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ON THE DENSITY OF SETS IN $(\mathbf{A}/\mathbb{Q})^n$ DEFINED BY POLYNOMIALS

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A theorem of Hermann Weyl (see [1]) states that if $\alpha_1, \ldots, \alpha_n$ are irrational, then the set

$$\{(\alpha_1 x, \alpha_2 x^2, \ldots, \alpha_n x^n) \bmod \mathbb{Z} : x \in \mathbb{N}\}$$

is dense in $(\mathbb{R}/\mathbb{Z})^n$. (The condition that the α_j all be irrational is also clearly necessary; for instance, if $\alpha_1 = p/q \in \mathbb{Q}$, then the points $(\alpha_1 x, \ldots, \alpha_n x^n)$ all lie on the hyperplanes $(1/q)\mathbb{Z} \times \mathbb{R}^{n-1}$.) Weyl's proof of his theorem relied on another well-known theorem of his. Say that the sequence $\{y_n : n \in \mathbb{N}\}$ is *uniformly distributed* mod 1 if for every interval $[a, b] \subseteq [0, 1]$,

$$\lim_{n\to\infty} \frac{1}{n} \operatorname{Card}\{y_j : j \le n \text{ and } y_j \in \mathbb{Z} + [a, b]\} = b - a.$$

That is, for every interval I, the probability of an element in the first n terms of the sequence belonging to I mod 1 converges to the length of I. Weyl proved that if $P(x) = \sum_{j=1}^{\infty} \alpha_j x^j$ is a polynomial such that at least one of the α_j is irrational, then $\{P(1), P(2), \ldots\}$ is uniformly distributed.

This paper is concerned with similar results in the case where $\alpha_1, \ldots, \alpha_n$ are in the adeles $\mathbf{A}$ (over the rationals), x takes on values in $\mathbb{Q}$, and we are interested in the compact group $\mathbf{A}/\mathbb{Q}$. The result analogous to Weyl's first theorem is:

THEOREM 1. *Let $\alpha_1, \ldots, \alpha_n$ be non-rational elements of $\mathbf{A}$. Then the set $\{(\alpha_1 x, \alpha_2 x^2, \ldots, \alpha_n x^n) \bmod \mathbb{Q} : x \in \mathbb{Q}\}$ is dense in $(\mathbf{A}/\mathbb{Q})^n$.*

This theorem is useful in the following setting: let G be the discrete group of $\mathbb{Q}$-rational points of a nilpotent algebraic group defined over $\mathbb{Q}$. The Lie algebra $\mathfrak{g}$ corresponding to G is a $\mathbb{Q}$-vector space, and it is natural to consider coadjoint orbits in the dual of $\mathfrak{g}$. A consequence of Theorem 1 is that the closure of any such orbit is "flat" (a coset of the annihilator of a subspace

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of $\mathfrak{g}$). We will show in a future paper how this can be used in studying the representations of G; Theorem 1 appears to have some independent interest, however. We prove it below.

It is harder to give a precise analogue to Weyl's theorem on uniform distributions, because $\mathbb{Q}$, unlike $\mathbb{N}$, does not have a natural order. (It is true that if a countable set R is dense in a separable compact group G, then R can be arranged in a sequence so that it is uniformly distributed; a proof is given on pp. 185–186 of [4]. However, the proof says nothing about the order, and, of course, such a result does not help us to prove anything about density.) Weyl used the following criterion: $\{y_n\}$ is uniformly distributed $\Leftrightarrow \lim_{N\to\infty} N^{-1}\sum_{n=1}^{N} e^{2\pi i r y_n} = 0$ for all non-zero $r \in \mathbb{Z}$. In our procedure, a similar role is played by:

Proposition 1. *Let G be any compact Abelian group. The countable set R is dense in G if for any finite set $\{X_1, \ldots, X_k\}$ of non-trivial characters of R and every $\varepsilon > 0$, there is a finite subset $\{z_1, \ldots, z_{N(\varepsilon)}\}$ of R such that*

$$\frac{1}{N(\varepsilon)}\Big|\sum_{n=1}^{N(\varepsilon)} X_j(z_n)\Big| < \varepsilon, \qquad 1 \le j \le k.$$

