.1. *The Bergman kernel* $B_\Omega(s,z)$ *of* Ω *is given by*

$$B_\Omega(\zeta,z)=c_n\Big[(\zeta_1-\overline{z}_1)(\zeta_2-\overline{z}_2)-\sum_{j=3}^{n}(\zeta_j-\overline{z}_j)^2\Big]^{-n}, \tag{1}$$

where $\zeta=(\zeta_1,\ldots,\zeta_n)$, $z=(z_1,\ldots,z_n)\in\Omega$.

DEFINITION 2.2. Let $k(t,y)$ denote the positive kernel defined on the cone Γ by

$$k(t,y)=\Big[(t_1+y_1)(t_2+y_2)-\sum_{j=3}^{n}(t_j+y_j)^2\Big]^{-n/2},$$
$$t=(t_1,\ldots,t_n),\ y=(y_1,\ldots,y_n).$$

Let T be the integral operator associated with this kernel. We call k (resp. T) the *Hilbert–Gindikin kernel* (resp. the *Hilbert–Gindikin operator*) in Γ.

We first prove the following proposition:

PROPOSITION 2.3. *For each* $p\ge 1$, *there exists a constant* C_p *such that for all* $y\in\Gamma$ *and* $\xi=s+it\in\Omega$,

$$\int\limits_{\mathbb{R}^n}|B_\Omega(\zeta,x+iy)|^p\,dx=C_p[k(t,y)]^{2p-1}. \tag{2}$$

Moreover, *there exists a constant* c_p *such that*, *for each* $y\in\Gamma$ *such that* $|y|<1/100$ *and each* $\zeta=s+it\in\Omega$ *such that* $|\zeta|<1/100$,

$$\int\limits_{I\times\ldots\times I}|B_\Omega(\zeta,x+iy)|^p\,dx\ge C_p[k(t,y)]^{2p-1}, \tag{3}$$

where I *denotes the interval* $[-1,1]$.

Proof. Let us first prove (2). Setting $z=x+iy$, in view of (1) we get

$$\begin{aligned}|B_\Omega(\zeta,z)|^p&=c_n^p|\zeta_2-\overline{z}_2|^{-np}\bigg|\zeta_1-\overline{z}_1-\frac{\sum_{j=3}^n(\zeta_j-\overline{z}_j)^2}{\zeta_2-\overline{z}_2}\bigg|^{-np}\\&=c_n^p|\zeta_2-\overline{z}_2|^{-np}\bigg\{\bigg[s_1-x_1-\operatorname{Re}\frac{\sum_{j=3}^n(\zeta_j-\overline{z}_j)^2}{\zeta_2-\overline{z}_2}\bigg]^2\\&\quad+\bigg[t_1+y_1-\operatorname{Im}\frac{\sum_{j=3}^n(\zeta_j-\overline{z}_j)^2}{\zeta_2-\overline{z}_2}\bigg]^2\bigg\}^{-np/2}.\end{aligned} \tag{4}$$

Integrating first with respect to x_1 in $\mathbb{R}$, we easily obtain

$$\int_{\mathbb{R}} |B_\Omega(\zeta, x+iy)|^p\,dx_1 = c_p|\zeta_2-\overline{z}_2|^{-np}\left[t_1+y_1-\operatorname{Im}\frac{\sum_{j=3}^n(\zeta_j-\overline{z}_j)^2}{\zeta_2-\overline{z}_2}\right]^{-np+1}.$$

Next, we notice that

$$(15)\quad t_1+y_1-\operatorname{Im}\frac{\sum_{j=3}^n(\zeta_j-\overline{z}_j)^2}{\zeta_2-\overline{z}_2} = \frac{1}{|\zeta_2-\overline{z}_2|^2}\bigg\{(t_2+y_2)\sum_{j=3}^n\left[s_j-x_j-\frac{(t_j+y_j)(s_2-x_2)}{t_2+y_2}\right]^2 + |\zeta_2-\overline{z}_2|^2\left[t_1+y_1-\frac{\sum_{j=3}^n(t_j+y_j)^2}{t_2+y_2}\right]^2\bigg\}.$$

Integrating with respect to $dx_3\ldots dx_n$ then yields

$$\int_{\mathbb{R}^{n-1}} |B_\Omega(\zeta, x+iy)|^p\,dx_1\,dx_3\ldots dx_n$$
$$= c_p|\zeta_2-\overline{z}_2|^{-np+n-2}(t_2+y_2)^{-n/2+1}\left[t_1+y_1-\frac{\sum_{j=3}^n(t_j+y_j)^2}{t_2+y_2}\right]^{-np+n/2}.$$

We integrate finally with respect to x_2 in $\mathbb{R}$; it is easy to check that

$$\int_{\mathbb{R}} |\zeta_2-\overline{z}_2|^{-np+n-2}\,dx_2 = c_p(t_2+y_2)^{-np+n-1}.$$

This yields (2).

