COLLOQUIUM MATHEMATICUM

VOL. LXVII 1994 FASC. 2

ON POSITIVE ROCKLAND OPERATORS

BY

PASCAL AUSCHER (RENNES), A. F. M. TER ELST (EINDHOVEN) AND DEREK W. ROBINSON (CANBERRA)

Let G be a homogeneous Lie group with a left Haar measure dg and L the action of G as left translations on $L_p(G;dg)$. Further, let $H = dL(C)$ denote a homogeneous operator associated with L. If H is positive and hypoelliptic on L_2 we prove that it is closed on each of the L_p-spaces, $p \in \langle 1,\infty\rangle$, and that it generates a semigroup S with a smooth kernel K which, with its derivatives, satisfies Gaussian bounds. The semigroup is holomorphic in the open right half-plane on all the L_p-spaces, $p \in [1,\infty]$.

Further extensions of these results to nonhomogeneous operators and general representations are also given.

1. Introduction. A differential operator H on a homogeneous group G is defined to be a *Rockland operator* if it is right-invariant, homogeneous and injective in each nontrivial irreducible unitary representation. If H is positive on $L_2(G;dg)$ then it generates a holomorphic semigroup with a kernel and recently Dziubański [Dzi] proved that the kernel, together with its derivatives, decreases exponentially on the right half-plane. Dziubański also raised the question whether one could establish stronger decrease properties than exponential. Stronger bounds have been derived by Hebisch [Heb] for Rockland operators which are sums of even powers and in fact Hebisch showed that the kernel, and its derivatives, satisfy "Gaussian" bounds on the positive real line. The aim of this paper is to show that the kernel of the semigroup generated by a general positive Rockland operator has "Gaussian" bounds on the right half-plane. Similar bounds are also established for the derivatives.

The theory of Rockland operators began with Rockland's analysis of differential operators on the Heisenberg group [Roc]. Helffer and Nourrigat [HeN1] proved that a Rockland operator on a graded group is hypoelliptic and in addition they derived several inequalities between the norm on the C^n spaces and the operator norm. Then Miller [Mil] showed that one can replace

1991 *Mathematics Subject Classification*: 43A85, 22E45, 35B45.

a graded group by a homogeneous group in the Helffer–Nourrigat theorem. Subsequently, Folland and Stein [FoS] used the proof of an earlier theorem of Nelson and Stinespring [NeS] to deduce that a positive Rockland operator is essentially self-adjoint on the space $C_c^\infty(G)$. Moreover, they established that the closure generates a continuous semigroup with a kernel which is in the Schwartz space over the group. Dziubański [Dzi] then extended their result and showed that on the right half-plane one has an exponential decay for the kernel and all its derivatives. Our results give more precise "Gaussian" bounds, more detailed analyticity properties of the semigroup and stronger regularity properties of the Rockland operators. Consequently, we are able to extend many of the previous results to nonhomogeneous operators and to general representations of the group.

Throughout the sequel we adopt the general notation of [Rob]. Let G be a (connected, simply connected) homogeneous group with Lie algebra $\mathfrak{g}$ and let $(\mathcal{X}, G, U)$ denote a strongly, or weak*, continuous representation of G on the Banach space $\mathcal{X}$ by bounded operators $g \mapsto U(g)$. If $a_i \in \mathfrak{g}$ then A_i $(= dU(a_i))$ will denote the generator of the one-parameter subgroup $t \mapsto U(\exp(-ta_i))$ of the representation. Let $(\gamma_t)_{t>0}$ be a family of *dilations* on $\mathfrak{g}$, i.e., a one-parameter group of automorphisms of the form

$$\gamma_t(a_i) = t^{w_i} a_i$$

for some basis $a_1, \ldots, a_d$ of $\mathfrak{g}$ and some positive numbers $w_1, \ldots, w_d$, which we call *weights*. We always assume that the smallest weight is at least one. Let $D = \sum_{i=1}^d w_i$ be the *homogeneous dimension* of G. For the multi-indices we introduce the following notation. If $n \in \mathbb{N}_0$ let

$$J_n(d) = \bigoplus_{k=0}^{n} \{1, \ldots, d\}^k$$

and set

$$J(d) = \bigcup_{n=0}^{\infty} J_n(d).$$

Then if $\alpha = (i_1, \ldots, i_n) \in J(d)$ we denote the *Euclidean length* n of α by $|\alpha|$ and the *weighted length* by

$$\|\alpha\| = \sum_{k=1}^{n} w_{i_k}.$$

If $n \in \mathbb{N}$ we define $\mathcal{X}_n = \mathcal{X}_n(U) = \bigcap_{\alpha \in J_n(d)} D(A^\alpha)$ and

