

## A SARD TYPE THEOREM FOR BOREL MAPPINGS

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We find a condition for a Borel mapping  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$  which implies that the Hausdorff dimension of  $f^{-1}(y)$  is less than or equal to  $m - n$  for almost all  $y \in \mathbb{R}^n$ .

**1. Introduction and statement of result.** Sard's theorem ([14], [13], [15]) implies that if  $f \in C^\infty(\mathbb{R}^m, \mathbb{R}^n)$ , then for almost all  $y \in \mathbb{R}^n$ ,  $f^{-1}(y)$  is a smooth manifold of dimension  $m - n$ . In general this type of result is no longer true for continuous mappings. Peano curve type examples show that the set  $f^{-1}(y)$  can have Hausdorff dimension greater than  $m - n$  for all  $y \in \mathbb{R}^n$ . We will be concerned here with Borel mappings  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ , without any smoothness assumptions, hence we cannot expect that  $f^{-1}(y)$  is a manifold. However, we can ask what conditions (different than smoothness conditions)  $f$  should satisfy in order to get the estimate  $\dim_{\text{H}}(f^{-1}(y)) \leq m - n$  for almost all  $y \in \mathbb{R}^n$  ( $\dim_{\text{H}}$  stands for the Hausdorff dimension).

The Eilenberg inequality ([3], [4], [2], [5, Theorem 2.10.25]) implies that the above estimate holds a.e. provided  $f$  is Lipschitz. We will generalize this result.

This article grew out from the author's interest in the theory of Sobolev mappings. Related Sard type theorems for Sobolev mappings are presented in [7], [6] and in a slightly different form in [1]. Namely, a Sobolev mapping satisfies a similar assumption to that of Theorem 1 below. Hence by the method presented in the proof of Theorem 1 we get a similar estimate of the Hausdorff dimension of  $f^{-1}(y)$ , where  $f$  is a suitable Sobolev mapping. This estimate is of crucial importance for the validity of the so-called *co-area* formula (see [7]). It is also important in the context of nonlinear elasticity (see [6]). It seems that the method of the proof of Theorem 1 is more important than the theorem itself. This is because we can use the same method to produce a large class of theorems by slightly modifying the assumptions and the claim.

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Let  $H^k$  denote Hausdorff measure. We aim to prove the following

**THEOREM 1.** *Let  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ , where  $n \leq m$ , be a Borel mapping. Assume that there exists a constant  $C$  such that*

$$(1) \quad H^n(f(A)) \leq CH^n(A)$$

for all Borel subsets  $A \subseteq \mathbb{R}^m$ . Then  $\dim_{\mathbb{H}}(f^{-1}(y)) \leq m - n$  for almost all  $y \in \mathbb{R}^n$ .

**Remarks.** 1) Note that if  $A$  is a Borel set, then  $f(A)$  is measurable—see Proposition 2 below.

2) The condition (1) is satisfied by Lipschitz mappings.

3) In fact, we will prove a slightly stronger result—see the remark at the end of the paper.

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**2. Basic notion.** Before we proceed to the proof, let us recall some basic notion and results which will be used in the sequel.

Let  $0 < k < m$  be integers. By  $O^*(m, k)$  we denote the manifold of all orthogonal projections from  $\mathbb{R}^m$  onto  $k$ -dimensional linear subspaces.  $O^*(m, k)$  is equipped with the unique invariant measure  $\vartheta_{m,k}^*$  normalized to have total mass 1. As usual by  $N(p|B, y)$  we denote the Banach indicatrix, i.e. the number of points in  $f^{-1}(y) \cap A$ . The *integral-geometric measure* is defined for Borel sets by

$$I^k(B) = \frac{1}{\beta(m, k)} \int_{p \in O^*(m, k)} \int_{y \in \text{image } p} N(p|B, y) dH^k(y) d\vartheta_{m,k}^*(p)$$

( $\beta(m, k)$  is a normalizing coefficient).

We will use the following theorem of Mattila.

**PROPOSITION 1** ([12, Theorem 4.7]). *If  $E \subseteq \mathbb{R}^m$  is such that  $I^k(E) < \infty$ , then  $\dim_{\mathbb{H}}(E) \leq k$ .*

The well known theorem of Lusin and Sierpiński ([5, Theorem 2.2.13], [8, Lemma 39.2, Theorem 94], [9], [10], [11, p. 44]) states that if  $A \subseteq \mathbb{R}^{k+l}$  is a Borel subset and  $\pi_k : \mathbb{R}^{k+l} \rightarrow \mathbb{R}^k$  is the standard projection, then the set  $\pi_k(A)$  is  $H^k$ -measurable. It is also known that not only projections, but an arbitrary Borel image of a Borel set is measurable (see [9, Chap. 3.38.III, Th. 2; Chap. 3.38.IV, Remark 1]). However, for the reader's convenience we present a “one-line” proof of this fact.

**PROPOSITION 2.** *If  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$  is a Borel mapping, and  $A \subseteq \mathbb{R}^m$  is a Borel subset, then  $f(A)$  is measurable.*

**Proof.** Let  $\pi_n : \mathbb{R}^{m+n} \rightarrow \mathbb{R}^n$  and  $\pi_m : \mathbb{R}^{m+n} \rightarrow \mathbb{R}^m$  denote the standard projections. Let  $G_f \subseteq \mathbb{R}^{m+n}$  denote the graph of  $f$ . Since  $f$  is Borel,  $G_f$  is a Borel set and hence by Lusin–Sierpiński’s theorem  $f(A) = \pi_n(G_f \cap \pi_m^{-1}(A))$  is measurable.