Let us next prove inequality (3). We keep s, y, t fixed and we denote by $J_2, \ldots, J_n$ the subintervals of $I = [-1,1]$ defined by

$$J_2 = \big\{x_2\in\mathbb{R} : |s_2-x_2| < \tfrac{1}{5}(t_2+y_2)\big\},$$
$$J_j = \big\{x_j\in\mathbb{R} : |s_j-x_j| < \tfrac{1}{100n}\sqrt{(t_1+y_1)(t_2+y_2)}\big\},\qquad j=3,\ldots,n.$$

Then for each $(x_2,\ldots,x_n)\in J_2\times\ldots\times J_n$, we deduce from (5) that

$$0 < t_1+y_1-\frac{\sum_{j=3}^n(t_j-y_j)^2}{t_2+y_2} < t_1+y_1-\operatorname{Im}\frac{\sum_{j=3}^n(\zeta_j-\overline{z}_j)^2}{\zeta_2-\overline{z}_2} < \frac{1}{10}.$$

On the other hand, we have

$$\operatorname{Re}\frac{\sum_{j=3}^n(\zeta_j-\overline{z}_j)^2}{\zeta_2-\overline{z}_2} = \frac{1}{|\zeta_j-\overline{z}_j|^2}\Big\{(s_2-x_2)\Big[\sum_{j=3}^n(s_j-x_j)^2-\sum_{j=3}^n(t_j+y_j)^2\Big] + 2(t_2+y_2)\sum_{j=3}^n(s_j-x_j)(t_j+y_j)\Big\}.$$

Then it is easy to check that for each $(x_2,\dots,x_n)\in J_2\times\dots\times J_n$,

$$\left|\operatorname{Re}\frac{\sum_{j=3}^n(\zeta_j-\overline{z}_j)^2}{\zeta_2-\overline{z}_2}\right|\le\frac{1}{10}$$

and the interval J_1 defined by

$$J_1=\left\{x_1\in\mathbb{R}:\left|s_1-x_1-\operatorname{Re}\frac{\sum_{j=3}^n(\zeta_j-\overline{z}_j)^2}{\zeta_2-\overline{z}_2}\right|\right.\\ \left.\le t_1+y_1-\operatorname{Im}\frac{\sum_{j=3}^n(\zeta_j-\overline{z}_j)^2}{\zeta_2-\overline{z}_2}\right\}$$

is contained in I. Hence, we get

$$\int_{I^n}|B_\Omega(\zeta,z)|^p\,dx\ge\int_{J_1\times\dots\times J_n}|B_\Omega(\zeta,z)|^p\,dx.$$

Now we deduce from (4) that for each $(x_2,\dots,x_n)\in J_2\times\dots\times J_n$,

$$(6)\quad\int_{J_1}|B_\Omega(\zeta,z)|^p\,dx_1=C_p|\zeta_2-\overline{z}_2|^{-np}\left[t_1+y_1-\operatorname{Im}\frac{\sum_{j=3}^n(\zeta_j-\overline{z}_j)^2}{\zeta_2-\overline{z}_2}\right]^{-np+1}.$$

We notice next that for each $x_2\in J_2$, the set E defined by

$$E=\left\{(x_3,\dots,x_n)\in\mathbb{R}^{n-2}:\sum_{j=3}^n\left[s_j-x_j-\frac{(t_j+y_j)(s_2-x_2)}{t_2+y_2}\right]^2\right.\\ \left.<\frac{t_2+y_2}{10^4}\left[t_1+y_1-\frac{\sum_{j=3}^n(t_j+y_j)^2}{t_2+y_2}\right]\right\}$$

is contained in $J_3\times\dots\times J_n$. Hence, in view of (5) and (6), it is easy to see that for each $x_2\in J_2$,

$$(7)\quad\int_{J_1\times J_3\times\dots\times J_n}|B_\Omega(\zeta,z)|^p\,dx_1\,dx_3\dots dx_n\\ \ge C_p'(t_2+y_2)^{-np+n/2-1}\left[t_1+y_1-\frac{\sum_{j=3}^n(t_j+y_j)^2}{t_2-y_2}\right]^{-np+n/2}.$$

Integrating finally with respect to x_2 in J_2 immediately yields (3). This concludes the proof of Proposition 2.3. ■

The next step is the following lemma:

LEMMA 2.4. *Let $p\ge1$. Then for each $t\in\Gamma$, the Hilbert–Gindikin kernel $k(t,y)$ belongs to $L^p(\Gamma,dy)$ if and only if $p>(2n-2)/n$. In this case, there exists a constant C_p such that for each $t\in\Gamma$,*

$$\int_\Gamma[k(t,y)]^p\,dy=C_p[k(t,t)]^{p-1}.$$

Proof. We use the following identity:

$$t_1 + y_1 - \frac{\sum_{j=3}^n (t_j - y_j)^2}{t_2 + y_2} = y_1 - \frac{\sum_{j=3}^n y_j^2}{y_2} + \varphi(t, y_2, \ldots, y_n), \tag{8}$$

where

$$\varphi(t, y_2, \ldots, y_n) = t_1 - \frac{\sum_{j=3}^n t_j^2}{t_2} + \frac{t_2 \sum_{j=3}^n (y_j - y_2 t_j/t_2)^2}{y_2(t_2 + y_2)}.$$

In view of (8), integrating first with respect to y_1 yields

$$\int_{\Sigma_{j=3}^n y_j^2/y_2}^{\infty} [k(t,y)]^p \, dy_1 = C_p (t_2 + y_2)^{-np/2} [\varphi(t, y_2, \ldots, y_n)]^{-np/2+1}.$$

We next integrate with respect to $dy_3 \ldots dy_n$ in $\mathbb{R}^{n-2}$; it is easy to obtain

$$\int_{\mathbb{R}^{n-2}} [\varphi(t, y_2, \ldots, y_n)]^{-np/2+1} dy_3 \ldots dy_n$$

$$= C_p' \left[\frac{y_2(t_2 + y_2)}{t_2} \right]^{n/2-1} \left(t_1 - \frac{\sum_{j=3}^n t_j^2}{t_2} \right)^{-n(p-1)/2}$$

if $p > 1$, and is ∞ if $p = 1$. Finally, integrating with respect to y_2 in $(0, \infty)$ yields the desired conclusion because

$$\int_0^\infty y_2^{n/2-1} (t_2 + y_2)^{-n(p-1)/2-1} \, dy_2 = C_p t_2^{-np/2+n-1},$$

where

$$C_p = \int_0^\infty y_2^{n/2-1} (1 + y_2)^{-n(p-1)/2-1} \, dy_2,$$

and this last integral converges if and only if $p > (2n-2)/n$. ∎

Proof of Theorem 1. The Bergman projection P_Ω is a self-adjoint operator; thus it suffices to prove that P_Ω is unbounded on $L^p(\Omega, dV)$ when $p \in \left[1, \frac{3n-2}{2n}\right]$. More precisely, we shall exhibit a function f_0 in all $L^p(\Omega, dV), p \geq 1$, but such that for all $p \in \left[1, \frac{3n-2}{2n}\right]$, $P_\Omega f_0$ does not belong to $L^p(\Omega, dV)$.

