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ON DYADIC SPACES AND ALMOST MILYUTIN SPACES

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Introduction. In order to determine whether a compact space K is an almost Milyutin space, it is necessary to study if an arbitrary continuous map from a Cantor cube $2^{\mathfrak{m}}$ onto K can admit an averaging operator. In [2], S. Z. Ditor obtains a sufficient condition in order that the norm of an averaging operator for a continuous onto map $\phi : S \to T$, where S and T are compact spaces, has a lower bound. This condition is given in terms of the topological structure of T and the decomposition induced by ϕ on S. Here we give another condition which relies only upon the topological structure of the space T and which allows us to apply Ditor's theorem to an arbitrary continuous map from $2^{\mathfrak{m}}$ onto K. As an application of our results, two problems posed by A. Pełczyński in [3] are solved.

The results. We first state Ditor's result in the formulation given by W. G. Bade in [1]. For this we need the following terminology.

Let S and T be compact Hausdorff spaces and let $\phi : S \to T$ be a continuous onto map. Suppose that $\{t_\alpha\}$ is a net in T converging to t. We define

$$\overline{\lim}\, \phi^{-1}(t_\alpha) = \{s \in S \mid \text{for each } \alpha_0 \text{ and each neighborhood } U \text{ of } s \text{ there is an } \alpha \geq \alpha_0 \text{ with } \phi^{-1}(t_\alpha) \cap U \neq \emptyset\}.$$

The set $\overline{\lim}\, \phi^{-1}(t_\alpha)$ is a non-empty compact subset of $\phi^{-1}(t)$.

Let $M_\phi^{(0)} = T$. We inductively define, for each positive integer n,

$$M_\phi^{(n)} = \{t \in T \mid \text{there are nets } \{t_\alpha\} \text{ and } \{t_\beta\} \text{ of points of } M_\phi^{(n-1)} \text{ converging to } t \text{ such that } \overline{\lim}\, \phi^{-1}(t_\alpha) \text{ and } \overline{\lim}\, \phi^{-1}(t_\beta) \text{ are disjoint}\}.$$

An *averaging operator* for a continuous onto map $\phi : S \to T$ is a continuous linear operator $u : C(S) \to C(T)$ satisfying $u(f \circ \phi) = f$ for all $f \in C(T)$.

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THEOREM 1 (Ditor [2]). *If the set $M_\phi^{(n)}$ is non-empty and u is an averaging operator for ϕ, then $\|u\| \geq n+1$.*

Let A be a subset of a space K. The G_δ-*closure* of A is the set $G_\delta A$ of all points x in K such that every G_δ-set containing x meets A. It is clear that $G_\delta A$ is contained in $\overline{A}$.

In the following $\mathfrak{m}$ *will be an uncountable cardinal* and $2^{\mathfrak{m}}$ will be the generalized Cantor cube with $\mathfrak{m}$ factors.

DEFINITION 2. Let T be a compact space. We set $D^{(0)} = T$ and we inductively define for each positive integer n the sets $D^{(n+1)} = \{p \in T \mid$ there are two disjoint open subsets U and V of T such that $p \in G_\delta(U \cap D^{(n)}) \cap \overline{V \cap D^{(n)}}\}$.

THEOREM 3. *Let T be a compact space. Then $D^{(n)} \subseteq M_\phi^{(n)}$ for every continuous onto map $\phi : 2^{\mathfrak{m}} \to T$.*

Proof. By induction on n. For $n = 0$ the statement is obvious, so we suppose that $D^{(n)} \subseteq M_\phi^{(n)}$ and we prove it for $n+1$. Take a point $p \in D^{(n+1)}$. This means that there are two open subsets U_1 and U_2 of T such that $U_1 \cap U_2 = \emptyset$ and $p \in G_\delta(U_1 \cap D^{(n)}) \cap \overline{U_2 \cap D^{(n)}}$.

