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GEODESIC SPHERES AND ISOMETRIC FLOWS

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1. Introduction. *Locally Killing-transversally symmetric spaces* (briefly, *locally* KTS-*spaces*) are Riemannian manifolds (M, g) equipped with an isometric flow generated by a unit Killing vector field ξ such that the local reflections with respect to the flow lines are isometries. These spaces have been introduced and studied in [5], [6] where several characterizations and a lot of examples are provided.

On the one hand, these spaces form a subclass of the class of *transversally symmetric Riemannian foliations* studied in [13], [14] but on the other hand, they extend the class of φ-*symmetric Sasakian spaces* introduced in [11]. These last spaces are the analogues of Hermitian symmetric spaces in contact geometry. (See [2] for more details and further references.) Several characterizations for these spaces have been given by using the geometry, extrinsic and intrinsic, of small geodesic spheres [4], [13], in particular by focussing on the shape operator and the Ricci operator of these spheres.

Although the class of locally KTS-spaces is much broader than that of φ-symmetric spaces, their geometries are reasonably similar and this leads to a list of analogous characteristic properties. The main purpose of this paper is to derive these properties.

In Section 2 we collect a series of definitions and results which will be needed to derive our theorems. We refer to [5], [6] for more details. Section 3 is devoted to the proof of a new characterization which will show to be very useful in Sections 4 and 5 where we derive the characteristic properties which use the extrinsic and intrinsic geometry of geodesic spheres, respectively.

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2. Preliminaries. Let (M,g) be a smooth, n-dimensional Riemannian manifold with $n \geq 2$ which is supposed to be connected where necessary. Further, let ∇ denote the Levi Civita connection of (M,g) and R the corresponding Riemannian curvature tensor given by

$$R_{UV} = \nabla_{[U,V]} - [\nabla_U, \nabla_V]$$

for $U, V \in \mathfrak{X}(M)$, the Lie algebra of smooth vector fields on M. ϱ denotes the Ricci tensor of type (0,2) and Q the corresponding Ricci endomorphism field. The scalar curvature will be denoted by τ.

A tangentially oriented foliation of dimension one on (M,g) is called a *flow*. The leaves of this foliation are the integral curves of a non-singular vector field on M and hence, by normalizing length, a flow is also given by a unit vector field with respect to g. In particular, a non-singular Killing vector field defines a *Riemannian flow* and such a flow is said to be an *isometric* flow. We refer to [12] for more information.

In the rest of this paper we shall denote by $\mathfrak{F}_\xi$ an isometric flow generated by a *unit* Killing vector field ξ. Then the flow lines of $\mathfrak{F}_\xi$ are geodesics and a geodesic which is orthogonal to ξ at one of its points is orthogonal to it at all of its points. Such geodesics are called *transversal geodesics*. Moreover, the Riemannian foliation determined by $\mathfrak{F}_\xi$ is locally a Riemannian submersion $\pi : \mathcal{U} \to \widetilde{\mathcal{U}} = \mathcal{U}/\xi$. Here $\mathcal{U}$ is a small neighborhood of a point on which ξ is regular [9]. The $(n-1)$-dimensional horizontal distribution on $\mathcal{U}$ is determined by $\eta = 0$ where η is the 1-form on M given by $\eta(U) = g(U,\xi)$ for all $U \in \mathfrak{X}(M)$. Further, the induced metric $\widetilde{g}$ on $\widetilde{\mathcal{U}}$ is determined by

$$\widetilde{g}(\widetilde{X}, \widetilde{Y}) = g(\widetilde{X}^*, \widetilde{Y}^*) \tag{2.1}$$

for $\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(\widetilde{\mathcal{U}})$ and where $\widetilde{X}^*, \widetilde{Y}^*$ are the horizontal lifts of $\widetilde{X}, \widetilde{Y}$. Its Levi Civita connection $\widetilde{\nabla}$ is related to ∇ by

$$\widetilde{\nabla}_{\widetilde{X}} \widetilde{Y} = \pi_* (\nabla_{\widetilde{X}^*} \widetilde{Y}^*)$$

for all $\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(\widetilde{\mathcal{U}})$.

Vector fields which are orthogonal to ξ are called *horizontal* and the transversal geodesics are also said to be *horizontal geodesics*.

Since we have locally Riemannian submersions, we may use the O'Neill tensors A and T [8], [1, Chapter 9] which are in fact globally defined for the Riemannian foliation [10], [12]. In our case, $T = 0$ since the leaves are geodesics. Further, for the integrability tensor A we have

$$\begin{cases} A_U\xi = \nabla_U\xi, \quad A_\xi U = 0, \\ A_X Y = (\nabla_X Y)^{\mathcal{V}} = -A_Y X, \quad g(A_X Y, \xi) = -g(A_X\xi, Y), \end{cases} \tag{2.2}$$

where $U \in \mathfrak{X}(M)$, X, Y are horizontal vector fields and $\mathcal{V}$ denotes the vertical component. Note that $A = 0$ if and only if the horizontal distribution is integrable. In that case, since $T = 0$, (M, g) is locally a Riemannian product of an $(n-1)$-dimensional space and a line.

Next, put

$$HU = -A_U \xi \tag{2.3}$$

for $U \in \mathfrak{X}(M)$, and define the $(0,2)$-tensor h by

$$h(U, V) = g(HU, V)$$

for $U, V \in \mathfrak{X}(M)$. Since ξ is a Killing field, we have

$$h(U, V) + h(V, U) = 0, \tag{2.4}$$

that is, H is a skew-symmetric endomorphism. Further, for all horizontal $X, Y \in \mathfrak{X}(M)$, we obtain easily

$$A_X Y = h(X, Y)\xi = \tfrac{1}{2}\eta([X, Y])\xi \tag{2.5}$$

and hence

$$h = -d\eta. \tag{2.6}$$

All this leads to

Lemma 2.1. *Let X, Y, Z be horizontal vector fields on $(M, g, \mathfrak{F}_\xi)$. Then*

$$(\nabla_\xi h)(X, Y) = g((\nabla_\xi A)_X Y, \xi) = 0, \tag{2.7}$$

$$R(X, Y, Z, \xi) = (\nabla_Z h)(X, Y), \tag{2.8}$$

$$R(X, \xi, Y, \xi) = g(HX, HY) = g(-H^2 X, Y). \tag{2.9}$$

This lemma yields

Theorem 2.1. *Let (M, g) be a Riemannian manifold equipped with a unit Killing vector field ξ. Then the following holds*:

(i) *the sectional curvature $K(X, \xi)$ is non-negative for all (horizontal) X*;

(ii) *$K(X, \xi) = 0$ for all $X \in \mathfrak{X}(M)$ if and only if $h = 0$ or equivalently, $A = 0$*;

(iii) *$K(X, \xi) > 0$ for all $X \in \mathfrak{X}(M)$ if and only if the endomorphism H is of maximal rank $n-1$. (In this case n is necessarily odd and η is a contact form on M.)*

This result leads to

Definition 2.1. An isometric flow on (M, g) is said to be a *contact* flow if η is a contact form or equivalently, if H is of maximal rank.