Proof. Let dx be normalized Haar measure on G. If R is not dense in G, then there is a continuous non-negative function ϕ on G such that $\int_G \phi(x)\,dx = 1$ and $R \cap \operatorname{supp}\phi = 0$. By Stone–Weierstrass, we can find a function $f(x) = \sum_{j=1}^{n} c_j\chi_j(x)$ ($\chi_j \in \widehat{G}$, $\forall j$) such that $\|\phi - f\|_\infty < 1/3$. Then

$$\Big|\int_G (f(x) - \phi(x))\,dx\Big| \le \int_G |f(x) - \phi(x)|\,dx < 1/3.$$

Let χ_1 be the trivial character. Since $\int_G \chi_j(x)\,dx = 0$ for $j > 1$, we have

$$|c_1 - 1| < 1/3\,, \qquad |c_1| > 2/3.$$

The hypothesis says that for $f_1 = f - c_1\chi_1$, there exists a finite subset $\{z_1, \ldots, z_N\}$ of R such that

$$N^{-1}\Big|\sum_{n=1}^{N} f_1(z_n)\Big| < 1/3.$$

Then

$$N^{-1}\Big|\sum_{n=1}^{N} f(z_n)\Big| \ge c_1 - 1/3 > 1/3.$$

However, $\phi(z_n) = 0$ for all n and $\|f - \phi\| < 1/3$; hence

$$N^{-1}\Big|\sum_{n=1}^{N} f(z_n)\Big| \le 1/3,$$

a contradiction. This proves Proposition 1.

We now consider the case where $G = (\mathbf{A}/\mathbb{Q})^n$. A standard reference for the facts we need about harmonic analysis on $\mathbf{A}$ is Tate's thesis, in [2]. We recall that $\mathbf{A} = \mathbb{R} \times \prod'_{p \text{ prime}}(\mathbb{Q}_p; \mathbb{Z}_p)$; this means that a typical element of $\mathbf{A}$ is

$$\mathbf{x} = (x_\infty, x_2, x_3, \ldots),$$
$$x_\infty \in \mathbb{R},\ x_p \in \mathbb{Q}_p,\ x_p \in \mathbb{Z}_p \text{ except for finitely many } p.$$

We write $\mathbf{A} = \mathbb{R} \times \mathbf{A}_{\mathrm{f}}$, $\mathbf{A}_{\mathrm{f}} = \prod'_{p \text{ prime}}(\mathbb{Q}_p; \mathbb{Z}_p)$; $\mathbf{A}_{\mathrm{f}}$ is topologized by decreeing that $\prod_{p \text{ prime}} \mathbb{Z}_p$ is open. Then $\mathbf{A}$ is a topological ring. We embed $\mathbb{Q}$ in $\mathbf{A}$ diagonally, by $x \mapsto (x, x, \ldots)$. Then $\mathbb{Q}$ is discrete and cocompact in $\mathbf{A}$.

We define fundamental characters χ_p on $\mathbb{Q}_p$ (where $\mathbb{Q}_\infty = \mathbb{R}$) by

$$\chi_\infty(x) = e^{-2\pi i x}; \quad \chi_p(a/p^n) = e^{2\pi i a/p^n} \quad \text{for } a \in \mathbb{Z}; \quad \chi_p = 1 \quad \text{on } \mathbb{Z}_p\,.$$

Define $\chi \in \widehat{A}$ by

$$\chi(\mathbf{x}) = \prod_p \chi_p(x_p).$$

(All but finitely many terms in the product are 1.) A fundamental result is:

Theorem ([2]). (a) *The map* $\mathbf{y} \mapsto \chi_{\mathbf{y}}$, $\chi_{\mathbf{y}}(\mathbf{x}) = \chi(\mathbf{xy})$, *is a topological isomorphism of* $\mathbf{A}$ *onto* $\widehat{\mathbf{A}}$.