$$\|x\|_n = \max_{\substack{\alpha \in J(d)\\ |\alpha| \le n}} \|A^\alpha x\|,$$

where $A^\alpha = A_{i_1} \ldots A_{i_n}$ if $\alpha = (i_1, \ldots, i_n)$. Similarly we define the weighted spaces

$$\mathcal{X}'_n = \mathcal{X}'_n(U) = \bigcap_{\substack{\alpha\in J(d)\\ \|\alpha\|\le n}} D(A^\alpha)$$

for all $n \in \mathbb{R}$ with $n > 0$. Now, however, it can happen for a given n that there are no multi-indices α such that $\|\alpha\| = n$. Therefore the corresponding norms and seminorms are given by

$$\|x\|'_n = \begin{cases} \max_{\alpha\in J(d),\|\alpha\|\le n} \|A^\alpha x\| & \text{if there exist } \alpha \in J(d) \text{ with } \|\alpha\| = n, \\ 0 & \text{otherwise,} \end{cases}$$

$$N'_n(x) = \begin{cases} \max_{\alpha\in J(d),\|\alpha\|= n} \|A^\alpha x\| & \text{if there exist } \alpha \in J(d) \text{ with } \|\alpha\| = n, \\ 0 & \text{otherwise,} \end{cases}.$$

Moreover, let $\mathcal{X}_\infty = \mathcal{X}_\infty(U) = \bigcap_{n=1}^\infty \mathcal{X}_n$. It follows by a line by line extension of Lemma 2.4 of [ElR2] that the Gårding space, and in particular the space $\mathcal{X}_\infty$, is dense in $\mathcal{X}'_n$ for all $n > 0$. The density is with respect to the weak, or weak*, topology. If U is the left regular representation on $L_p(G; dg)$ then we denote the corresponding spaces by $L_{p;n}$, $L'_{p;n}$, $L_{p;\infty}$ and the norms and seminorms by $\|\cdot\|_{p;n}$ etc. Further, we let L denote the left regular representation on L_2. If U is the left regular representation in the space $C_{\rm b}(G)$ of bounded continuous functions on G we similarly use the notation $C_{{\rm b};\infty}(G)$.

Let $m \in \langle 0, \infty\rangle$ and let $C : J(d) \to \mathbb{C}$ be such that $C(\alpha) = 0$ if $\|\alpha\| > m$ and there exists at least one $\alpha \in J(d)$ with $\|\alpha\| = m$ and $C(\alpha) \neq 0$. We call C a *form* of order m. We write $c_\alpha = C(\alpha)$. The *principal part* P of C is the form

$$P(\alpha) = \begin{cases} C(\alpha) & \text{if } \|\alpha\| = m, \\ 0 & \text{if } \|\alpha\| < m. \end{cases}$$

The *formal adjoint* $C^\dagger$ of C is the function $C^\dagger : J(d) \to \mathbb{C}$ defined by

$$C^\dagger(\alpha) = (-1)^{|\alpha|}\overline{C(\alpha_*)},$$

where $\alpha_* = (i_n, \ldots, i_1)$ if $\alpha = (i_1, \ldots, i_n)$. We consider the operators

$$dU(C) = \sum_{\alpha\in J(d)} c_\alpha A^\alpha$$

with domain $D(dU(C)) = \mathcal{X}'_m$. If $(\mathcal{F}, G, U_*)$ is the dual representation of $(\mathcal{X}, G, U)$ then $dU_*(C^\dagger)$ is called the *dual operator* and denoted by $H^\dagger$.

If P is the principal part of a form C we call P a *Rockland form* if the operator $dU(P)$ is injective on the space $\mathcal{X}_\infty(U)$ for every nontrivial irreducible unitary representation U of G. It then follows from the Helffer–Nourrigat theorem [HeN1] that $dL(P)|_{C_c^\infty}$ is a hypoelliptic operator. In fact, the Helffer–Nourrigat theorem is formulated for graded groups. But it follows from Propositions 1.3 and 1.4 of [Mil] that the existence of a

Rockland form ensures that the order m of P is an integer multiple of the smallest weight and all weights are rational multiples of this smallest weight. Therefore G is a graded group if one rescales the original weights by a large enough constant.

A Rockland form P is called a *positive Rockland form* if $dL(P)$ is symmetric and $(\varphi, dL(P)\varphi) \geq 0$ for all φ in the Schwartz space on G (see [FoS], p. 129). Throughout this paper we assume that C is a form of order m and that the principal part P of C is a positive Rockland form. We call $dL(P)$ a *positive Rockland operator*.

An important feature of the class of Rockland operators that we will use repeatedly is that it is closed under the operation of taking powers. If H is a Rockland operator then H^n is a Rockland operator for all $n \in \mathbb{N}$. Moreover, if H is a positive Rockland operator then all the powers, H^n, are also positive Rockland operators.