Further, on $(\widetilde{\mathcal{U}}, \widetilde{g})$ we have

$$\nabla_{\widetilde{X}^*} \widetilde{Y}^* = (\widetilde{\nabla}_{\widetilde{X}} \widetilde{Y})^* + h(\widetilde{X}^*, \widetilde{Y}^*)\xi \tag{2.10}$$

for all $\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(\widetilde{\mathcal{U}})$ and the corresponding Riemannian curvature tensor $\widetilde{R}$ is given by

$$(2.11)\qquad \begin{aligned}(\widetilde{R}_{\widetilde{X}\widetilde{Y}}\widetilde{Z})^* = R_{\widetilde{X}^*\widetilde{Y}^*}\widetilde{Z}^* &+ 2h(\widetilde{X}^*, \widetilde{Y}^*)H\widetilde{Z}^* \\ &+ \{(\nabla_{\widetilde{X}^*}h)(\widetilde{Y}^*, \widetilde{Z}^*) - (\nabla_{\widetilde{Y}^*}h)(\widetilde{X}^*, \widetilde{Z}^*)\}\xi \\ &+ h(\widetilde{X}^*, \widetilde{Z}^*)H\widetilde{Y}^* - h(\widetilde{Y}^*, \widetilde{Z}^*)H\widetilde{X}^*\end{aligned}$$

for all $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \mathfrak{X}(\widetilde{\mathcal{U}})$. Hence, this yields

$$(2.12)\qquad (\widetilde{\varrho}(\widetilde{X}, \widetilde{Y}))^* = \varrho(\widetilde{X}^*, \widetilde{Y}^*) + 2g(H\widetilde{X}^*, H\widetilde{Y}^*),$$

$$(2.13)\qquad \widetilde{\tau}^* = \tau + \varrho(\xi, \xi),$$

and for the corresponding sectional curvatures we get

$$(2.14)\qquad (\widetilde{K}_{\widetilde{p}}(\widetilde{u}, \widetilde{w}))^* = K_p(\widetilde{u}^*, \widetilde{w}^*) + 3(h_p(\widetilde{u}^*, \widetilde{w}^*))^2$$

where $\{\widetilde{u}, \widetilde{w}\}$ is an orthonormal pair of $T_{\widetilde{p}}\widetilde{\mathcal{U}}$, $\widetilde{p} = \pi(p)$. So,

$$(\widetilde{\varrho}(\widetilde{X}, \widetilde{X}))^* \geq \varrho(\widetilde{X}^*, \widetilde{X}^*), \quad \widetilde{\tau}^* \geq \tau, \quad (\widetilde{K}_{\widetilde{p}}(\widetilde{u}, \widetilde{w}))^* \geq K_p(\widetilde{u}^*, \widetilde{w}^*),$$

where the equalities hold if and only if the horizontal distribution is integrable.

In the rest of this paper an important role will be played by a special type of isometric flow introduced in [6].

Definition 2. An isometric flow $\mathfrak{F}_\xi$ on (M, g) is said to be *normal* if the curvature tensor R satisfies

$$(2.15)\qquad R(X, Y, X, \xi) = 0$$

for all horizontal vectors X, Y. (This is equivalent to $R(X, Y, Z, \xi) = 0$ for all horizontal X, Y, Z.)

From Lemma 2.1 we then get

Proposition 2.1. *Let (M, g) be a Riemannian manifold and $\mathfrak{F}_\xi$ an isometric flow on it. Then $\mathfrak{F}_\xi$ is normal if and only if*

$$(2.16)\qquad (\nabla_U H)V = g(HU, HV)\xi + \eta(V)H^2U$$

for all $U, V \in \mathfrak{X}(M)$.

Moreover, for a normal flow we have

$$(2.17)\qquad \begin{cases} R_{UV}\xi = \eta(V)H^2U - \eta(U)H^2V, \\ R_{U\xi}V = g(HU, HV)\xi + \eta(V)H^2U. \end{cases}$$

Further, $\varrho(X, \xi) = 0$ for each horizontal X, and from Lemma 2.1 and Proposition 2.1 we see that $\varrho(\xi, \xi)$ is a non-negative constant.

Next, for a normal flow $\mathfrak{F}_\xi$, (2.11) reduces to

$$(2.18)\qquad (\widetilde{R}_{\widetilde{X}\widetilde{Y}}\widetilde{Z})^* = R_{\widetilde{X}^*\widetilde{Y}^*}\widetilde{Z}^* - g(H\widetilde{Y}^*,\widetilde{Z}^*)H\widetilde{X}^* + g(H\widetilde{X}^*,\widetilde{Z}^*)H\widetilde{Y}^* + 2g(H\widetilde{X}^*,\widetilde{Y}^*)H\widetilde{Z}^*$$

for all $\widetilde{X},\widetilde{Y},\widetilde{Z}\in\mathfrak{X}(\widetilde{\mathcal{U}})$. Hence, using (2.10), (2.16) and (2.17) we obtain

$$(2.19)\qquad ((\widetilde{\nabla}_{\widetilde{V}}\widetilde{R})_{\widetilde{X}\widetilde{Y}}\widetilde{Z})^* = ((\nabla_{\widetilde{V}^*}R)_{\widetilde{X}^*\widetilde{Y}^*}\widetilde{Z}^*)^{\mathcal{H}}$$

where $(\ )^{\mathcal{H}}$ denotes the horizontal part, and from (2.12) we get

$$(2.20)\qquad ((\widetilde{\nabla}_{\widetilde{X}}\widetilde{\varrho})(\widetilde{Y},\widetilde{Z}))^* = (\nabla_{\widetilde{X}^*}\varrho)(\widetilde{Y}^*,\widetilde{Z}^*).$$

Finally, let $\widetilde{H}$ be the $(1,1)$-tensor field on $\widetilde{\mathcal{U}}$ defined by

$$(2.21)\qquad \widetilde{H}\widetilde{X} = \pi_* H\widetilde{X}^*.$$

Then $\widetilde{H}$ is skew-symmetric with respect to $\widetilde{g}$, and moreover,

$$(2.22)\qquad (\nabla_{\widetilde{X}^*}H)\widetilde{Y}^* = ((\widetilde{\nabla}_{\widetilde{X}}\widetilde{H})\widetilde{Y})^* + g(H\widetilde{X}^*,H\widetilde{Y}^*)\xi.$$

So, making use of (2.8), we have

PROPOSITION 2.2. *An isometric flow $\mathfrak{F}_\xi$ on (M,g) is normal if and only if*

$$(2.23)\qquad \widetilde{\nabla}\widetilde{H} = 0$$

for each $(\widetilde{\mathcal{U}},\widetilde{g})$.

Finally, we introduce the spaces we want to study in more detail. So, let $\mathfrak{F}_\xi$ be an isometric flow on (M,g). Let $m\in M$ and denote by σ the geodesic flow line through m. A local diffeomorphism s_m of M defined in a neighborhood $\mathcal{U}$ of m is said to be a (*local*) *reflection with respect to* σ if for every transversal geodesic $\gamma(s)$, where $\gamma(0)$ lies in the intersection of $\mathcal{U}$ with σ, we have

$$(s_m\cdot\gamma)(s) = \gamma(-s)$$

for all s with $\gamma(\pm s)\in\mathcal{U}$, s being the arc length. Note that in this case

$$(s_{m*})(m) = (-I + 2\eta\otimes\xi)(m).$$

DEFINITION 2.3. A *locally Killing-transversally symmetric space* (briefly, a *locally* KTS-*space*) is a Riemannian manifold equipped with an isometric flow such that the reflection s_m with respect to the flow line through m is an isometry for all $m\in M$.

These spaces may be characterized as follows:

THEOREM 2.2 [6]. *Let $\mathfrak{F}_\xi$ be an isometric flow on (M,g). Then $(M,g,\mathfrak{F}_\xi) = (M,g,\xi)$ is a locally* KTS-*space if and only if $\mathfrak{F}_\xi$ is normal and*

$$(\nabla_X R)(X,Y,X,Y) = 0$$

for all horizontal vector fields X, Y.

Using (2.19) we then get

THEOREM 2.3. *Let $\mathfrak{F}_\xi$ be a normal flow on (M,g). Then (M,g,ξ) is a locally* KTS-*space if and only if each base space $\widetilde{\mathcal{U}}$ of a local Riemannian submersion $\pi : \mathcal{U} \to \widetilde{\mathcal{U}} = \mathcal{U}/\xi$ is a locally symmetric space.*

3. A new characterization of locally KTS-spaces. The main purpose of this section is to give a useful characterization for a class of locally KTS-spaces by using contact normal flows and a property for ∇R. Specifically, we shall prove

THEOREM 3.1. *Let $\mathfrak{F}_\xi$ be a contact flow on (M,g). Then (M,g,ξ) is a locally* KTS-*space if and only if $\mathfrak{F}_\xi$ is normal and*

$$(\nabla_X R)(X,HX,X,HX) = 0$$

for all horizontal X.