(b) *Under this identification of* $\mathbf{A}$ *with* $\widehat{\mathbf{A}}$, $\mathbb{Q}^\perp = \mathbb{Q}$. *Therefore* $(\mathbf{A}/\mathbb{Q})^\wedge \simeq \mathbb{Q}$. *Similarly,* $((\mathbf{A}/\mathbb{Q})^n)^\wedge \simeq \mathbb{Q}^n$; *for* $\mathbf{x} = (x_1, \ldots, x_n) \in (\mathbf{A}/\mathbb{Q})^n$ *and* $q = (q_1, \ldots, q_n) \in \mathbb{Q}^n$, $\chi_q(\mathbf{x}) = \prod_{j=1}^n \chi_{q_j}(x_j)$.

We now apply Proposition 1 (and its corollary) to prove Theorem 1. Since any character χ' of $(\mathbf{A}/\mathbb{Q})^n$ satisfies $\chi'((\mathbf{a}r, \mathbf{a}r^2, \ldots, \mathbf{a}r^n)) = \chi(f(r))$ for some non-trivial polynomial $f : \mathbb{Q} \to \mathbf{A}$ without constant term and with at least one non-rational coefficient, it suffices to show that if $f_1, \ldots, f_k$ are such polynomials and χ is the standard character, then for every positive integer n there is a subset $R_n \subseteq \mathbb{Q}$ such that

$$|R_n|^{-1} \Big| \sum_{x \in R_n} \chi(f_j(x)) \Big| < n^{-1} \quad \text{for all } j = 1, \ldots, k.$$

The coefficients of the polynomials are determined mod $\mathbb{Q}$; we normalize most of them by letting the real components be 0 whenever they are rational. We assume that all real components of f_j are 0 for $1 \le j \le k_1$ and that some real component is irrational for $j > k_1$; we deal with the $j \le k_1$ first. The estimates that we need are consequences of the following statements, all easy to verify:

(a) Write $f_{p,j}$ for the $\mathbb{Q}_p$-component of f_j. Suppose that $\chi_{p,j}$ is not trivial; then for every $m \ge 0$ there is an $M > 0$ such that $\chi_p(f_{p,j}(a/p^m + p^M)) = \chi_p(f_{p,j}(a/p^m))$, for all $a \in \mathbb{Z}$. (For χ_p is trivial on $\mathbb{Z}_p$, and Taylor's Theorem shows that one can choose M such that $f_{p,j}(a/p^m + p^M) - f_{p,j}(a/p^m) \in \mathbb{Z}_p$.)

(b) Let m, M be as in (a), and let q be prime to p. Then for any $b \in \mathbb{Z}$,

$$\sum_{a=1}^{p^M} \chi_p\left(f_{p,j}\left(\frac{a}{p^m}\right)\right) = \sum_{a=1}^{p^M} \chi_p\left(f_{p,j}\left(\frac{a}{p^m} + \frac{b}{q}\right)\right).$$

(For there is an integer r such that $b/q - r \in (p)^{M+m}$; from (a), we may replace b/q in the sum with r. But it is also clear from (a) that the sum is independent of r.)

(c) Write

$$A(j;p,m,M) = \left|p^{-(m+M)} \sum_{a=1}^{p^{m+M}} \chi_p\left(f_{p,j}\left(\frac{a}{p^m}\right)\right)\right|,$$

where m, M are related as in (a). Let $S = \{p_1, \ldots, p_\nu\}$ be a finite set of primes (with $\infty \notin S$) such that $p \in S$ if $f_{p,j}$ has a coefficient not in $\mathbb{Z}_p$. For $p_\sigma \in S$, let m_σ, N_σ correspond as in (a), let $p_S^m = \prod_{p_\sigma \in S} p_\sigma^{m_\sigma}$ (and similarly for p_S^M, p_S^{m+M}). Then

$$(p_S^{M+m})^{-1} \sum_{a=1}^{p_S^{m+M}} \chi\left(f_j\left(\frac{a}{p_S^m}\right)\right) = \prod_{\sigma \in S} A(j; p_\sigma, m_\sigma, M_\sigma).$$

(For if $p \notin S$, then $f_{p,j}(a/p_S^m) \in \mathbb{Z}_p$ and $\chi_p|_{\mathbb{Z}_p} \equiv 1$. Therefore $\chi(f_j(a/p_S^m)) = \prod_{\sigma \in S} \chi_{p_\sigma}(f_{p_\sigma,j}(a/p_S^m))$. Now the claim follows from (b).)