The group G can be equipped with two distances. The first is the Euclidean modulus $|\cdot|$, as defined in [Rob], p. 256. In addition it has a homogeneous modulus $|\cdot|'$, which also induces a right-invariant metric on G by $d'(g;h) = |gh^{-1}|'$ (see [HeS]). The balls with radius s are denoted by B_s and B'_s.

The main theorem of this paper is the following.

THEOREM 1.1. *Let P be a positive Rockland form, $(\mathcal{X}, G, U)$ a continuous representation of G and $H = dU(P)$ the associated operator. Then*

I. *The closure $\overline{H}$ of H generates a continuous semigroup S.*

II. *The semigroup S is holomorphic in the open right half-plane.*

III. *The semigroup S has a representation independent kernel K in $L_{1;\infty}(G; dg) \cap C_{0;\infty}(G)$ such that*

$$A^\alpha S_t x = \int_G dh\, (A^\alpha K_t)(h) U(h) x$$

for all $\alpha \in J(d)$, $x \in \mathcal{X}$ and $g \in G$.

IV. *For each $\varepsilon \in \langle 0, \pi/2\rangle$ and all $\alpha \in J(d)$ there exist $a, b > 0$ such that*

$$|(A^\alpha K_z)(g)| \leq a|z|^{-(D+\|\alpha\|)/m} e^{-b((|g|')^m |z|^{-1})^{1/(m-1)}}$$

for all $g \in G$ and $z \in \Lambda(\pi/2 - \varepsilon) = \{z \in \mathbb{C}\backslash\{0\} : |\arg z| < \pi/2 - \varepsilon\}$.

The outline of this paper is as follows. First we prove the theorem on L_2 and then we lift the result to a general representation by a transference argument. The L_2 proof is based on the examination of high powers $(\lambda I + H)^{-n}$ of the resolvent of H. Using Davies' method we obtain bounds on the kernels of these operators and then convert these to Gaussian bounds on the kernel of the semigroup by Cauchy integral techniques. The key point is that powers of the resolvent have improved regularity and boundedness properties. The technique of using powers of the resolvent to improve regularity was implicit in the iterative argument of Nirenberg [Nir] and was used explicitly by Nelson and Stinespring [NeS]. It also occurs in a slightly different context in Agmon's work, [Agm], pp. 250–260.

2. Positive Rockland operators on L_2. Let P be a positive Rockland form of order m and $H = dL(P)$ the associated positive Rockland operator on L_2. In this section we prepare for the proof of Theorem 1.1 by establishing analogous results for the left regular representation L on $L_2(G; dg)$.

PROPOSITION 2.1. *Let H be a positive Rockland operator which is homogeneous of degree m. Then*

I. *The operator H is self-adjoint.*

II. *For all $n \in \mathbb{N}$ one has $D(H^n) = L'_{2;nm}$, with equivalent norms. There exists a $c > 0$ such that*

$$cN'_{2;nm}(\varphi) \le \|H^n\varphi\|_2$$

for all $\varphi \in D(H^n)$.

III. *The spaces $L_{2;\infty}$ and $C_c^\infty(G)$ are cores for H^n, for all $n \in \mathbb{N}$.*

IV. *If $n \in \mathbb{N}$ and $k \in \langle 0, nm\rangle$ then there exists a $c > 0$ such that*

$$N'_{2;k}(\varphi) \le \varepsilon^{nm-k} N'_{2;nm}(\varphi) + c\varepsilon^{-k}\|\varphi\|_2$$

for all $\varepsilon > 0$ and $\varphi \in L'_{2;nm}$.

V. *If $n \in \mathbb{N}$ and $k \in \langle 0, nm\rangle$ then there exists a $c > 0$ such that*

$$\|\varphi\|'_{2;k} \le \varepsilon^{nm-k}\|\varphi\|'_{2;nm} + c\varepsilon^{-k}\|\varphi\|_2$$

for all $\varepsilon > 0$ and $\varphi \in L'_{2;nm}$.

Proof. The operator $H^n|_{C_c^\infty(G)}$ is essentially self-adjoint by the arguments of [NeS]. So

$$\overline{H^n} \subseteq (\overline{H^n})^* \subseteq \overline{H^n|_{C_c^\infty}}^* = \overline{H^n|_{C_c^\infty}} \subseteq \overline{H^n}$$

and $\overline{H^n|_{C_c^\infty}} = \overline{H^n}$. Therefore $\overline{H^n}$ is self-adjoint and $C_c^\infty(G)$ is a core for $\overline{H^n}$.