The proof will be given by using a series of lemmas which we consider first. We start with a well-known result from linear algebra.

LEMMA 3.1. *Let V be an n-dimensional vector space with a positive definite inner product and let $A : V \to V$ be a skew-symmetric endomorphism. Then the rank of A is an even number $2k \le n$, and there is an orthonormal basis $\{X_1, \ldots, X_n\}$ and real non-vanishing numbers $\lambda_1, \ldots, \lambda_k$ such that*

$$\begin{cases} AX_1 = \lambda_1 X_2, \quad AX_2 = -\lambda_1 X_1, \ldots, \\ AX_{2k-1} = \lambda_k X_{2k}, \quad AX_{2k} = -\lambda_k X_{2k-1}, \\ AX_{2k+1} = \ldots = AX_n = 0. \end{cases} \tag{3.1}$$

Further, we have from [6]:

LEMMA 3.2. *Let $\mathfrak{F}_\xi$ be a normal flow on (M,g). Then, for every $U,V,W \in \mathfrak{X}(M)$, we have*

$$\begin{aligned} R_{HUV}W + R_{UHV}W = {} & g(HV,W)H^2U - g(HU,W)H^2V \\ & - g(H^2U,W)HV + g(H^2V,W)HU \\ & + \eta(V)R_{HU\xi}W - \eta(U)R_{HV\xi}W. \end{aligned} \tag{3.2}$$

This leads to

LEMMA 3.3. *Let $\mathfrak{F}_\xi$ be a normal flow on (M, g). Then on each base space $(\widetilde{\mathcal{U}}, \widetilde{g})$ we have*

$$\widetilde{R}_{\widetilde{H}\widetilde{X}\widetilde{Y}} = \widetilde{R}_{\widetilde{H}\widetilde{Y}\widetilde{X}} \tag{3.3}$$

for all $\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(\widetilde{U})$.

Proof. (2.18) yields

$$\begin{aligned}\widetilde{R}_{\widetilde{H}\widetilde{X}\widetilde{Y}}\widetilde{Z} + \widetilde{R}_{\widetilde{X}\widetilde{H}\widetilde{Y}}\widetilde{Z} = \pi_*(&R_{(\widetilde{H}\widetilde{X})^*\widetilde{Y}^*}\widetilde{Z}^* + R_{\widetilde{X}^*(\widetilde{H}\widetilde{Y})^*}\widetilde{Z}^*)\\ &- \widetilde{g}(\widetilde{H}\widetilde{Y}, \widetilde{Z})\widetilde{H}^2\widetilde{X} + \widetilde{g}(\widetilde{H}^2\widetilde{X}, \widetilde{Z})\widetilde{H}\widetilde{Y}\\ &- \widetilde{g}(\widetilde{H}^2\widetilde{Y}, \widetilde{Z})\widetilde{H}\widetilde{X} + \widetilde{g}(\widetilde{H}\widetilde{X}, \widetilde{Z})\widetilde{H}^2\widetilde{Y}.\end{aligned}$$

Now the result follows at once by using (3.2). ∎

Proof of Theorem 3.1. First, let (M, g) be a locally KTS-space. Then the result follows from Theorem 2.2.

Next, we prove that the condition is sufficient. Therefore we shall prove that on each $(\widetilde{\mathcal{U}}, \widetilde{g})$,

$$(\widetilde{\nabla}_{\widetilde{X}}\widetilde{R})(\widetilde{X}, \widetilde{H}\widetilde{X}, \widetilde{X}, \widetilde{H}\widetilde{X}) = 0 \tag{3.4}$$

and

$$(\widetilde{\nabla}_{\widetilde{X}}\widetilde{R})(\widetilde{X}, \widetilde{Y}, \widetilde{X}, \widetilde{Y}) = 0$$

are equivalent. Then the result follows from Theorem 2.2 and (2.19).

So, let $\widetilde{p} \in \widetilde{\mathcal{U}}$ and $\widetilde{x} \in T_{\widetilde{p}}\widetilde{\mathcal{U}}$. Put $\widetilde{x} = \alpha\widetilde{y} + \beta\widetilde{z}$ for arbitrary α, β in (3.4). From (3.3) and Proposition 2.2 we then conclude that the coefficient of $\alpha\beta^4$ yields the following condition:

$$(\widetilde{\nabla}_{\widetilde{y}}\widetilde{R})(\widetilde{z}, \widetilde{H}\widetilde{z}, \widetilde{z}, \widetilde{H}\widetilde{z}) + 4(\widetilde{\nabla}_{\widetilde{z}}\widetilde{R})(\widetilde{y}, \widetilde{H}\widetilde{z}, \widetilde{z}, \widetilde{H}\widetilde{z}) = 0. \tag{3.5}$$

Applying the second Bianchi identity for the second term yields

$$5(\widetilde{\nabla}_{\widetilde{y}}\widetilde{R})(\widetilde{z}, \widetilde{H}\widetilde{z}, \widetilde{z}, \widetilde{H}\widetilde{z}) - 4(\widetilde{\nabla}_{\widetilde{H}\widetilde{z}}\widetilde{R})(\widetilde{z}, \widetilde{y}, \widetilde{z}, \widetilde{H}\widetilde{z}) = 0. \tag{3.6}$$

Replacing $\widetilde{z}$ by $\widetilde{H}\widetilde{z}$ in (3.6) then gives

$$5(\widetilde{\nabla}_{\widetilde{y}}\widetilde{R})(\widetilde{H}\widetilde{z}, \widetilde{H}^2\widetilde{z}, \widetilde{H}\widetilde{z}, \widetilde{H}^2\widetilde{z}) - 4(\widetilde{\nabla}_{\widetilde{H}^2\widetilde{z}}\widetilde{R})(\widetilde{H}\widetilde{z}, \widetilde{y}, \widetilde{H}\widetilde{z}, \widetilde{H}^2\widetilde{z}) = 0. \tag{3.7}$$

Now, put $\widetilde{z} = \alpha\widetilde{u} + \beta\widetilde{v}$ in (3.7) for arbitrary α, β. Then the coefficient of $\alpha^3\beta$, again by (3.3), yields the condition

$$\begin{aligned}&5(\widetilde{\nabla}_{\widetilde{y}}\widetilde{R})(\widetilde{H}\widetilde{u}, \widetilde{H}^2\widetilde{u}, \widetilde{H}\widetilde{u}, \widetilde{H}^2\widetilde{v}) - 3(\widetilde{\nabla}_{\widetilde{H}^2\widetilde{u}}\widetilde{R})(\widetilde{H}\widetilde{u}, \widetilde{y}, \widetilde{H}\widetilde{u}, \widetilde{H}^2\widetilde{v})\\ &- (\widetilde{\nabla}_{\widetilde{H}^2\widetilde{v}}\widetilde{R})(\widetilde{H}\widetilde{u}, \widetilde{y}, \widetilde{H}\widetilde{u}, \widetilde{H}^2\widetilde{u}) + (\widetilde{\nabla}_{\widetilde{H}^2\widetilde{u}}\widetilde{R})(\widetilde{H}^2\widetilde{v}, \widetilde{u}, \widetilde{y}, \widetilde{H}^2\widetilde{u}) = 0.\end{aligned}$$