Since $A(j; p_\sigma, m_\sigma, M_\sigma) \leq 1$ in any case, we can make

$$A(j;S,m,M) = \left|(p_S^{M+m})^{-1} \sum_{a=1}^{p_S^{m+M}} \chi\left(f_j\left(\frac{a}{p_S^m}\right)\right)\right|$$

smaller than any prescribed $\varepsilon > 0$ by making one $A(j; p_\sigma, m_\sigma, M_\sigma) \leq 1$ for each j. This is possible by a theorem of Hua [3]: for any integer $n > 0$ and any $\delta > 0$, there is a constant $C_{n,\delta}$ such that if $\varphi(x) = \sum_{j=1}^n a_j x^j$, with $a_j \in \mathbb{Z}$ for all j, and if $q \in \mathbb{Z}$ satisfies $(a_1, \ldots, a_n, q) = 1$, then

$$q^{-1}\left|\sum_{x=1}^{q} \exp(2\pi i \varphi(x)/q)\right| < C_{n,\delta} q^{\delta - 1/n}.$$

The application to the present setting is immediate.

We still need to deal with the f_j such that $j > k_1$ (so that the real character is non-trivial). The simplest procedure seems to be the following: given ε, suppose that we have selected S, m, and M such that $A(j;S,m,M) < \varepsilon$ for all $j \leq k_1$. We may now choose the coefficients of each f_j with $j > k_1$ such that $f_{j,p}(a/p_S^m) \in \mathbb{Z}_p$ for all finite p when $j > k_1$. (Recall that we are free to change each coefficient by an element of $\mathbb{Q}$.) Then for all $l \in \mathbb{Z}$ and $j > k_1$,

$$\chi\left(f_j\left(\frac{a}{p_S^m} + lp_S^m\right)\right) = \chi_\infty\left(f_{j,\infty}\left(\frac{a}{p_S^m} + lp_S^m\right)\right).$$

From Weyl's original result, we know that there is a K such that

$$K^{-1}\left|\sum_{l=1}^{K} \chi_\infty\left(f_{j,\infty}\left(\frac{a}{p_S^m} + lp_S^m\right)\right)\right| < \varepsilon \quad \text{for } 1 \le a \le p_S^{m+M},\ j > k_1,$$

since the above expression tends to 0 as $K \to \infty$. Hence

$$(Kp_S^{m+M})^{-1}\left|\sum_{l=1}^{K} \sum_{a=1}^{p_S^{m+M}} \chi\left(f_j\left(\frac{a}{p_S^m} + lp_S^m\right)\right)\right| < \varepsilon \quad \text{if } j > k_1.$$

For $j < k_1$,

$$(Kp_S^{m+M})^{-1}\left|\sum_{l=1}^{K} \sum_{a=1}^{p_S^{m+M}} \chi\left(f_j\left(\frac{a}{p_S^m} + lp_S^m\right)\right)\right| = A(j; S, m, M) < \varepsilon,$$

by (a) and the previous assumption. Thus the hypotheses of Proposition 1 are satisfied, and Theorem 1 is proved.

In the course of the proof, we have also proved the first part of the following theorem, and the second part has the same proof as Theorem 1:

THEOREM 2. (a) *Let* $f : \mathbf{A} \to \mathbf{A}$ *be a polynomial with adelic coefficients, and assume that at least one coefficient other than the constant term is not in* $\mathbb{Q}$. *Then the set* $\{f(x) \bmod \mathbb{Q} : x \in \mathbb{Q}\}$ *is dense in* $\mathbf{A}/\mathbb{Q}$.

(b) *Let* $f_1, \ldots, f_n : \mathbf{A} \to \mathbf{A}$ *be linearly independent polynomials without constant term, each with a non-rational coefficient. Then the set* $\{(f_1(x), \ldots$ $\ldots, f_n(x)) \bmod \mathbb{Q} : x \in \mathbb{Q}\}$ *is dense in* $(\mathbf{A}/\mathbb{Q})^n$.

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