If $\overline{\phi^{-1}(U_1)} \cap \overline{\phi^{-1}(U_2)} = \emptyset$ the result follows, because if $\{t_\alpha\}$ is a net in $U_i \cap D^{(n)}$ converging to p, then $\overline{\lim}\, \phi^{-1}(t_\alpha) \subset \overline{\phi^{-1}(U_i)}$.

Consequently, we can suppose that $G = \overline{\phi^{-1}(U_1)} \cap \overline{\phi^{-1}(U_2)} \neq \emptyset$.

Since the closure of an open set in $2^{\mathfrak{m}}$ depends on a countable set of coordinates ([4]), we conclude that G is a compact G_δ-set in $2^{\mathfrak{m}}$. So there is a family $\{W_k\}_{k=1}^\infty$ of clopen subsets of $2^{\mathfrak{m}}$ such that $G = \bigcap_{k=1}^\infty W_k$ and $W_{k+1} \subset W_k$ for all k.

Suppose, towards a contradiction, that for each positive integer k, there exists a neighborhood V_k of p such that for all $y \in V_k \cap U_1 \cap D^{(n)}$ the intersection $\phi^{-1}(y) \cap W_k$ is non-empty.

Since $V = \bigcap_{k=1}^\infty V_k$ is a G_δ-set containing p, from our hypothesis we have a point $y_0 \in V \cap U_1 \cap D^{(n)}$. So for each positive integer k, there is a point $x_k \in \phi^{-1}(y_0) \cap W_k$. Let z be a cluster point of the sequence $\{x_k\}_{k=1}^\infty$. Thus $z \in \phi^{-1}(y_0) \cap G$ and hence $\phi(z) = y_0 \in U_1$, but on the other hand $\phi(z) \in \phi(G) \subset \overline{U}_1 \cap \overline{U}_2$, which is absurd.

We conclude that there exists a positive integer k such that for each neighborhood V of p there is a point $y_V \in V \cap U_1 \cap D^{(n)}$ and $\phi^{-1}(y_V) \cap W_k = \emptyset$. The net $\{y_V\}$, where V runs over all neighborhoods of p, converges to p and

$$\overline{\lim}\, \phi^{-1}(y_V) \subset (2^{\mathfrak{m}} \setminus W_k) \cap \overline{\phi^{-1}(U_1)}.$$

Since $G \subset W_k$, $\overline{\lim}\,\phi^{-1}(y_V) \cap \overline{\phi^{-1}(U_2)} = \emptyset$. Therefore for every net $\{z_\alpha\}$ in $U_2 \cap D^{(n)}$ converging to p the sets $\overline{\lim}\,\phi^{-1}(y_V)$ and $\overline{\lim}\,\phi^{-1}(z_\alpha)$ are disjoint. Since by the inductive hypothesis $D^{(n)} \subset M_\phi^{(n)}$, we conclude that $p \in M_\phi^{(n+1)}$.

We now state two simple facts about averaging operators whose proofs are left to the reader.

LEMMA 4. *Let $T = T_1 \times T_2$ be a product of compact spaces.*

(a) *If $\phi_i : S_i \to T_i$, $i = 1, 2$, are continuous onto maps for which there are averaging operators $u_i : C(S_i) \to C(T_i)$ and $\phi : S_1 \times S_2 \to T$ is given by $\phi(u, v) = (\phi_1(u), \phi_2(v))$, then there exists an averaging operator u for ϕ such that $\|u\| = \|u_1\|\|u_2\|$.*

(b) *If $\phi : S \to T$ is a continuous onto map and $u : C(S) \to C(T)$ is an averaging operator for ϕ, then there are averaging operators u_i for the maps $\phi_i = \pi_i \circ \phi$ such that $\|u_i\| \le \|u\|$, $i = 1, 2$.*