Since $\widetilde{H}$ is skew-symmetric and of maximal rank $n-1$ there exists on $T_{\tilde{p}}\widetilde{\mathcal{U}}$ an orthonormal basis X_i, $i=1,\dots,n-1$, such that (3.1) holds. Then setting $\widetilde{v}=X_i$ in the last expression yields, with $\widetilde{u}=\sum x_iX_i$,

$$(3.8)\quad 5(\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{H}\widetilde{u},\widetilde{H}^2\widetilde{u},\widetilde{H}\widetilde{u},\widetilde{u}) + 3(\widetilde{\nabla}_{\widetilde{H}^2\tilde{u}}\widetilde{R})(\widetilde{y},\widetilde{H}\widetilde{u},\widetilde{H}\widetilde{u},\widetilde{u}) - (\widetilde{\nabla}_{\tilde{u}}\widetilde{R})(\widetilde{H}\widetilde{u},\widetilde{y},\widetilde{H}\widetilde{u},\widetilde{H}^2\widetilde{u})=0.$$

Next, we put $\widetilde{u}=\alpha\widetilde{a}+\beta\widetilde{b}$ in (3.8) and use (3.3) to obtain, from the coefficient of $\alpha^3\beta$,

$$\begin{aligned}(3.9)\quad & 10\{(\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{H}\widetilde{a},\widetilde{H}^2\widetilde{a},\widetilde{H}\widetilde{a},\widetilde{b}) + (\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{H}\widetilde{a},\widetilde{H}^2\widetilde{b},\widetilde{H}\widetilde{a},\widetilde{a})\}\\ &+6(\widetilde{\nabla}_{\widetilde{H}^2\tilde{a}}\widetilde{R})(\widetilde{y},\widetilde{H}\widetilde{a},\widetilde{H}\widetilde{b},\widetilde{a}) + 3(\widetilde{\nabla}_{\widetilde{H}^2\tilde{a}}\widetilde{R})(\widetilde{y},\widetilde{H}\widetilde{b},\widetilde{H}\widetilde{a},\widetilde{a})\\ &+3(\widetilde{\nabla}_{\widetilde{H}^2\tilde{b}}\widetilde{R})(\widetilde{y},\widetilde{H}\widetilde{a},\widetilde{H}\widetilde{a},\widetilde{a}) - 2(\widetilde{\nabla}_{\tilde{a}}\widetilde{R})(\widetilde{H}\widetilde{a},\widetilde{y},\widetilde{H}\widetilde{a},\widetilde{H}^2\widetilde{b})\\ &-(\widetilde{\nabla}_{\tilde{a}}\widetilde{R})(\widetilde{H}\widetilde{b},\widetilde{y},\widetilde{H}\widetilde{a},\widetilde{H}^2\widetilde{a}) - (\widetilde{\nabla}_{\tilde{b}}\widetilde{R})(\widetilde{H}\widetilde{a},\widetilde{y},\widetilde{H}\widetilde{a},\widetilde{H}^2\widetilde{a})=0.\end{aligned}$$

Now, using the first and second Bianchi identity and (3.3) in (3.9) implies

$$\begin{aligned}(3.10)\quad & 10(\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{H}\widetilde{a},\widetilde{H}^2\widetilde{b},\widetilde{H}\widetilde{a},\widetilde{a}) + 6(\widetilde{\nabla}_{\widetilde{H}\tilde{a}}\widetilde{R})(\widetilde{H}\widetilde{a},\widetilde{H}^2\widetilde{b},\widetilde{y},\widetilde{a})\\ &+4(\widetilde{\nabla}_{\tilde{a}}\widetilde{R})(\widetilde{a},\widetilde{H}^2\widetilde{b},\widetilde{H}\widetilde{y},\widetilde{H}\widetilde{a}) + 4(\widetilde{\nabla}_{\widetilde{H}\tilde{a}}\widetilde{R})(\widetilde{H}\widetilde{a},\widetilde{a},\widetilde{y},\widetilde{H}^2\widetilde{b})\\ &+(\widetilde{\nabla}_{\widetilde{H}\tilde{a}}\widetilde{R})(\widetilde{a},\widetilde{H}^2\widetilde{b},\widetilde{y},\widetilde{H}\widetilde{a}) + 3(\widetilde{\nabla}_{\widetilde{H}^2\tilde{b}}\widetilde{R})(\widetilde{y},\widetilde{H}\widetilde{a},\widetilde{H}\widetilde{a},\widetilde{a})\\ &+3(\widetilde{\nabla}_{\tilde{a}}\widetilde{R})(\widetilde{H}^2\widetilde{b},\widetilde{H}\widetilde{a},\widetilde{H}\widetilde{a},\widetilde{y})=0.\end{aligned}$$

Again we put, as before, $\widetilde{b}=X_i$, $i=1,\dots,n-1$, in (3.10). Then with $\widetilde{z}=\sum z_iX_i$ we obtain

$$(3.11)\quad 5(\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{z},\widetilde{H}\widetilde{z},\widetilde{z},\widetilde{H}\widetilde{z}) - 4(\widetilde{\nabla}_{\tilde{z}}\widetilde{R})(\widetilde{H}\widetilde{z},\widetilde{y},\widetilde{H}\widetilde{z},\widetilde{z})=0.$$

This together with (3.5) yields

$$(3.12)\quad (\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{z},\widetilde{H}\widetilde{z},\widetilde{z},\widetilde{H}\widetilde{z})=0.$$

Next, we use the same method for (3.12) to obtain

$$(3.13)\quad (\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{u},\widetilde{H}\widetilde{z},\widetilde{z},\widetilde{H}\widetilde{z})=0.$$

Linearizing this once again we obtain

$$(3.14)\quad (\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{u},\widetilde{H}\widetilde{w},\widetilde{z},\widetilde{H}\widetilde{z}) + (\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{u},\widetilde{H}\widetilde{z},\widetilde{w},\widetilde{H}\widetilde{z}) + (\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{u},\widetilde{H}\widetilde{z},\widetilde{z},\widetilde{H}\widetilde{w})=0.$$

Now, put $\widetilde{u}=\widetilde{w}$ in (3.14) to obtain

$$2(\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{u},\widetilde{H}\widetilde{z},\widetilde{u},\widetilde{H}\widetilde{z}) + (\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{u},\widetilde{H}\widetilde{u},\widetilde{z},\widetilde{H}\widetilde{z})=0.$$

Using the first Bianchi identity for the second term yields

$$(3.15)\quad 3(\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{u},\widetilde{H}\widetilde{z},\widetilde{u},\widetilde{H}\widetilde{z}) + (\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{u},\widetilde{z},\widetilde{H}\widetilde{u},\widetilde{H}\widetilde{z})=0$$

and with (3.3) we get

$$3(\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{u},\widetilde{H}\widetilde{z},\widetilde{u},\widetilde{H}\widetilde{z})-(\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{u},\widetilde{z},\widetilde{u},\widetilde{H}^2\widetilde{z})=0. \tag{3.16}$$

Further, put $\widetilde{z}=\alpha\widetilde{a}+\beta\widetilde{b}$ in (3.16). From the coefficient $\alpha\beta$ we get

$$\begin{aligned}6(\widetilde{\nabla}_{\tilde{y}}\widetilde{R})&(\widetilde{u},\widetilde{H}\widetilde{a},\widetilde{u},\widetilde{H}\widetilde{b})\\ &-(\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{u},\widetilde{a},\widetilde{u},\widetilde{H}^2\widetilde{b})-(\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{u},\widetilde{b},\widetilde{u},\widetilde{H}^2\widetilde{a})=0.\end{aligned} \tag{3.17}$$

Using (3.3) for the last term leads to

$$3(\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{u},\widetilde{H}\widetilde{a},\widetilde{u},\widetilde{H}\widetilde{b})-(\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{u},\widetilde{a},\widetilde{u},\widetilde{H}^2\widetilde{b})=0. \tag{3.18}$$