Let e denote the point of Ω given by $e = (i, i, 0, \ldots, 0)$, let β be the Euclidean ball of radius $1/n$, centered at e, and let f_0 be the characteristic function of β. Since β is contained in Ω, by the mean value formula, there exists a constant C_n such that for each $\zeta \in \Omega$, $P_\Omega f_0(\zeta) = C_n B_\Omega(\zeta, e)$. Equality (2) and Lemma 2.4 then yield the desired conclusion. This concludes the proof of Theorem 1 in Ω. ∎

2.2. *Proof of Theorem* 2 *in* Ω. *Sufficiency*. We first prove the following lemma:

LEMMA 2.5. *Let $g_{\gamma,\delta}$ be the positive function in Γ given by*
$$g_{\gamma,\delta}(y) = y_2^\gamma (y_1 y_2 - y_3^2 - \ldots - y_n^2)^\delta.$$
Under the conditions $-1 < \delta < 0$ and $-n/2 < \gamma+\delta < -n/2+1$, there exists a constant $C(\gamma,\delta)$ such that for each $t \in \Gamma$,
$$\int_\Gamma k(t,y) g_{\gamma,\delta}(y)\, dy = C(\gamma,\delta) g_{\gamma,\delta}(t).$$

P r o o f. The proof is very similar to that of Lemma 2.4. We integrate first with respect to y_1 using (8), next with respect to $dy_3 \ldots dy_n$ in $\mathbb{R}^{n-2}$ and lastly with respect to y_2 in $(0,\infty)$.

The next step is to prove the sufficiency of the condition $p \in \left(\frac{2n-2}{n}, \frac{2n-2}{n-2}\right)$ for the boundedness of T on $L^p(\Gamma)$. In view of Schur's Lemma (cf. [7]), it is enough to show that for such a p, there exists a positive test function g in Γ and constants C_1 and C_2 such that

(i) for each $t \in \Gamma$,
$$\int_\Gamma k(t,y)[g(y)]^{p'}\, dy \le C_1 [g(t)]^{p'}, \tag{9}$$

(ii) for each $y \in \Gamma$,
$$\int_\Gamma k(t,y)[g(t)]^{p}\, dt \le C_2 [g(y)]^{p}. \tag{10}$$

For the test function $g_{\gamma,\delta}$ given by Lemma 2.5, inequality (9) (resp. (10)) holds when
$$-\frac{1}{p'} < \delta < 0 \quad \text{and} \quad -\frac{n}{2p'} < \gamma+\delta < -\frac{n-2}{2p'},$$
respectively when
$$-\frac{1}{p} < \delta < 0 \quad \text{and} \quad -\frac{n}{2p'} < \gamma+\delta < -\frac{n-2}{2p}.$$
The two conditions on $\gamma+\delta$ may be simultaneously satisfied if
$$p \in \left(\frac{2n-2}{n}, \frac{2n-2}{n-2}\right). \ \blacksquare$$

Necessity. Again, we first prove the necessity of the condition $p \in \left(\frac{2n-2}{n}, \frac{2n-2}{n-2}\right)$ for the boundedness of T on $L^p(\Gamma)$. By Lemma 2.4, for each $y \in \Gamma$, the Hilbert–Gindikin kernel $k(t,y)$ belongs to $L^p(\Gamma, dt)$ if and only if $p > (2n-2)/n$. Now, the conclusion follows as in the proof of Theorem 1. The test function here is the characteristic function of the Euclidean ball b in $\mathbb{R}^n$, of radius $1/n$, centered at $(1,1,0,\ldots,0)$. Here, the mean value formula is replaced by the following fact, whose proof is easy: for each $t \in \Gamma$ and each $y \in b$, $k(t,y) \ge k(t,(2,2,0,\ldots,0))$.

We next prove that the condition $p \in \left(\frac{2n-2}{n}, \frac{2n-2}{n-2}\right)$ is necessary for the boundedness of P^*_Ω on $L^p(\Omega, dV)$.

Under the assumption that P^*_Ω is bounded on $L^p(\Omega, dV)$, there exists a constant C_p such that for each positive function f in Γ, supported in $\{y : |y| < 1/100\}$,

$$\int\limits_{\{s\in\Omega:|s|<1/50\}} \Big\{ \int\limits_{\{y\in\Gamma:|y|<1/100\}} \Big(\int\limits_{|x_j|<1, j=1,\dots,n} |B_\Omega(\zeta, z)| dx \Big) f(y) dy \Big\}^p dV(s) \le C_p \int\limits_{\{y\in\Gamma:|y|<1/100\}} [f(y)]^p\, dy.$$

Furthermore, in view of (3),

$$\int\limits_{\{t\in\Gamma:|t|<1/100\}} \Big(\int\limits_{\{y\in\Gamma:|y|<1/100\}} k(t,y) f(y)\, dy \Big)^p dt \le C'_p \int\limits_{\{y\in\Gamma:|y|<1/100\}} [f(y)]^p\, dy.$$