Next for all $\alpha \in J(d)$ and $n \in \mathbb{N}$ with $nm \ge \|\alpha\|$ there exists, by Helffer–Nourrigat [HeN1], Proposition 6.4, a $c > 0$ such that

$$\|A^\alpha\varphi\|_2 \le c(\|H^n\varphi\|_2 + \|\varphi\|_2)$$

for all $\varphi \in C_c^\infty(G)$. Since $C_c^\infty(G)$ is dense in $D(\overline{H^n})$ and $L'_{2;k}$ is complete it follows that $D(\overline{H^n}) \subseteq L'_{2;k}$ if $k \le nm$. But $L'_{2;nm} \subseteq D(H^n) \subseteq D(\overline{H^n})$, so $D(\overline{H^n}) = D(H^n) = L'_{2;nm}$, with equivalent norms. This proves Statements I, III and the first part of II.

It follows by scaling that there exists a $c > 0$ such that

$$\|A^\alpha\varphi\|_2 \le \varepsilon^{nm-\|\alpha\|}\|H^n\varphi\|_2 + c\varepsilon^{-\|\alpha\|}\|\varphi\|_2$$

for all $\varphi \in C_c^\infty(G)$ and $\varepsilon > 0$, if $\|\alpha\| < nm$. In particular, for all $k < nm$,

$$N'_{2;k}(\varphi) \le \varepsilon^{nm-k}\|H^n\varphi\|_2 + c\varepsilon^{-k}\|\varphi\|_2$$

and in addition

$$N'_{2;nm}(\varphi) \le c\|H^n\varphi\|_2$$

for all $\varphi \in C_c^\infty(G)$. This proves Statements IV, V and the last part of II by closure and density. ∎

Our aim is to bound the semigroup kernel associated with H. To this end we adopt the tactic of [AMT]. First, we obtain appropriate bounds on the kernel of a large power of the resolvent kernel where the power is large relative to the largest weight r. Secondly, we derive the semigroup estimates from the resolvent estimates by a Cauchy contour integration technique.

Note that with the choice of r one has the continuous embedding $L'_{2;kr} \subseteq L_{2;k}$ for each $k \in \mathbb{N}$.

Proposition 2.2. *If $n \in \mathbb{N}$, $k, \varepsilon > 0$ and $n \ge (dr + k)/m$ then $(\lambda I + H)^{-2n}$ has an integral kernel $R_\lambda^{(2n)} \in L'_{\infty;k}$ for all $\lambda \in \Lambda(\pi - \varepsilon)$. Moreover,*

$$|(A^\alpha R_\lambda^{(2n)})(g)| \le a|\lambda|^{-2n+(D+\|\alpha\|)/m} e^{-b|\lambda|^{1/m}|g|'}$$

for all $g \in G$ and $\alpha \in J(d)$ with $\|\alpha\| \le k$, where $a, b > 0$ are independent of λ.

Proof. It follows from the Helffer–Nourrigat estimates (Proposition 2.1.II) and the spectral theorem that there exists $c_1 > 0$ such that

$$(\|(\lambda I + H)^{-n}\|_{L_2 \to L'_{2;nm}})^{-1} \ge c_1^{-1}$$

uniformly for all $\lambda \in \Lambda(\pi - \varepsilon)$ with $|\lambda| = 1$.

Next, fix a real $\psi \in C_{b;\infty}(G)$. Let

$$n_1(\psi) = \sup\{|(A_i\psi)(g)| : g \in G, i \in \{1, \ldots, d\}\}$$

and for $j \in \mathbb{N}$ define recursively

$$n_{j+1}(\psi) = \sup_{i \in \{1,\ldots,d\}} n_j(A_i\psi) \vee n_j(\psi).$$

If $j \le t < j + 1$, with $j \in \mathbb{N}$, define $n_t(\psi) = n_j(\psi)$. Next for $\varrho \in \mathbb{R}$ define $U_\varrho : L_2 \to L_2$ by

$$(U_\varrho\varphi)(g) = e^{-\varrho\psi(g)}\varphi(g).$$

Then U_ϱ maps $L_{2;l}$ continuously into $L_{2;l}$ for all $l \in \mathbb{N}_0$. Let

$$H^{(\varrho)} = U_\varrho H U_\varrho^{-1}.$$

The starting point is the observation that

$$U_\varrho A_i U_\varrho^{-1} = A_i + \varrho M_{\psi_i}$$

for all $i \in \{1, \ldots, d\}$, where $\psi_i = A_i \psi$ and M_{ψ_i} is the multiplication operator with ψ_i. Therefore $U_\varrho A_i U_\varrho^{-1}$ is a perturbation of A_i.