Now we put $\widetilde{b}=X_{2j}, j=1,\dots,(n-1)/2$ in (3.18) to obtain

$$3(\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{u},\widetilde{H}\widetilde{a},\widetilde{u},\widetilde{H}X_{2j})+(\lambda_j)^2(\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{u},\widetilde{a},\widetilde{u},X_{2j})=0. \tag{3.19}$$

For $\widetilde{b}=X_{2j-1}$ we get

$$3(\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{u},\widetilde{H}\widetilde{a},\widetilde{u},\widetilde{H}X_{2j-1})+(\lambda_j)^2(\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{u},\widetilde{a},\widetilde{u},X_{2j-1})=0. \tag{3.20}$$

Next, we replace $\widetilde{z}$ by $\widetilde{H}\widetilde{z}$ in (3.16):

$$3(\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{u},\widetilde{H}^2\widetilde{z},\widetilde{u},\widetilde{H}^2\widetilde{z})-(\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{u},\widetilde{H}\widetilde{z},\widetilde{u},\widetilde{H}^3\widetilde{z})=0.$$

Also here we put $\widetilde{z}=\alpha\widetilde{a}+\beta\widetilde{b}$ and consider the coefficient of $\alpha\beta$ to obtain

$$\begin{aligned}&6(\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{u},\widetilde{H}^2\widetilde{a},\widetilde{u},\widetilde{H}^2\widetilde{b})-(\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{u},\widetilde{H}\widetilde{a},\widetilde{H}^2\widetilde{b},\widetilde{H}\widetilde{u})\\ &-(\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{H}\widetilde{a},\widetilde{H}^2\widetilde{b},\widetilde{H}\widetilde{u},\widetilde{u})-(\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{H}^2\widetilde{b},\widetilde{u},\widetilde{H}^2\widetilde{a},\widetilde{u})=0.\end{aligned} \tag{3.21}$$

Putting $\widetilde{b}=X_i, i=1,\dots,n-1$, in (3.21) and then considering $\widetilde{a}=\sum a_iX_i$ gives

$$5(\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{u},\widetilde{H}^2\widetilde{a},\widetilde{u},\widetilde{a})-(\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{u},\widetilde{H}\widetilde{a},\widetilde{a},\widetilde{H}\widetilde{u})-(\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{H}\widetilde{a},\widetilde{a},\widetilde{H}\widetilde{u},\widetilde{u})=0.$$

Using the first Bianchi identity for the last term, we obtain

$$3(\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{u},\widetilde{H}^2\widetilde{a},\widetilde{u},\widetilde{a})-(\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{u},\widetilde{H}\widetilde{a},\widetilde{u},\widetilde{H}\widetilde{a})=0. \tag{3.22}$$

Further, a new linearization in (3.22) leads to

$$3(\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{u},\widetilde{H}^2\widetilde{b},\widetilde{u},\widetilde{a})-(\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{u},\widetilde{H}\widetilde{a},\widetilde{u},\widetilde{H}\widetilde{b})=0. \tag{3.23}$$

Now, put $\widetilde{b}=X_{2j}$, $j\in\{1,\dots,(n-1)/2\}$, in (3.23) to get

$$3(\lambda_j)^2(\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{u},X_{2j},\widetilde{u},\widetilde{a})+(\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{u},\widetilde{H}\widetilde{a},\widetilde{u},\widetilde{H}X_{2j})=0. \tag{3.24}$$

For $\widetilde{b}=X_{2j-1}$ we get

$$3(\lambda_j)^2(\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{u},X_{2j-1},\widetilde{u},\widetilde{a})+(\widetilde{\nabla}_{\tilde{y}}\widetilde{R})(\widetilde{u},\widetilde{H}\widetilde{a},\widetilde{u},\widetilde{H}X_{2j-1})=0. \tag{3.25}$$

So, comparing (3.19) with (3.24) and (3.20) with (3.25), taking into account that $\widetilde{H}$ is non-singular, we obtain

$$(\widetilde{\nabla}_{\widetilde{y}}\widetilde{R})(\widetilde{u}, X_i, \widetilde{u}, \widetilde{a}) = 0, \qquad i = 1, \dots, n-1.$$

So, in particular, $(\widetilde{\nabla}_{\widetilde{u}}\widetilde{R})(\widetilde{u}, \widetilde{a}, \widetilde{u}, \widetilde{a}) = 0$, and this proves the result since the converse is trivial. ∎

Now we will express Theorem 3.1 in a more geometrical way. Therefore we note that when the flow $\mathfrak{F}_\xi$ is normal, (2.9) and Proposition 2.1 imply that the sectional curvature $K(\gamma', \xi)$, or equivalently, $\|H\gamma'\|$ is *constant along any transversal geodesic* γ. Then we get

COROLLARY 3.1. *Let $\mathfrak{F}_\xi$ be a contact normal flow on (M, g). Then (M, g, ξ) is a locally* KTS*-space if and only if the sectional curvature $K(\gamma', H\gamma')$ is constant along the transversal geodesic γ for all γ.*

Proof. Let γ be a unit speed transversal geodesic. Then Proposition 2.1 implies that $(\nabla_{\gamma'} H)\gamma'$ is vertical and hence

$$\gamma' R(\gamma', H\gamma', \gamma', H\gamma') = (\nabla_{\gamma'} R)(\gamma', H\gamma', \gamma', H\gamma').$$

Thus, $\gamma' K(\gamma', H\gamma') = \|H\gamma'\|^{-2}(\nabla_{\gamma'} R)(\gamma', H\gamma', \gamma', H\gamma')$. The required result now follows at once from Theorem 3.1. ∎

Note that for a normal contact flow $\mathfrak{F}_\xi$ the sectional curvature $K(H\gamma', \xi)$ is constant along a transversal geodesic γ.

4. Extrinsic geometry of geodesic spheres. The shape operator. For our applications we start by considering the shape operator of small geodesic spheres. Let $m \in M$ and denote by $G_m(r)$ the geodesic sphere centered at m and with sufficiently small radius r such that $G_m(r)$ is contained in a normal neighborhood. The extrinsic geometry of this hypersurface is described by the *shape operator* T_m. Let $p \in G_m(r)$ and let γ denote the unit speed geodesic through p with $\gamma(0) = m$ and $\gamma'(0) = u$. Since $\frac{\partial}{\partial r}(\gamma(r))$ is a unit normal vector of $G_m(r)$ at $p = \gamma(r) = \exp_m(ru)$, T_m is defined by

$$T_m(p)X = \left(\nabla_X \frac{\partial}{\partial r}\right)(p)$$

where $X \in T_p G_m(r)$. See [3], [7], [15] for more details and for further references about the geometry of small geodesic spheres. Jacobi vector fields may be used to derive the following power series expansion for the shape operator:

$$T_m(p) = \frac{1}{r} I - \frac{r}{3} R_m - \frac{r^2}{4} R'_m - \frac{r^3}{90}(9R''_m + 2R_m^2) + O(r^4) \tag{4.1}$$

where $R_m = R(m) = R_{u.}u$, $R'_m = R'(m) = (\nabla_u R)_{u.}u$, etc. Moreover, it should be noted that for this expression the spaces $\{\gamma'(0)\}^\perp$ and $\{\gamma'(r)\}^\perp$ are identified via a parallel orthonormal basis along γ.

Now we prove our first characterization.

THEOREM 4.1. *Let $\mathfrak{F}_\xi$ be an isometric flow on (M,g). Then (M,g,ξ) is a locally* KTS*-space if and only if for every point $m \in M$ and every transversal geodesic γ through m we have the following property: for every $p \in \gamma$ such that p and $s_m(p)$ lie in a normal neighborhood of m the shape operators $T_p(m)$ and $T_{s_m(p)}(m)$ at m of the geodesic spheres of radius $d(m,p)$ centered at p and at $s_m(p)$ commute with $(s_{m*})(m)$, that is,*

$$(s_{m*})(m) \cdot T_p(m) = T_{s_m(p)}(m) \cdot (s_{m*})(m). \tag{4.2}$$

P r o o f. First, let (M,g,ξ) be a locally KTS-space. Then each s_m is an isometry and hence (4.2) follows.