Dilating the balls $100N$ times and using the homogeneity of the kernel $k(t, y)$ easily yields that for each positive function f in Γ,

$$\int\limits_{\{t\in\Gamma:|t|<N\}} \Big(\int\limits_{\{y\in\Gamma:|y|<N\}} k(t,y) f(y)\, dy \Big)^p dt \le C'_p \int\limits_{\{y\in\Gamma:|y|<N\}} [f(y)]^p\, dy.$$

When we let N tend to infinity, we get the conclusion that T is bounded on $L^p(\Gamma)$ and hence, according to the first part of the proof, the condition $p \in \left(\frac{2n-2}{n}, \frac{2n-2}{n-2}\right)$ is necessary. This concludes the proof of Theorem 2 in Ω. ■

3. Proofs of Theorems 1 and 2 in the Lie ball: a transfer principle

3.1. *Preliminaries.* Let $z = \Phi(z')$ be the linear fractional mapping from ω onto Ω which is given in [5]. In particular, we assume that $\Phi(0) = e$, where $e = (i, i, 0, \dots, 0)$ and Φ is holomorphic outside $Z = \{z \in \mathbb{C}^n : Q(z) = 0\}$, where Q is a polynomial such that $Q(0) = 1$. In view of the change of variable formula, the Bergman kernel $B_\omega(\zeta', z')$ of ω has the following expression in terms of that of Ω:

$$B_\omega(\zeta', z') = B_\Omega(\Phi(\zeta'), \Phi(z')) J\Phi(\zeta') \overline{J\Phi(z')}. \tag{11}$$

On the other hand, since ω is a circular domain, for each real number θ,

$$B_\omega(e^{i\theta}\zeta', e^{i\theta}z') = B_\omega(\zeta', z') \tag{12}$$

and thus, there exists a constant C such that $B_\omega(\zeta', 0) = C$ for each $\zeta' \in \omega$. Hence, from (11), we get

$$J\Phi(\zeta') = C'[B_\Omega(\Phi(\zeta'), e)]^{-1}. \tag{13}$$

The following lemma is a straightforward consequence of (4) and (5):

LEMMA 3.1. *For all z and ζ in Ω,*

$$|B_\Omega(\zeta, z)| \le B_\Omega(z, z).$$

In the sequel, we let K be the closed unit ball of $\mathbb{C}^n$ and we set $S = \Phi^{-1}(K \cap \Omega)$. We shall use the following lemma:

LEMMA 3.2. *There exist constants c and C such that for each $\zeta' \in S$,*

$$c \le |J\Phi(\zeta')| \le C. \tag{14}$$

P r o o f. The latter inequality follows easily from (13) and formula (1) for B_ω. The former inequality is the particular case of Lemma 3.1 where $z = e$. ■

We shall also use the following lemma:

LEMMA 3.3. *For all ζ' and z' in the closure $\overline{\omega}$ of ω, there exists a real number $\theta = \theta(\zeta', z')$ and there exist bounded open neighborhoods $\mathcal{O}^1(\zeta')$ and $\mathcal{O}^2(z')$ of $e^{i\theta}\zeta'$ and $e^{i\theta}z'$ respectively, such that neither $\mathcal{O}^1(\zeta')$ nor $\mathcal{O}^2(z')$ intersects Z.*

P r o o f. By an obvious argument, it suffices to prove that for all ζ' and z' in $\overline{\omega}$, there exists a real number θ such that neither $e^{i\theta}\zeta'$ nor $e^{i\theta}z'$ belongs to Z. Keeping ζ' and z' fixed, let p and q denote the analytic polynomials in $\mathbb{C}$ given by $p(\lambda) = Q(\lambda z')$ and $q(\lambda) = Q(\lambda\zeta')$. By a contradiction argument, we assume that the product polynomial pq is identically zero on the unit circle. It follows that one polynomial, say p, is identically zero in the complex plane $\mathbb{C}$; but this contradicts the hypothesis $p(0) = q(0) = 1$. ■

3.2. *Proof of Theorem* 1 *and of the necessity part of Theorem* 2 *in ω.* In the sequel, for each compact set Δ in $\mathbb{C}^n$, we let $L^p_\Delta(D)$ denote the subspace of $L^p(D, dV)$ consisting of functions supported in Δ.

We assume that P_ω (resp. P^*_ω) is bounded on $L^p(\omega, dV)$. Then by (14), it is easy to deduce that P_Ω (resp. P^*_Ω) is bounded from $L^p_K(\Omega)$ to $L^p(K \cap \Omega, dV)$. In view of Theorems 1 and 2 in Ω, it is then enough to prove the following lemma:

LEMMA 3.4. *Assume that P_Ω (resp. P^*_Ω) is bounded from $L^p_K(\Omega)$ to $L^p(K \cap \Omega, dV)$. Then P_Ω (resp. P^*_Ω) is bounded on $L^p(\Omega, dV)$.*

P r o o f. At the end of the proof of Theorem 2 in Ω, we proved the analogous result in Γ for the kernel $k(t, y)$; we again use the same argument.

Let $\mathcal{P}$ denote either P_Ω or P^*_Ω and let $Q(\cdot,\cdot)$ be its kernel. Since $\mathcal{P}$ is bounded from $L^p_K(\Omega)$ to $L^p(K\cap\Omega, dV)$, there exists a constant C_p such that for each $\mathcal{C}^\infty$ function f in Ω with compact support,

$$\int_{\{\zeta\in\Omega:|\zeta|\le 1\}} \Big| \int_{\{z\in\Omega:|z|\le 1\}} Q(\zeta,z)f(z)\,dV(z)\Big|^p\,dV(\zeta) \le C_p \int_{\{z\in\Omega:|z|\le 1\}} |f(z)|^p\,dV(z).$$