The operator $(\lambda I + H^{(\varrho)})^n$ can be expressed in the form

$$(\lambda I + H^{(\varrho)})^n = (\lambda I + H)^n + \sum_{\substack{\beta \in J(d) \\ \|\beta\| \le nm-1}} M_\beta(\varrho, \lambda, \psi) A^\beta$$

where $M_\beta(\varrho, \lambda, \psi)$ are bounded multiplication operators satisfying

$$\|M_\beta(\varrho, \lambda, \psi)\| \le c_2 |\varrho| (1 + |\varrho|^{nm-1})(1 + n_{nm}(\psi))^{nm}$$

uniformly for all $\varrho \in \mathbb{R}$, $\lambda \in \Lambda(\pi - \varepsilon)$ with $|\lambda| = 1$, $\beta \in J(d)$ with $\|\beta\| \le nm - 1$ and real $\psi \in C_{\mathrm{b};\infty}(G)$. Therefore

$$\|(\lambda I + H^{(\varrho)})^n - (\lambda I + H)^n\|_{L'_{2;nm} \to L_2} \le nmd^{nm-1} c_2 |\varrho| (1 + |\varrho|^{nm-1})(1 + n_{nm}(\psi))^{nm}.$$

So if $|\varrho| \le 1$, $2^{nm+1} c_2 d^{nm-1} nm |\varrho| < (2c_1)^{-1}$, $\lambda \in \Lambda(\pi - \varepsilon)$ with $|\lambda| = 1$ and real $\psi \in C_{\mathrm{b};\infty}(G)$ with $n_{nm}(\psi) \le 1$ then $(\lambda I + H^{(\varrho)})^n$ is invertible as an operator from $L'_{2;nm}$ into L_2 and

$$\|(\lambda I + H^{(\varrho)})^{-n} \varphi\|'_{2;nm} \le 2c_1 \|\varphi\|_2$$

for all $\varphi \in L_2$. Hence there exist $c_3, c_4 > 0$ such that

$$\|U_\varrho A^\alpha U_\varrho^{-1} (\lambda I + H^{(\varrho)})^{-n} \varphi\|'_{2;nm-k} \le c_3 \|\varphi\|_2$$

for all $\varrho \in [-c_4, c_4]$, $\lambda \in \Lambda(\pi - \varepsilon)$ with $|\lambda| = 1$, $\alpha \in J(d)$ with $\|\alpha\| \le k$, real $\psi \in C_{\mathrm{b};\infty}(G)$ with $n_{nm}(\psi) \le 1$ and $\varphi \in L_2$. Thus the operator $U_\varrho A^\alpha (\lambda I + H)^{-n} U_\varrho^{-1}$ maps L_2 continuously into $L'_{2;nm-k}$. But $L'_{2;nm-k}$ is continuously embedded in $L_{2;d}$ by the condition on n. Moreover, by the Sobolev embedding theorem for Lie groups ([Rob], B2.2), the space $L_{2;d}$ is continuously embedded in C_0. Therefore there exists $R^{(n,\varrho)}_{\alpha,\lambda} \in L_2$ such that

$$(U_\varrho A^\alpha U_\varrho^{-1} (\lambda I + H^{(\varrho)})^{-n} \check{\varphi})(e) = \int_G dh\, R^{(n,\varrho)}_{\alpha,\lambda}(h) \varphi(h)$$

for all $\varphi \in L_2$, where $\check{\varphi}(g) = \varphi(g^{-1})$. Moreover, $\|R^{(n,\varrho)}_{\alpha,\lambda}\|_2 \le c_5$ for some constant $c_5 > 0$ uniformly for all $|\varrho| \le c_4$, $\lambda \in \Lambda(\pi - \varepsilon)$ with $|\lambda| = 1$, $\alpha \in J(d)$ with $\|\alpha\| \le k$ and real $\psi \in C_{\mathrm{b};\infty}(G)$ with $n_{nm}(\psi) \le 1$. Then, by right invariance,

$$(A^\alpha (\lambda I + H)^{-n} \varphi)(g) = \int_G dh\, R^{(n,0)}_{\alpha,\lambda}(h) \varphi(h^{-1} g)$$

for all $\varphi \in L_2$ and $g \in G$. Let $R^{(n)}_{\alpha,\lambda} = R^{(n,0)}_{\alpha,\lambda}$ and define $R^{(n)}_\lambda = R^{(n,0)}_{\beta,\lambda}$ if $\|\beta\| = 0$.