Conversely, suppose (4.2) holds. Using (4.1) we have

$$T_p(m) = \frac{1}{r}I - \frac{r}{3}R_p + \frac{r^2}{4}R'_p - \frac{r^3}{90}(9R''_p + 2R_p^2) + O(r^4)$$

or

$$T_p(m) = \frac{1}{r}I - \frac{r}{3}R_m - \frac{r^2}{12}R'_m - \frac{r^3}{180}(3R''_m + 4R_m^2) + O(r^4). \tag{4.3}$$

Similarly,

$$T_{s_m(p)}(m) = \frac{1}{r}I - \frac{r}{3}R_m + \frac{r^2}{12}R'_m - \frac{r^3}{180}(3R''_m + 4R_m^2) + O(r^4). \tag{4.4}$$

Next, let $x \in \{u,\xi\}^\perp$ at T_mM. Then (4.2)–(4.4) yield at once $R(u,x,u,\xi) = 0$ and $(\nabla_u R)(u,x,u,x) = 0$, for all horizontal u and all $m \in M$. Hence, the result follows from Theorem 2.2. ■

Further, we have

THEOREM 4.2. *Let (M,g) be a Riemannian manifold and let $\mathfrak{F}_\xi$ be a contact flow on it. Then (M,g,ξ) is a locally* KTS*-space if and only if for all unit horizontal u, all $m \in M$ and all sufficiently small r the operator*

$$(T_p + T_{s_m(p)})(m), \qquad p = \exp_m(ru),$$

preserves the horizontal subspace of $T_m(M)$.

P r o o f. For a locally KTS-space the result follows at once from Theorem 4.1.

Conversely, the hypothesis implies together with the expansions (4.3) and (4.4),

$$\begin{aligned} &g((T_p + T_{s_m(p)})(m)x,\xi) \\ &\quad = -\tfrac{2}{3}rR(u,x,u,\xi) - \tfrac{1}{90}r^3(3(\nabla^2_{uu}R)(u,x,u,\xi) + 4R(u,R_{ux}u,u,\xi)) + O(r^4) \end{aligned}$$

for all $x \in \{u, \xi\}^{\perp}$. Consequently,

$$R(u, x, u, \xi) = 0, \tag{4.5}$$

which shows that $\mathfrak{F}_\xi$ is normal. Further, we also have

$$3(\nabla^2_{uu} R)(u, x, u, \xi) + 4R(u, R_{ux}u, u, \xi) = 0. \tag{4.6}$$

But from (4.5) we get

$$\begin{cases} R_{u\xi}u = \|Hu\|^2 \xi, \\ (\nabla_u R)_{u\xi}u = R_{uHu}u - \|Hu\|^2 Hu, \\ (\nabla^2_{uu} R)_{u\xi}u = 2(\nabla_u R)_{uHu}u, \end{cases} \tag{4.7}$$

which, together with (4.6), yields $(\nabla_u R)(u, Hu, u, x) = 0$. In particular,

$$(\nabla_u R)(u, Hu, u, Hu) = 0 \tag{4.8}$$

for all horizontal u and all $m \in M$. Now the result follows from Theorem 3.1. ∎

Similarly we get

THEOREM 4.3. *Let $\mathfrak{F}_\xi$ be a contact normal flow on (M, g). Then (M, g, ξ) is a locally* KTS*-space if and only if for all $m \in M$, all horizontal unit u and all sufficiently small r we have, for $p = \exp_m(ru)$,*

(i) $(T_p + T_{s_m(p)})(m)\xi = \alpha_1 \xi$, *or*
(ii) $(T_p + T_{s_m(p)})(m)Hu$ *is horizontal*, *or*
(iii) $(T_{s_m(p)} - T_p)(m)Hu = \alpha_2 \xi$.

Proof. For a locally KTS-space the result follows again from Theorem 4.1. Conversely, using (4.7) we get, from (4.3) and (4.4),

$$\begin{aligned} (T_p + T_{s_m(p)})(m)\xi &= \alpha\xi - \tfrac{1}{15} r^3 (\nabla_u R)_{uHu}u + O(r^4), \\ g((T_p + T_{s_m(p)})(m)Hu, \xi) &= -\tfrac{1}{15} r^3 (\nabla_u R)(u, Hu, u, Hu) + O(r^4), \\ (T_{s_m(p)} - T_p)(m) &= \tfrac{1}{6} r^2 R'_m + O(r^4). \end{aligned}$$

Then each of the conditions implies (4.8) and the result follows again. ∎

Next, we will consider some functions of a geometrical nature instead of operators and investigate their behavior under the geodesic reflections. Let $\mathfrak{F}_\xi$ be a contact normal flow on (M, g) and let γ be a transversal geodesic tangent to the unit horizontal vector u. We also put $\gamma' = u$. Further, denote by σ the geodesic on $G_m(r)$ tangent to Hu at $p = \exp_m(ru)$ and by $\kappa_m(p)$ its curvature at that point, that is,

$$\kappa_m(p) = \|Hu\|^{-2} g(T_m(p)Hu, Hu).$$

Note that since $\nabla_u(Hu) = \|Hu\|^2\xi$, Hu is not parallel along γ but the 2-plane $\{\xi, Hu\}$ is parallel along γ. So, let (E_1, E_2) be an orthonormal basis of that plane spanned by the parallel vectors E_1, E_2 and with initial values

$$E_1(0) = \xi(m), \qquad E_2(0) = \|Hu\|^{-1}(Hu)(m).$$

Then at $p = \exp_m(ru)$ we have

$$\begin{cases} \xi(p) = E_1(r)\cos(r\|Hu\|) - E_2(r)\sin(r\|Hu\|), \\ (Hu)(p) = \|Hu\|\{E_1(r)\sin(r\|Hu\|) + E_2(r)\cos(r\|Hu\|)\}. \end{cases} \tag{4.9}$$

Now we prove

THEOREM 4.4. *Let $\mathfrak{F}_\xi$ be a contact normal flow on (M, g). Then (M, g, ξ) is a locally* KTS-*space if and only if*

$$\kappa_m(p) = \kappa_m(s_m(p)), \qquad p = \exp_m(ru), \tag{4.10}$$

for all $m \in M$, all horizontal unit u and all sufficiently small r.

Proof. First, suppose (M, g, ξ) is a locally KTS-space. Then each reflection s_m is an isometry which preserves ξ and so, it also preserves H. Hence, (4.10) follows.

Conversely, suppose (4.10) holds. Using (4.9) and (4.1) we obtain

$$\begin{aligned} \kappa_m(p) = r^{-1} &- \tfrac{1}{3}\|Hu\|^{-2} r(R(u, Hu, u, Hu)(m) \\ &- \tfrac{1}{4}\|Hu\|^{-2} r^2 (\nabla_u R)(u, Hu, u, Hu)(m) + O(r^3) \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} \kappa_m(s_m(p)) = r^{-1} &- \tfrac{1}{3}\|Hu\|^{-2} r R(u, Hu, u, Hu)(m) \\ &+ \tfrac{1}{4}\|Hu\|^{-2} r^2 (\nabla_u R)(u, Hu, u, Hu)(m) + O(r^3). \end{aligned} \tag{4.12}$$

Then we obtain again $(\nabla_u R)(u, Hu, u, Hu) = 0$ for all horizontal u and all $m \in M$. The desired result follows then also from Theorem 3.1. ∎

Using (4.3) and (4.4) we obtain in a similar way

THEOREM 4.5. *Let $\mathfrak{F}_\xi$ be a contact normal flow on (M, g). Then (M, g, ξ) is a locally* KTS-*space if and only if*

$$\kappa_p(m) = \kappa_{s_m(p)}(m), \qquad p = \exp_m(ru), \tag{4.13}$$

for all $m \in M$, all unit horizontal u and all sufficiently small r.