Dilating the balls N times and using the homogeneity of the kernel Q yields

$$\int_{\{\zeta\in\Omega:|\zeta|\le N\}} \Big| \int_{\{z\in\Omega:|z|\le N\}} Q(\zeta,z)f(z)\,dV(z)\Big|^p\,dV(\zeta) \le C_p \int_{\{z\in\Omega:|z|\le N\}} |f(z)|^p\,dV(z)$$

for each $\mathcal{C}^\infty$ function f with compact support. When we let N tend to infinity, we conclude that $\mathcal{P}$ is bounded on $L^p(\Omega)$. ■

3.3. *Proof of the sufficiency part of Theorem 2 in ω.* Since $\Phi^{-1}(\{\infty\})$ is obviously contained in Z, the following lemma is a straightforward consequence of Theorem 2 in Ω and (14):

LEMMA 3.5. *Let K' be a compact set in $\mathbb{C}^n$ such that $K'\cap Z=\emptyset$ and the interior of $K'\cap\omega$ is nonempty. Then for each $p\in\big(\frac{2n-2}{n},\frac{2n-2}{n-2}\big)$, P^*_ω is bounded from $L^p_K(\omega)$ to $L^p(K\cap\omega, dV)$.*

Now, in view of Lemma 3.3, since $\overline{\omega}\times\overline{\omega}$ is compact, its open covering

$$\{e^{-i\theta(\zeta',z')}(\mathcal{O}^1(\zeta')\times\mathcal{O}^2(z')) : (\zeta',z')\in\overline{\omega}\times\overline{\omega}\}$$

contains a finite covering $\{e^{-i\theta_j}(\mathcal{O}^1_j\times\mathcal{O}^2_j) : j=1,\ldots,N\}$ and the set $K'=\bigcup_{j=1}^N(\overline{\mathcal{O}^1_j}\cup\overline{\mathcal{O}^2_j})$ is a compact set in $\mathbb{C}^n$ such that $K'\cap Z=\emptyset$.

Thus, for all positive functions f and g, we get

$$\begin{aligned}\iint_{\omega\times\omega} & |B_\omega(\zeta',z')|f(z')g(\zeta')\,dV(z')\,dV(\zeta')\\ &\le \sum_{j=1}^N \iint_{e^{-i\theta_j}\mathcal{O}^1_j\times e^{-i\theta_j}\mathcal{O}^2_j} |B_\omega(\zeta',z')|f(z')g(\zeta')\,dV(z')\,dV(\zeta')\\ &\le \sum_{j=1}^N \int_{K'\cap\omega}\int_{K'\cap\omega} |B_\omega(\zeta',z')|f(e^{-i\theta_j}z')g(e^{-i\theta_j}\zeta')\,dV(z')\,dV(\zeta'),\end{aligned}$$

since ω is circular. By Lemma 3.5, it is then easy to conclude that P^*_ω is bounded on $L^p(\omega, dV)$ when $p\in\big(\frac{2n-2}{n},\frac{2n-2}{n-2}\big)$. This concludes the proof of Theorem 2 in ω. ■

4. Proofs of Theorems 3 and 4

4.1. *Proof of Theorem* 4. Since P_ω is a self-adjoint operator, it suffices to prove that P_ω is unbounded from $L^{p'}(\omega, dV)$ to $L^1(\omega, dV)$ when $p' \in \big(1, \frac{4n}{3n+2}\big)$. Furthermore, as at the beginning of the proof of Theorem 1 in ω (cf. 3.2), it is enough to prove that for such a p', P_Ω is unbounded from $L^{p'}_K(\Omega)$ to $L^1(K\cap\Omega, dV)$. Let $b_\tau, \tau \in (0,1/2)$, denote the Euclidean ball of radius $\tau/(100n)$, centered at $(i\tau/16, i\tau, 0, \ldots, 0)$. This ball is contained in Ω; then by the mean value formula, there exists a constant C_n such that for each $s \in \Omega$ and each $\tau \in (0,1/2)$,

$$P_\Omega \chi_{b_\tau}(\zeta) = C_n \tau^{2n} B_\Omega(\zeta, (i\tau/16, i\tau, 0, \ldots, 0)).$$

Hence, by (3), we get

$$\int\limits_{K\cap\Omega} |P_\Omega \chi_{b_\tau}(\zeta)|\, dV(\zeta) \ge C'_n \tau^{2n} I(\tau/16, \tau, 0, \ldots, 0), \tag{15}$$

where, for each $y \in \Gamma$, we set

$$I(y) = \int\limits_{\{t\in\Gamma: t_1<1,\, t_2<1\}} k(t,y)\, dt. \tag{16}$$

The key lemma is the following:

Lemma 4.1. *There exists a constant C_n such that, for each $y \in \Gamma$ such that $y_1 \le y_2/16$ and $y_2 < 1/2$,*

$$I(y) \ge C_n y_2^{-(n/2-1)}. \tag{17}$$

Proof. Let b denote the ball in $\mathbb{R}^{n-2}$ given by

$$b = \Big\{(t_3, \ldots, t_n) : \sum_{j=3}^n t_j^2/t_2 < 1 - y_1 + \sum_{j=3}^n y_j^2/y_2\Big\}.$$

Then for each $(t_3, \ldots, t_n) \in b$, the interval $\{t_1 \in \mathbb{R} : \sum_{j=3}^n t_j^2/t_2 < t_1 < 1\}$ contains the interval $\{t_1 : 0 < t_1 - \sum_{j=3}^n t_j^2/t_2 < y_1 - \sum_{j=3}^n y_j^2/y_2\}$. Now, in view of (8), we get

$$I(y) \ge C_n \bigg(y_1 - \frac{\sum_{j=3}^n y_j^2}{y_2}\bigg) \times \int_0^1 (t_2 + y_2)^{-n/2} \Big(\int_b \varphi(y, t_2, \ldots, t_n)\, dt_3 \ldots dt_n\Big)^{-n/2} dt_2.$$