Since $(\overline{\lambda}I + H^{(-\varrho)})^{-n}$ is bounded from L_2 into L_∞ it follows by duality that $(\lambda I + H^{(\varrho)})^{-n}$ is bounded from L_1 into L_2. Therefore, by composition,

$$R^{(2n,\varrho)}_{\alpha,\lambda} = R^{(n,\varrho)}_{\alpha,\lambda} * R^{(n,\varrho)}_{\beta,\lambda} \in C_0(G),$$

where β is again the multi-index with length zero. Using the fact that $H^{(\varrho)} = U_\varrho H U_\varrho^{-1}$, it then follows that

$$R^{(2n,\varrho)}_{\alpha,\lambda}(g) = R^{(2n)}_{\alpha,\lambda}(g) e^{\varrho(\psi(g^{-1})-\psi(e))}$$

for all $g \in G$, $|\varrho| \le c_4$, $\lambda \in \Lambda(\pi-\varepsilon)$ with $|\lambda| = 1$, $\alpha \in J(d)$ with $\|\alpha\| \le k$ and real $\psi \in C_{\mathrm{b};\infty}(G)$ with $n_{nm}(\psi) \le 1$. By [Rob], Chapter III, Equation (4.31), there exists a $c_6 > 0$ such that $|\psi(g^{-1}) - \psi(e)| \ge c_6|g|$ uniformly for all $g \in G$ and real $\psi \in C_{\mathrm{b};\infty}(G)$ with $n_{nm}(\psi) \le 1$. So

$$|R^{(2n)}_{\alpha,\lambda}(g)| \le \|R^{(2n,\varrho)}_{\alpha,\lambda}\|_\infty e^{-c_4 c_6 |g|} \le c_5^2 e^{-c_4 c_6 |g|}$$

uniformly for all $g \in G$. By [VSC], Proposition III.4.2, there exists a $c_7 > 0$ such that $|g| \ge c_7 |g|'$ for all $g \in G$ with $|g|' \ge 1$. So there exist $a, b > 0$ such that

$$|R^{(2n)}_{\alpha,\lambda}(g)| \le a e^{-b|g|'}$$

uniformly for all $g \in G$ and $\lambda \in \Lambda(\pi - \varepsilon)$ with $|\lambda| = 1$.

Next we use a scaling argument to remove the restriction on λ. Let $\lambda \in \Lambda(\pi-\varepsilon)$. Since $H(\varphi \circ \gamma_u) = u^m (H\varphi) \circ \gamma_u$ and $(A^\alpha \varphi)\gamma_u^{-1} = u^{\|\alpha\|} A^\alpha(\varphi \circ \gamma_u^{-1})$ for all $u > 0$, one readily deduces that

$$(A^\alpha(\lambda I + H)^{-2n}\varphi) \circ \gamma_u^{-1} = u^{-2nm+\|\alpha\|} A^\alpha(\lambda u^{-m} I + H)^{-2n}(\varphi \circ \gamma_u^{-1}).$$

Choose $u = |\lambda|^{1/m}$. Then

$$(A^\alpha(\lambda I + H)^{-2n}\varphi(g) = |\lambda|^{-2n+(D+\|\alpha\|)/m} \int_G dh\, R^{(2n)}_{\alpha,\lambda}(\gamma_u(h))\varphi(h^{-1}g).$$

So $A^\alpha(\lambda I + H)^{-2n}$ has a kernel $R^{(2n)}_{\alpha,\lambda}$ and $R^{(2n)}_{\alpha,\lambda}(h) = |\lambda|^{-2n+(D+\|\alpha\|)/m} \times R^{(2n)}_{\alpha,\lambda}(\gamma_u(h))$ for all $h \in G$. Then

$$|R^{(2n)}_{\alpha,\lambda}(h)| \le a|\lambda|^{-2n+(D+\|\alpha\|)/m} e^{-b|\lambda|^{1/m}|h|'}$$

for all $h \in G$.

Finally, we prove that $R^{(2n)}_\lambda$ is k times differentiable and that $A^\alpha R^{(2n)}_\lambda = R^{(2n)}_{\alpha,\lambda}$ for all $\alpha \in J(d)$ with $\|\alpha\| \le k$. It suffices to consider the case $|\lambda| = 1$. Then

$$R^{(2n)}_{\alpha,\lambda} = R^{(n)}_{\alpha,\lambda} * R^{(n)}_\lambda = A^\alpha(\lambda I + H)^{-n} R^{(n)}_\lambda = A^\alpha((\lambda I + H)^{-n} R^{(n)}_\lambda) = A^\alpha R^{(2n)}_\lambda.$$

This completes the proof of the proposition. ∎

COROLLARY 2.3. *The holomorphic self-adjoint semigroup S generated by H has a kernel $K \in L_{1;\infty}(G;dg) \cap C_{0;\infty}(G)$ such that*

$$(A^\alpha S_z \varphi)(g) = \int_G dh\, (A^\alpha K_z)(h)\varphi(h^{-1}g)$$

for all $\alpha \in J(d)$, $z \in \Lambda(\pi/2)$, $\varphi \in L_2$ and $g \in G$. Moreover, the function $z \mapsto K_z(g)$ is analytic on $\Lambda(\pi/2)$, uniformly for $g \in G$, and for each $\alpha \in J(d)$ and $\varepsilon \in \langle 0, \pi/2\rangle$ there exist $a, b > 0$ such that

$$|(A^\alpha K_z)(g)| \le a|z|^{-(D+\|\alpha\|)/m} e^{-b((|g|')^m |z|^{-1})^{1/(m-1)}}$$

for all $z \in \Lambda(\pi/2 - \varepsilon)$ and all $g \in G$.