**3. Proof of Theorem 1.** Since the problem is local it suffices to consider the mapping  $f$  restricted to a ball  $B^m$ .

**Step 1:**  $m = n$ . For each  $p \in \mathbb{N}$ , let  $H_p = \{A_{i,p}\}_{i=1}^p$  be a decomposition of  $B^n$  into Borel sets (i.e.  $B^n = \bigcup_{i=1}^p A_{i,p}$ ,  $A_{i,p} \cap A_{j,p} = \emptyset$  for  $i \neq j$ ) such that each  $A_{i,p} \in H_p$  is a union of elements from  $H_{p+1}$  and  $\sup_i \text{diam } A_{i,p} \rightarrow 0$  as  $p \rightarrow \infty$ . Evidently

$$N(f|B^n, y) = \lim_{p \rightarrow \infty} \sum_{i=1}^p \chi_{f(A_{i,p})}(y)$$

( $\chi_E$  stands for the characteristic function of  $E$ ). Note that the above limit exists, since the sequence is non-decreasing. It follows directly from Proposition 2 (or, in a more elementary way, from (1)) that  $N(f|B^n, \cdot)$  is a measurable function. Hence by (1) we obtain

$$\begin{aligned} (2) \quad \int_{\mathbb{R}^n} N(f|B^n, y) dH^n(y) &= \lim_{p \rightarrow \infty} \sum_{i=1}^p \int_{\mathbb{R}^n} \chi_{f(A_{i,p})}(y) dH^n(y) \\ &= \lim_{p \rightarrow \infty} \sum_{i=1}^p H^n(f(A_{i,p})) \\ &\leq C \lim_{p \rightarrow \infty} \sum_{i=1}^p H^n(A_{i,p}) = CH^n(B^n). \end{aligned}$$

This inequality completes the proof.

**Step 2:** General case. Fix  $p \in O^*(m, m - n)$ . Let  $B_{p,x}^n = p^{-1}(x) \cap B^m$  for  $x \in \text{image } p$ . Assume for the moment that we have proved the measurability of  $N(f|B_{p,x}^n, y)$  as a function of  $p, x$  and  $y$ . Applying (2) to  $f|B_{p,x}^n$  we have

$$\int_{\mathbb{R}^n} N(f|B_{p,x}^n, y) dH^n(y) \leq CH^n(B_{p,x}^n).$$

Integrating both sides over  $p \in O^*(m, m - n)$  and  $x \in \text{image } p$  we get

$$\int_{\mathbb{R}^n} \int_{p \in O^*(m, m - n)} \int_{x \in \text{image } p} N(f|B_{p,x}^n, y) dH^{m-n}(x) d\vartheta_{m, m-n}^*(p) dH^n(y) \leq C_1.$$

Since  $N(f|B_{p,x}^n, y) = N(p|f^{-1}(y), x)$ , by Fubini’s theorem we get  $\int_{\mathbb{R}^n} N(p|f^{-1}(y), x) dH^{m-n}(x) < \infty$  for  $H^n$ -almost all  $y \in \mathbb{R}^n$ . Now the theorem follows from Proposition 1.

To complete our argument it remains to prove the measurability of the function  $N(f|B_{p,x}^n, y)$ . We first describe precisely its domain of definition. Set

$$\begin{aligned} \pi : B^m \times \mathbb{R}^n \times O^*(m, m-n) &\longrightarrow B^m \times \mathbb{R}^n \times O^*(m, m-n), \\ (x, y, p) &\longmapsto (p(x), y, p). \end{aligned}$$

Note that the image  $W$  of  $\pi$  is a manifold equipped with the measure  $H^{m-n} \otimes H^n \otimes \vartheta_{m, m-n}^*$ . This is the domain of definition of the considered function  $N$ . Let  $G_A$  denote the graph of  $f|_A : A \rightarrow \mathbb{R}^n$ , where  $A \subseteq B^m$  is a Borel subset. We have  $G_A \times O^*(m, m-n) \subseteq B^m \times \mathbb{R}^n \times O^*(m, m-n)$ . Let  $S_A = \pi(G_A \times O^*(m, m-n))$ . Since  $G_A$  is a Borel set, being the graph of a Borel mapping, it follows from Proposition 2 that  $S_A$  is a measurable subset of  $W$ . Let  $B^m = \bigcup_{i=1}^p A_{i,p}$  be a Borel decomposition as in Step 1. Now it is clear that

$$N : W \rightarrow \mathbb{N}_0 \cup \{\infty\}, \quad N(f|B_{p,x}^n, y) = \lim_{p \rightarrow \infty} \sum_{i=1}^p \chi_{S_{A_{i,p}}}.$$

Hence  $N$  is measurable.

**Remark.** In fact, we have proved a slightly stronger result than stated in Theorem 1. Namely, it suffices to assume that (1) holds only for Borel subsets contained in  $n$ -dimensional affine subspaces of  $\mathbb{R}^m$  and as a result we conclude that for almost all  $y$  the set  $f^{-1}(y)$  has locally finite  $I^{m-n}$  measure (i.e.  $I^{m-n}(f^{-1}(y) \cap K) < \infty$  for all bounded sets  $K \subseteq \mathbb{R}^m$ ).

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