On the other hand, under the assumption $y_1 \le y_2/16$, the ball b contains the ball

$$b' = \bigg\{(t_3, \ldots, t_n) : \sum_{j=3}^n \Big(y_j - \frac{y_2 t_j}{t_2}\Big)^2 < \frac{y_2(t_2+y_2)}{t_2}\bigg(y_1 - \frac{\sum_{j=3}^n t_j^2}{y_2}\bigg)\bigg\};$$

thus,

$$\int_b \varphi^{-n/2}\, dt_3 \ldots dt_n \geq C_n \left(y_1 - \frac{\sum_{j=3}^n y_j^2}{y_2} \right)^{-1} \left[\frac{t_2(t_2+y_2)}{y_2} \right]^{n/2-1}.$$

Furthermore, since $y_2 < 1/2$, we get

$$I(y) \geq C_n y_2^{-n/2+1} \int_{1/2}^{1} (t_2+y_2)^{-1} t_2^{n/2-1}\, dt_2 \geq C_n' y_2^{-n/2+1}. \ ■$$

In view of (17), the left hand side of (15) is greater than $C_n \tau^{3n/2+1}$; thus, the boundedness of P_Ω from $L_K^{p'}(\Omega)$ to $L^1(K \cap \Omega, dV)$ implies the existence of a constant C_p such that, for each $\tau < 1/2$, $\tau^{3n/2+1} \leq C_p \tau^{2n/p'}$. Therefore the condition $p' > 4n/(3n+2)$ is necessary. This concludes the proof of Theorem 4.

4.2. *Proof of Theorem* 3. Let (E) denote the estimate

$$\text{(E)} \qquad \int_\omega \Big(\int_\omega |B_\omega(\zeta', z')|\, dV(\zeta') \Big)^p dV(z') < \infty.$$

It is easy to reduce Theorem 3 to the following equivalence: (E) holds if and only if $p \in \big(0, \frac{2n}{n-2}\big)$. Furthermore, in view of the end of the proof of Theorem 2 in ω (cf. 3.3), estimate (E) is equivalent to the following estimate: for each compact set K' in $\mathbb{C}^n$ such that $K' \cap Z = \emptyset$,

$$\text{(E}'\text{)} \qquad \int_{K'\cap\omega} \Big(\int_{K'\cap\omega} |B_\omega(\zeta', z')|\, dV(z') \Big)^p dV(\zeta') < \infty.$$

When carried over to the unbounded domain Ω, estimate (E$'$) takes the following form:

$$\text{(E}''\text{)} \qquad \int_{K\cap\Omega} \Big(\int_{K\cap\Omega} |B_\Omega(\zeta, z)|\, dV(z) \Big)^p dV(\zeta) < \infty,$$

where $K = \{z \in \mathbb{C}^n : |z| \leq 1\}$. But in view of Proposition 2.3, (E$''$) is equivalent to

$$(18) \qquad I_p = \int_{\{t\in\Gamma : t_1<1,\, t_2<1\}} (I(t))^p\, dt < \infty,$$

where $I(t)$ is the integral given by (16).

We first assume (18). In view of (17),

$$I_p \geq C_p \int_{\{t\in\Gamma : t_1 \leq t_2/16,\, t_2<1/2\}} t_2^{-(n/2-1)p}\, dt = C_p' \int_0^{1/2} t_2^{n-1-(n/2-1)p}\, dt_2.$$

This last integral converges only if $p < 2n/(n-2)$. This proves the necessity.

Conversely, assume that $p < 2n/(n-2)$. To get (18), the key lemma is the following:

LEMMA 4.2. *There exists a constant C_n such that for each $t \in \Gamma$ such that $t_1 < t_2 < 1$, the integral $I(t)$ given by* (16) *satisfies*

$$I(t) \le C_n t_2^{-(n/2-1)} \log 4\Big(t_1 - t_2^{-1}\sum_{j=3}^n t_j^2\Big)^{-1}.$$

Proof. In view of (8), we get

$$\int_{\Sigma_{j=3}^n y_j^2/y_2}^1 k(t,y)\,dy_1 \le \frac{2}{n-2}(t_2+y_2)^{-n/2}[\varphi(t,y_2,\dots,y_n)]^{-n/2+1}. \tag{19}$$

Integrating next with respect to $dy_3\dots dy_n$ gives

$$\int_{\Sigma_{j=3}^n y_j^2<y_2} [\varphi(t,y_2,\dots,y_n)]^{-n/2+1}\,dy_3\dots dy_n = I_1(t,y_2)+I_2(t,y_2), \tag{20}$$

where $I_1(t,y_2)$ is the integral over the set

$$E_1 = \Big\{(y_3,\dots,y_n) : \sum_{j=3}^n y_j^2 < y_2,\ \frac{t_2\sum_{j=3}^n(y_j - y_2t_j/t_2)^2}{y_2(t_2+u_2)} < t_1 - \frac{\sum_{j=3}^n t_j^2}{t_2}\Big\}$$

and $I_2(t,y_2)$ is the integral over the set

$$E_2 = \Big\{(y_3,\dots,y_n) : \sum_{j=3}^n y_j^2 < y_2,\ \frac{t_2\sum_{j=3}^n(y_j - y_2t_j/t_2)^2}{y_2(t_2+u_2)} > t_1 - \frac{\sum_{j=3}^n t_j^2}{t_2}\Big\}.$$