Proof. The semigroup S can be constructed from the resolvent $(\lambda I + H)^{-1}$ using the standard Cauchy integral representation of the exponential. Alternatively, one can use the Cauchy representation for the derivatives of the exponential. Let $n \in \mathbb{N}$ and $k > 0$ and suppose that $n \ge (dr + k)/m$. Assume that $z \in \Lambda(\pi/2 - \varepsilon)$. Then

$$S_z = (2\pi i)^{-1}(2n-1)!z^{-(2n-1)} \int_{\Gamma_R} d\lambda\, e^{\lambda z}(\lambda I + H)^{-2n}$$

where Γ_R is the contour in the complex plane formed by connecting the two line segments $L_{R,\pm} = \{\lambda \in \mathbb{C} : \arg\lambda = \pm(\pi - \varepsilon/2), |\lambda| \ge R|z|^{-1}\}$ and the arc $A_R = \{\lambda \in \mathbb{C} : \arg\lambda \in [-\pi + \varepsilon/2, \pi - \varepsilon/2], |\lambda| = R|z|^{-1}\}$. We will deduce below, from the bounds of Proposition 2.2, that the semigroup kernel K exists,

$$K_z = (2\pi i)^{-1}(2n-1)!z^{-(2n-1)} \int_{\Gamma_R} d\lambda\, e^{\lambda z} R_\lambda^{(2n)}$$

and $z \mapsto K_z(g)$ is analytic for all $g \in G$.

First, assume $R \ge 1$. If $\lambda \in L_{R,\pm}$ then $\mathrm{Re}(\lambda z) < -|\lambda||z|\delta$, where $\delta = \sin(\varepsilon/2)$. Therefore the foregoing integral exists pointwise and

$$\begin{aligned}|K_z(g)| &\le a(2\pi)^{-1}(2n-1)!|z|^{-(2n-1)} \int_{\Gamma_R} d\lambda\, e^{-|\lambda||z|\delta}|\lambda|^{-(2n-D/m)} \\ &\le a'|z|^{-D/m}\end{aligned}$$

for some $a' > 0$ and for all $g \in G$. Moreover, a similar estimate bounds the derivative of $z \mapsto K_z(g)$, uniformly for $g \in G$, and this establishes uniform analyticity of the function.

Secondly, $|K_z(g)|$ can be estimated more accurately as follows. Assume $|z| = 1$. The integral over the arc A_R gives a contribution to $K_z(g)$ which is bounded by

$$B_z^{(A)}(g) = a(2\pi)^{-1}(2n-1)! \int_{-\pi+\varepsilon/2}^{\pi-\varepsilon/2} d\theta\, R^{-(2n-1-D/m)} e^{R} e^{-bR^{1/m}|g|'}$$
$$\le a' e^{R-bR^{1/m}|g|'},$$

where we have used $2n-1 \ge D/m$ and $R \ge 1$. But the exponential is minimized by choosing $R = (b|g|'m^{-1})^{m/(m-1)}$ and hence if $b|g|' \ge m$ then

$$B_z^{(A)}(g) \le a'' e^{-b'(|g|')^{m/(m-1)}}.$$

Thirdly, the line segments $L_{R,\pm}$ in Γ_R both give contributions which can be bounded by

$$B_z^{(L)}(g) = a(2\pi)^{-1}(2n-1)! \int_R^{\infty} d\mu\, \mu^{-(2n-D/m)} e^{-\mu\delta - b\mu^{1/m}|g|'}.$$

Since $R = (b|g|'m^{-1})^{m/(m-1)} \ge 1$ one again has bounds of the form

$$B_z^{(L)}(g) \le a'' e^{-b'(|g|')^{m/(m-1)}}.$$

Therefore one concludes that

$$|K_z(g)| \le a e^{-b(|g|')^{m/(m-1)}}$$

for some $a, b > 0$, uniformly for all $z \in \Lambda(\pi/2-\varepsilon)$ with $|z| = 1$ and all $g \in G$ with $|g|'$ sufficiently large. But we have already established that K_z is uniformly bounded if $|z| = 1$. Therefore the foregoing bounds extend to all $g \in G$, and $z \in \Lambda(\pi/2-\varepsilon)$ with $|z| = 1$, by increasing the value of a.