5. Intrinsic geometry of geodesic spheres. The Ricci operator. In this final section we consider some aspects of the intrinsic geometry of small geodesic spheres. We take the same notations and conventions as in Section 4. Using the same identification as for (4.1), the power series expansion for $T_m(p)$ leads, via the Gauss equation for the hypersurfaces

$G_m(r)$ and via contraction, to the following expansion for the Ricci operator $\widetilde{Q}_m$ of $G_m(r)$:

$$
\begin{aligned}
(5.1)\quad & \widetilde{Q}_m(p) \\
&= \frac{n-2}{r^2} I + \Big\{ Q - \varrho(u,\cdot)u - \frac{1}{3}\varrho(u,u)I - \frac{n}{3}R \Big\}(m) \\
&\quad + r\Big\{ \nabla_u Q - (\nabla_u \varrho)(u,\cdot)u - \frac{1}{4}(\nabla_u\varrho)(u,u)I - \frac{n+1}{4}\nabla_u R \Big\}(m) \\
&\quad + r^2\Big\{ \frac{1}{2}\nabla^2_{uu}Q - \frac{1}{2}(\nabla^2_{uu}\varrho)(u,\cdot)u - \frac{1}{10}(\nabla^2_{uu}\varrho)(u,u)I \\
&\quad - \frac{n+2}{10}\nabla^2_{uu}R + \frac{1}{9}\varrho(u,u)R - \frac{1}{45}\sum_{a,b=1}^{n} R^2_{uaub} I - \frac{n+2}{45}R^2 \Big\}(m) + O(r^3)
\end{aligned}
$$

where $R_{uaub} = R(u, e_a, u, e_b)$ and $\{e_a : a = 1, \ldots, n\}$ is an arbitrary orthonormal basis of $T_m M$.

First, we have

THEOREM 5.1. *Let $\mathfrak{F}_\xi$ be an isometric flow on a Riemannian manifold (M, g) of dimension $n > 3$. Then (M, g, ξ) is a locally* KTS-*space if and only if for every $m \in M$, every transversal geodesic $\gamma : r \mapsto \exp_m(ru)$, u horizontal and $\|u\| = 1$, and every small r we have the following commutativity property for the Ricci operator $\widetilde{Q}$:*

$$
(5.2)\qquad (s_{m*})(m) \cdot \widetilde{Q}_p(m) = \widetilde{Q}_{s_m(p)}(m) \cdot (s_{m*})(m),
$$

where $p = \exp_m(ru)$.

Proof. Let (M, g, ξ) be a locally KTS-space. Then the result follows from Theorem 4.1 by using the Gauss equation and the fact that all s_m are isometries.

Conversely, from (5.1) we get

$$
\begin{aligned}
(5.3)\quad \widetilde{Q}_p(m) &= \frac{n-2}{r^2} I + \Big\{ Q - \varrho(u,\cdot)u - \frac{1}{3}\varrho(u,u)I - \frac{n}{3}R \Big\}(m) \\
&\quad - \frac{r}{12}\{(\nabla_u\varrho)(u,u)I + (n-3)\nabla_u R\}(m) \\
&\quad + r^2\Big\{ -\frac{1}{60}(\nabla^2_{uu}\varrho)(u,u)I - \frac{n-3}{60}\nabla^2_{uu}R + \frac{1}{9}\varrho(u,u)R \\
&\quad - \frac{1}{45}\sum_{a,b=1}^{n} R^2_{uaub} I - \frac{n+2}{45}R^2 \Big\}(m) + O(r^3),
\end{aligned}
$$

and

$$(5.4)\quad \widetilde{Q}_{s_m(p)}(m)$$
$$= \frac{n-2}{r^2}I + \Big\{Q - \varrho(u,\cdot)u - \frac{1}{3}\varrho(u,u)I - \frac{n}{3}R\Big\}(m)$$
$$+ \frac{r}{12}\{(\nabla_u\varrho)(u,u)I + (n-3)\nabla_u R\}(m)$$
$$+ r^2\Big\{ -\frac{1}{60}(\nabla^2_{uu}\varrho)(u,u)I - \frac{n-3}{60}\nabla^2_{uu}R + \frac{1}{9}\varrho(u,u)R$$
$$- \frac{1}{45}\sum_{a,b=1}^{n} R^2_{uaub}I - \frac{n+2}{45}R^2\Big\}(m) + O(r^3).$$

Next, let x be a unit vector in T_mM orthogonal to the plane $\{u,\xi\}$. Then, using (5.2), (5.3) and (5.4) we obtain

$$\varrho(x,\xi) = \tfrac{1}{3}nR(u,x,u,\xi), \tag{5.5}$$
$$(\nabla_u\varrho)(u,u) = (3-n)(\nabla_u R)(u,x,u,x). \tag{5.6}$$

Now, let a and b be arbitrary horizontal vectors. Then (5.5) and (5.6) imply, for a orthogonal to b,

$$\varrho(a,\xi) = \tfrac{1}{3}nR(b,a,b,\xi), \tag{5.7}$$
$$(\nabla_a\varrho)(a,a) = (3-n)(\nabla_a R)(a,b,a,b). \tag{5.8}$$

Next, let $b = e_i$ where $e_i, i = 1,\dots,n-2$, is an orthonormal basis of $\{a,\xi\}^\perp$. Then contraction in (5.7) yields easily, since $n > 3$, $R(b,a,b,\xi) = 0$, which means that the flow is normal. Together with (5.8), this implies by explicit computation that $(\nabla_a\varrho)(a,a) = 0$, and so $(\nabla_a R)(a,b,a,b) = 0$. The required result now follows from Theorem 2.2. ∎

In a similar way, using also (4.7) and proceeding as in Section 4, we get

THEOREM 5.2. *Let $\mathfrak{F}_\xi$ be a contact flow on (M,g) with* $\dim M > 3$. *Then (M,g,ξ) is a locally* KTS-*space if and only if for all unit horizontal u, all $m \in M$ and all sufficiently small r the operator*

$$(\widetilde{Q}_p + \widetilde{Q}_{s_m(p)})(m), \qquad p = \exp_m(ru),$$

preserves the horizontal subspace of T_mM.

THEOREM 5.3. *Let $\mathfrak{F}_\xi$ be a contact normal flow on (M,g) with* $\dim M>3$. *Then (M,g,ξ) is a locally* KTS-*space if and only if for all unit horizontal u, all $m \in M$ and all sufficiently small r we have, for $p = \exp_m(ru)$,*

(i) $(\widetilde{Q}_p + \widetilde{Q}_{s_m(p)})(m)\xi = \alpha\xi$, *or*

(ii) $(\widetilde{Q}_p + \widetilde{Q}_{s_m(p)})(m)Hu$ *is horizontal.*

We finish this section by deriving two additional characterizations by means of properties of special *sectional curvatures* of small geodesic spheres.

Let $\mathfrak{F}_\xi$ be again a contact normal flow on (M,g), $m \in M$, and γ a transversal unit speed geodesic with $\gamma(0) = m$, $\gamma'(0) = u$. We recall that the two-plane spanned by ξ and $H\gamma'$ is parallel along γ and tangent to the geodesic spheres $G_m(r)$ at $p = \exp_m(ru)$. Let $K_m^G(p)$ denote the corresponding sectional curvature of $G_m(r)$. Then we have

THEOREM 5.4. *Let $\mathfrak{F}_\xi$ be a contact normal flow on (M,g). Then (M,g,ξ) is a locally* KTS*-space if and only if*

$$K_m^G(p) = K_m^G(s_m(p)), \qquad p = \exp_m(ru), \tag{5.9}$$

for all $m \in M$, all unit horizontal u and all sufficiently small r.