Clearly $I_1(t,y_2)$ is bounded by $(t_1 - \sum_{j=3}^n t_j^2/t_2)^{-n/2+1}|E_1|$, which gives

$$I_1(t,y_2) \le C_n\Big[\frac{y_2(t_2+y_2)}{t_2}\Big]^{n/2-1}. \tag{21}$$

Since $0 < t_1 < t_2 < 1$ implies that $\sum_{j=3}^n t_j^2 < t_2$, we get

$$I_2(t,y_2) \le \Big[\frac{y_2(t_2+u_2)}{t_2}\Big]^{n/2-1}\int\Big[\sum_{j=3}^n\Big(y_j - \frac{y_2t_j}{t_2}\Big)^2\Big]^{-n/2+1} dy_3\dots dy_n,$$

where the integral on the right hand side is taken over

$$\Big\{(y_3,\dots,y_n) : \frac{(t_1 - \sum_{j=3}^n t_j^2/t_2)y_2(t_2+y_2)}{t_2} < \sum_{j=3}^n\Big(y_j - \frac{y_2t_j}{t_2}\Big)^2 < 4y_2\Big\}.$$

Thus,

$$I_2(t,y_2) \le C_n\Big[\frac{y_2(t_2+y_2)}{t_2}\Big]^{n/2-1}\log 4\Big(t_1 - t_2^{-1}\sum_{j=3}^n t_j^2\Big)^{-1}. \tag{22}$$

Now, from (18), (20), (21) and (22), we conclude the proof by integrating over y_2. ■

We can now prove (18) under the assumption that $p < 2n/(n-2)$. In view of Lemma 4.2, it is enough to prove that, for such a p,

$$(23) \qquad \int\limits_{\{t\in\Gamma:0<t_1<t_2<1\}} t_2^{-(n/2-1)p}\Big(t_1 - \frac{\sum_{j=3}^n t_j^2}{t_2}\Big)^{\varepsilon p} dt < \infty$$

for some $\varepsilon > 0$. But for $\varepsilon < 1/p$, this integral is equal to

$$C_{\varepsilon,p}\int\limits_0^1 t_2^{n-1-\varepsilon p-(n/2-1)p}\,dt_2.$$

For $p < 2n/(n-2)$, we may take $\varepsilon < \inf\{1/p, n/p - n/2 + 1\}$ to have convergence. ■

Remark. In view of the homogeneity of the Bergman kernel of the unbounded domain Ω, it is easy to show that the statements analogous to Theorems 3 and 4 are false in Ω. However, the following local statements hold:

THEOREM 4.3. *The operator* P_Ω^* *is bounded from* $L_K^\infty(\Omega)$ *to* $L^p(K\cap\Omega, dV)$ *if and only if* $p < 2n/(n-2)$.

THEOREM 4.4. *The Bergman projection* P_Ω *is unbounded from* $L_K^\infty(\Omega)$ *to* $L^p(K\cap\Omega, dV)$ *when* $p > 4n/(n-2)$.

5. Proofs of Theorems 6 and 7

5.1. *The Szegö projection of* ω: *preliminary results.* The Shilov boundary $\partial_0\omega$ of the Lie ball ω of $\mathbb{C}^n$, $n \ge 3$, is given by

$$\partial_0\omega = \{e^{i\theta}x : \theta\in[0,2\pi), x\in S_{n-1}\},$$

where S_{n-1} denotes the unit sphere of $\mathbb{R}^n$. An invariant measure on $\partial_0\omega$ is $d\sigma(e^{i\theta}x) = d\theta\,d\mu(x)$, where $d\mu$ is the Lebesgue measure on S_{n-1}. The Szegö and Bergman kernels of ω are respectively given by the following formulae [9]: for $e^{i\theta}x\in\partial_0\omega$ and $\zeta'\in\overline{\omega}$ such that $\zeta'\ne e^{i\theta}x$,

$$S_\omega(\zeta', e^{i\theta}x) = \tau_n\Big[1 - 2e^{-i\theta}x\cdot\zeta' + e^{-2i\theta}\Big(\sum_{j=1}^n \zeta_j'^2\Big)\Big]^{-n/2};$$

respectively for ζ' and z' in $\mathbb{C}^n$,

$$B_\omega(\zeta', z') = \tau_n'\Big[1 - 2\zeta'\overline{z}' + \Big(\sum_{j=1}^n \zeta_j'^2\Big)\Big(\sum_{j=1}^n \overline{z}_j'^2\Big)\Big]^{-n}.$$

On the other hand, the Szegö kernel of Ω is given by (cf. [8])

$$S_\Omega(s+it,x) = C_n\Big[(s_1-x_1+it_1)(s_2-x_2+it_2) - \sum_{j=3}^{n}(s_j-x_j+it_j)^2\Big]^{-n/2},$$

where $s+it \in \Omega$ and $x\in\mathbb{R}^n$. So, in view of (11), for all $\zeta'\in\overline{\omega}\setminus Z$ and $e^{i\theta}x\in\partial_0\omega\setminus Z$ such that $\zeta'\neq e^{i\theta}x$,

$$[S_\omega(\zeta',e^{i\theta}x)]^2 = [S_\Omega(\Phi(\zeta),\Phi(e^{i\theta}x))]^2 J\Phi(\zeta')\overline{J\Phi(e^{i\theta}x)}. \tag{24}$$

We shall use the following lemma:

LEMMA 5.1. *Let E denote the complement of $\Phi^{-1}(\{\infty\})$ in $\partial_0\omega$. There exists a $\mathcal{C}^\infty$ real function η in $\mathbb{R}^n$ such that for each $e^{i\theta}x\in E$, if $s=\Phi(e^{i\theta}x)$, then*

$$d\sigma(e^{i\theta}x) = \eta(s)\,ds.$$

Moreover, if Δ is a compact set in $\mathbb{R}^n$, then there exist two positive constants c and C such that for each $s\in\Delta$,

$$c\le|\eta(s)|\le C.$$

Proof. $\Phi: E\to\mathbb{R}^n$ is a $\mathcal{C}^\infty$ diffeomorphism. ∎

5.2. *Proof of Theorem* 6. By a contradiction argument, we assume that there exists a $p\in(1,\infty)$, $p\neq 2$, such that the Szegö projection of ω is bounded on $L^p(\partial_0\omega, d\sigma)$. Then there exists a constant C_p such that for each compact set K' in $\mathbb{C}^n$ satisfying $K'\cap Z=\emptyset$ and for each $\mathcal{C}^\infty$ function f with compact support in $\partial_0\omega$,

$$\int\limits_{K'\cap\partial_0\omega}\Big|\lim_{r\to1}\int\limits_{K'\cap\partial_0\omega} S_\omega(re^{i\theta}\xi,e^{i\varphi}x)f(e^{i\theta}x)\,d\sigma(e^{i\theta}x)\Big|^p d\sigma(e^{i\varphi}\xi) \le C_p\int\limits_{K'\cap\partial_0\omega}|f(e^{i\theta}x)|^p\,d\sigma(e^{i\theta}x).$$

We are going to carry over this estimate to the Shilov boundary $\mathbb{R}^n$ of the tube Ω. We set $s=\Phi(e^{i\varphi}\xi)$, $u=\Phi(e^{i\theta}x)$ and we define a family of curves in Ω by $\gamma_r(s)=\Phi(re^{i\varphi}\xi)$, $r\in[0,1)$. Then $\gamma_0(s)=e$, and $\lim_{r\to1}\gamma_r(s)=s$. Moreover, in view of (24) and Lemma 5.1, one easily shows that there exists a constant C_p such that for each $\mathcal{C}^\infty$ function g with compact support in the unit ball of $\mathbb{R}^n$,

$$\int\limits_{|s|\le1}\Big|\lim_{r\to1}\int\limits_{|u|\le1} S_\Omega(\gamma_r(s),u)g(u)\,du\Big|^p ds \le C_p\int\limits_{|s|\le1}|g(s)|^p\,ds. \tag{25}$$

Then the analytic continuation to Ω of its Szegö projection belongs to $H^2(\Omega)$ and hence (cf. [13], p. 119), it has restricted nontangential limits for almost every $s\in\mathbb{R}^n$. Moreover, the boundary value function in this sense coincides with the Szegö projection $\mathbb{S}_\Omega g$ of g. On the other hand, for $s \in \mathbb{R}^n$ fixed, the curve $\{\gamma_r(s) : 0 \le r < 1\}$ is the image by Φ of a radius in ω; then it is easy to show that there exists a proper subcone Γ_0 of Γ and a positive number α such that for each s in the compact set $\{s \in \mathbb{R}^n : |s| \le 1\}$,

(i) the imaginary part of $\gamma(s)$ belongs to Γ_0 and
(ii) $|\mathrm{Re}\,\gamma_r(s) - s| < \alpha |\mathrm{Im}\,\gamma_r(s)|$.

(The curve $r \to \gamma_r(s)$, which goes inside Γ from s, has a tangent vector $\frac{d}{dr}\gamma(s)|_{r=1}$ whose imaginary part is $\neq 0$ and belongs to some proper subcone. So $\mathrm{Im}\,\gamma_r(s)$ is in some proper subcone of Γ for $1-r$ small by Taylor's formula. All constants may be uniformly bounded on compact sets.)

So, by restricted nontangential convergence, for $|s| \le 1$,

$$\lim_{r\to 1} \int_{\mathbb{R}^n} S_\Omega(\gamma_r(s), u) g(u)\, du = \mathbb{S}_\Omega g(s) \qquad \text{a.e.},$$

and by Fatou's lemma, we deduce from (25) that

$$\Big| \int_{\{s\in\mathbb{R}^n : |s|\le 1\}} S_\Omega g(s) \Big|^p ds \le C_p \int_{\mathbb{R}^n} |g(s)|^p\, ds. \tag{26}$$

Dilating the balls N times in (26) and using the homogeneity of the Szegö kernel yields

$$\int_{\{s\in\mathbb{R}^n : |s|\le N\}} |\mathbb{S}_\Omega g(s)|^p\, ds \le C_p \int_{\mathbb{R}^n} |g(s)|^p\, ds$$

for g compactly supported.

Now, when we let N tend to infinity, we conclude that for each $\mathcal{C}^\infty$ function g with compact support,

$$\int_{\mathbb{R}^n} |\mathbb{S}_\Omega g(s)|^p\, ds \le C_p \int_{\mathbb{R}^n} |g(s)|^p\, ds.$$

This contradicts the negative result for the tube Ω stated as Theorem 5 in the introduction. ■

5.3. *Proof of Theorem* 7. Assume that $\mathbb{S}_\omega$ is bounded from $L^\infty(\partial_0\omega)$ to $L^q(\partial_0\omega, d\sigma)$. We carry over this estimate to the Shilov boundary $\mathbb{R}^n$ of Ω. As in the proof of Theorem 6 (see 5.2), in view of (24) and Lemma 5.1, we have the following: there exists a constant C_p such that for each bounded function g supported in the closed ball $b = \{s \in \mathbb{R}^n : |s| \le \sqrt{n}\}$,

$$\int_b |\mathbb{S}_\Omega g(s)|^q\, ds \le C_q \|g\|_\infty. \tag{27}$$

Thus, in the particular case where g is the characteristic function of $b\cap(-\Gamma)$, estimate (27) implies

$$\int\limits_{\{t\in\Gamma:t_1<1,\,t_2<1\}} (I(t))^p\,dt<\infty,$$

where $I(t)$ is the integral given by (16). Now, one realizes that this last estimate is nothing but estimate (18) and we proved in 5.2 that the condition $p<2n/(n-2)$ is necessary for its validity. This concludes the proof of Theorem 7.

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