THEOREM 5. *Let $\{T_i\}_{i=1}^\infty$ be a family of compact spaces such that for each i there are disjoint open subsets U_i, V_i of T_i such that $G_\delta(U_i) \cap \overline{V}_i \neq \emptyset$. Let $P = \prod_{i=1}^\infty T_i$ and for each n let $P_n = \prod_{i=1}^n T_i$. Then*:

(a) *Every averaging operator for every continuous onto map $\phi : 2^{\mathfrak{m}} \to P_n$ has norm greater than or equal to $n + 1$.*

(b) *There is no averaging operator for any continuous onto map $\phi : 2^{\mathfrak{m}} \to P$.*

Proof. Take $p_i \in G_\delta(U_i) \cap \overline{V}_i$ and define

$$H_k = \{(x_1, \ldots, x_n) \mid x_i = p_i \text{ for } i = 1, \ldots, k\}.$$

We shall show that each H_k is contained in $D^{(k)}$ and so the set $D^{(n)}$ will be non-empty.

We proceed by induction on k. Suppose that this holds for k (the case $k = 1$ is similar).

Let $x = (p_1, \ldots, p_{k+1}, x_{k+2}, \ldots, x_n)$ be a point in H_{k+1}, and let $U = \pi_{k+1}^{-1}(U_{k+1})$, $V = \pi_{k+1}^{-1}(V_{k+1})$, where π_j is the projection onto the jth factor. Then U and V are disjoint open subsets of P_n. Let Z be a G_δ-subset of P_n containing x and let $q : T_{k+1} \to P_n$ be the map defined by

$$q(s) = (p_1, \ldots, p_k, s, x_{k+2}, \ldots, x_n).$$

Then $q^{-1}(Z)$ is a G_δ-subset of T_{k+1} containing p_{k+1}, by hypothesis there exists a point $s \in q^{-1}(Z) \cap U_{k+1}$ and by the inductive hypothesis $(p_1, \ldots, p_k, s, x_{k+2}, \ldots, x_n)$ is in $Z \cap U \cap D^{(k)}$. Thus $x \in G_\delta(U \cap D^{(k)})$. Similarly we can prove that $x \in \overline{V \cap D^{(k)}}$ and therefore $x \in D^{(k+1)}$.

Part (a) now follows from Theorems 1 and 3, and (b) is a consequence of (a) and Lemma 4(b), since for each positive integer n, the space P factors as $P = P_n \times X$ for some space X.

A compact space K is *dyadic* if it is a continuous image of a generalized Cantor cube. The space K is *almost Milyutin* if there exists a continuous map from a cube $2^{\mathfrak{m}}$ onto K which admits an averaging operator.

Let α be an uncountable cardinal and let T be the space obtained from 2^α by identification of two different points. We can decompose T as a disjoint union $T = U \cup V \cup \{p\}$, where U and V are open sets, p is the identified point and the subsets $U \cup \{p\}$ and $V \cup \{p\}$ are homeomorphic to the cube 2^α (cf. [3], p. 67).

The next corollary solves problems 6 and 7 posed by Pełczyński in [3].

COROLLARY 6. (a) *For each positive integer n, the space T^n is an almost Milyutin space with the property that every averaging operator for every continuous map from $2^{\mathfrak{m}}$ onto T^n has norm greater than or equal to $n+1$.*

(b) *The space $T^{\aleph_0}$ is dyadic but it is not an almost Milyutin space.*

Proof. The space T is almost Milyutin ([3], p. 67), and by Lemma 4(a) so is T^n. Let U, V and p be as in the preceding remarks. Since the points in 2^α are not G_δ-sets, it follows that $p \in G_\delta(U) \cap G_\delta(V)$. Thus the corollary follows from Theorem 5.

Remark. In Theorem 5 we cannot just suppose that the two disjoint open sets have non-disjoint closures. This condition holds for instance in the closed interval $[0, 1]$ but it is well known that there are continuous maps from $2^{\aleph_0}$ onto this space which admit norm-one averaging operators.

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