Proof. First, let (M,g,ξ) be a locally KTS-space. Then any reflection s_m is an isometry which preserves ξ and also H (Theorem 5.3 of [6]). Now (5.9) follows at once from this remark and from the Gauss equation for the hypersurface $G_m(r)$.

To prove the converse we proceed as follows. Let $(E_1,\dots,E_{n-1},E_n = \gamma')$ be a parallel orthonormal basis along γ and denote by R^G the Riemannian curvature tensor of $G_m(r)$. Then we have [3, (4.3)]

$$\begin{aligned} R^G_{abcd}(p) &\\ = r^{-2}&(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) \\ &+ \{R_{abcd} - \tfrac{1}{3}(R_{ubud}\delta_{ac} + R_{uauc}\delta_{bd} - R_{ubuc}\delta_{ad} - R_{uaud}\delta_{bc})\}(m) \\ &+ r\{\nabla_u R_{abcd} - \tfrac{1}{4}(\nabla_u R_{ubud}\delta_{ac} + \nabla_u R_{uauc}\delta_{bd} - \nabla_u R_{ubuc}\delta_{ad} \\ &- \nabla_u R_{uaud}\delta_{bc})\}(m) + O(r^2), \end{aligned} \tag{5.10}$$

where $R_{abcd} = R(E_a,E_b,E_c,E_d)$, etc. and $a,b,c,d = 1,\dots,n-1$.

Next, put

$$E_1(0) = \xi(m), \qquad E_2(0) = \|Hu\|^{-1}(Hu)(m).$$

Since the two-plane spanned by ξ and $H\gamma'$ is parallel, we have $K_m^G(p) = R^G_{1212}(p)$ and so (5.10) yields, after a straightforward computation,

$$\begin{aligned} K_m^G(p) = r^{-2} &+ \|Hu\|^{-2}R(\xi,Hu,\xi,Hu)(m) \\ &- \tfrac{1}{3}\{\|Hu\|^{-2}R(u,Hu,u,Hu) + \|Hu\|^2\}(m) \\ &- \tfrac{1}{4}r\|Hu\|^{-2}(\nabla_u R)(u,Hu,u,Hu)(m) + O(r^2). \end{aligned} \tag{5.11}$$

Hence (5.9) and (5.11) yield clearly $(\nabla_u R)(u,Hu,u,Hu) = 0$, and the required result follows from Theorem 3.1. ∎

With the same notations as above we also have

THEOREM 5.5. *Let $\mathfrak{F}_\xi$ be a contact normal flow on* (M,g). *Then* (M,g,ξ) *is a locally* KTS*-space if and only if*

$$K^G_p(m) = K^G_{s_m(p)}(m), \qquad p = \exp_m(ru),$$

for all $m \in M$, *all unit horizontal* u *and all sufficiently small* r.

Proof. For a locally KTS-space, (5.12) follows from Theorem 4.1 and the Gauss equation of the geodesic spheres.

To prove the converse, we first derive a power series expansion for $K^G_p(m)$. To do this, put

$$E_1(r) = \xi(p), \quad E_2(r) = \|Hu\|^{-1}(Hu)(p), \quad E_n(r) = u,$$

where $\gamma'(r)$ is also denoted by u. Then from (5.10) we have

$$\text{(5.13)} \quad K^G_p(m) = R^G_{1212}(m) = r^{-2} + R_{1212}(p) - \tfrac{1}{3}(R_{u2u2} + \|Hu\|^2)(p) + \tfrac{1}{4}r(\nabla_u R_{u2u2})(p) + O(r^2).$$

Since $R_{abcd}(p) = R_{abcd}(m) + r(\nabla_u R_{abcd})(m) + O(r^2)$, (5.13) takes the form

$$\text{(5.14)} \quad K^G_p(m) = r^{-2} + R_{1212}(m) - \tfrac{1}{3}(R_{u2u2} + \|Hu\|^2)(m) + r(\nabla_u R_{1212} - \tfrac{1}{12}\nabla_u R_{u2u2})(m) + O(r^2).$$

Further, we have

$$\xi(m) = E_1(0)\cos(r\|Hu\|) + E_2(0)\sin(r\|Hu\|),$$
$$(Hu)(m) = -\|Hu\|\{E_1(0)\sin(r\|Hu\|) - E_2(0)\cos(r\|Hu\|)\}$$

and hence

$$\text{(5.15)} \quad \begin{aligned} E_1(0) &= \xi(m) - r(Hu)(m) - \tfrac{1}{2}r^2\|Hu\|^2\xi(m) + O(r^3),\\ E_2(0) &= \|Hu\|^{-1}(Hu)(m) + r\|Hu\|\xi(m)\\ &\quad - \tfrac{1}{2}r^2\|Hu\|(Hu)(m) + O(r^3). \end{aligned}$$

Substitution of (5.15) in (5.14) yields

$$\text{(5.16)} \quad \begin{aligned} K^G_p(m) &= r^{-2} + \|Hu\|^{-2}R(\xi,Hu,\xi,Hu)(m)\\ &\quad - \tfrac{1}{3}\{\|Hu\|^{-2}R(u,Hu,u,Hu) + \|Hu\|^2\}(m)\\ &\quad - \tfrac{1}{12}r\|Hu\|^{-2}(\nabla_u R)(u,Hu,u,Hu)(m) + O(r^2). \end{aligned}$$

The corresponding expression for $K^G_{s_m(p)}(m)$ follows by replacing r by $-r$ in (5.16). So, the hypothesis (5.12) leads again to $(\nabla_u R)(u,Hu,u,Hu) = 0$, and the required result follows. ∎

REFERENCES

[1] A. L. Besse, *Einstein Manifolds*, Ergeb. Math. Grenzgeb. (3) 10, Springer, Berlin, 1987.

[2] P. Bueken, *Reflections and rotations in contact geometry*, doctoral dissertation, Catholic University Leuven, 1992.

[3] B. Y. Chen and L. Vanhecke, *Differential geometry of geodesic spheres*, J. Reine Angew. Math. 325 (1981), 28–67.

[4] M. Djorić and L. Vanhecke, *Geometry of geodesic spheres on Sasakian manifolds*, Rend. Sem. Mat. Univ. Politec. Torino 49 (1991), 329–357.

[5] M. C. González-Dávila, *Espacios transversalmente simétricos de tipo Killing*, doctoral dissertation, Universidad de La Laguna, 1992.

[6] J. C. González-Dávila, M. C. González-Dávila and L. Vanhecke, *Reflections and isometric flows*, Kyungpook Math. J., to appear.

[7] A. Gray and L. Vanhecke, *Riemannian geometry as determined by the volumes of small geodesic balls*, Acta Math. 142 (1979), 157–198.

[8] B. O'Neill, *The fundamental equations of a submersion*, Michigan Math. J. 13 (1966), 459–469.

[9] R. S. Palais, *A global formulation of the Lie theory of transformation groups*, Mem. Amer. Math. Soc. 22 (1957).

[10] B. L. Reinhart, *Differential Geometry of Foliations*, Ergeb. Math. Grenzgeb. 99, Springer, Berlin, 1983.

[11] T. Takahashi, *Sasakian φ-symmetric spaces*, Tôhoku Math. J. 29 (1977), 91–113.

[12] Ph. Tondeur, *Foliations on Riemannian Manifolds*, Universitext, Springer, Berlin, 1988.

[13] Ph. Tondeur and L. Vanhecke, *Transversally symmetric Riemannian foliations*, Tôhoku Math. J. 42 (1990), 307–317.

[14] —, —, *Jacobi fields, Riccati equation and Riemannian foliations*, to appear.

[15] L. Vanhecke, *Geometry in normal and tubular neighborhoods*, in: Proc. Workshop on Differential Geometry and Topology, Cala Gonone (Sardinia) 1988, Rend. Sem. Fac. Sci. Univ. Cagliari, Supplemento al vol. 58 (1988), 73–176.

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