Finally, since H is homogeneous one obtains the bounds

$$|K_z(g)| \le a|z|^{-D/m} e^{-b((|g|')^m |z|^{-1})^{1/(m-1)}}$$

for all $z \in \Lambda(\pi/2-\varepsilon)$ and all $g \in G$ by scaling.

It now follows straightforwardly that K is indeed the kernel of S.

If $\alpha \in J(d)$ with $\|\alpha\| \le k$ then one can define similarly the function $K_{\alpha,z} : G \to \mathbb{C}$ by

$$K_{\alpha,z}(g) = (2\pi i)^{-1}(2n-1)! z^{-(2n-1)} \int_{\Gamma_R} d\lambda\, e^{\lambda z} (A^\alpha R_\lambda^{(2n)})(g)$$

It follows as above that

$$|K_{\alpha,z}(g)| \le a|z|^{-(D+\|\alpha\|)/m} e^{-b((|g|')^m |z|^{-1})^{1/(m-1)}}$$

for all $z \in \Lambda(\pi/2-\varepsilon)$ and all $g \in G$. It then follows that K_z is pointwise differentiable and that $A^\alpha K_z = K_{\alpha,z}$ for all α with $\|\alpha\| \le k$. Hence K_z is infinitely often differentiable in both the L_1 and the L_∞ sense. ■

R e m a r k. The kernel K is usually not positive. It follows from [Rob], Chapter III, Section 5, that K is positive if and only if H is a second-order operator, in the unweighted sense, with real coefficients and with the principal coefficients satisfying an ellipticity condition.

3. Miscellany. In this section we extend the foregoing results to general positive Rockland operators as defined in the introduction and to general continuous representations. In particular, we complete the proof of Theorem 1.1. Subsequently, we examine properties of Rockland operators which are special to unitary representations or to translations on the L_p-spaces.

3.1. *Lower order terms*. In the previous section we derived Gaussian bounds for the kernel of the semigroup generated by H on L_2 under the assumption that H is homogeneous. Next we use perturbation theory to remove this homogeneity hypothesis.

PROPOSITION 3.1. *Let $H = dL(C)$ where C is a form of order m whose principal part is a positive Rockland form. Then*

I. *For all $n \in \mathbb{N}$ one has $D(H^n) = L'_{2;nm}$, with equivalent norms. Moreover, there exists a $c > 0$ such that*

$$N'_{2;nm}(\varphi) \le c(\|H^n\varphi\|_2 + \|\varphi\|_2)$$

for all $\varphi \in D(H^n)$.

II. *The spaces $L_{2;\infty}$ and $C_c^\infty(G)$ are cores for H^n, for all $n \in \mathbb{N}$.*

P r o o f. Let $H_0 = dL(P)$ denote the principal part of H.

Suppose $n = 1$. Then $D(H_0) = L'_{2;m}$ by Proposition 2.1. Moreover, since $H_1 = H - H_0$ is of degree at most $m - 1$ one has

$$\|H_1\varphi\|_2 \le \varepsilon\|H_0\varphi\|_2 + c\varepsilon^{-m+1}\|\varphi\|_2$$

for some $c > 0$ and all $\varepsilon \in \langle 0, 1]$ by Statements I and II of Proposition 2.1 applied to H_0. Thus H_1 is a relatively bounded perturbation of H_0 with relative bound zero. Hence $D(H) = D(H_0) = L'_{2;m}$ with equivalent norms. The rest of the proof then follows the proof of Proposition 2.1.

Next suppose $n > 1$. Then the principal part H_0^n of the operator H^n is a positive Rockland operator. Therefore we can repeat the foregoing argument with H_0 replaced by H_0^n, H_1 replaced by $H^n - H_0^n$ and the degree m replaced by nm. ∎

The semigroup results now follow by use of perturbation theory.

THEOREM 3.2. *Let $H = dL(C)$ where C is a form of order m whose principal part is a positive Rockland form. Then H generates a continuous semigroup S with a kernel $K \in L_{1;\infty}(G; dg) \cap C_{0;\infty}(G)$. Moreover, for each*

$\alpha \in J(d)$ and $\varepsilon \in \langle 0, \pi/2\rangle$ there exist $a, b > 0$ and an $\omega \geq 0$ such that

$$|(A^\alpha K_z)(g)| \leq a|z|^{-(D+\|\alpha\|)/m} e^{\omega|z|} e^{-b((|g|')^m|z|^{-1})^{1/(m-1)}}$$

for all $z \in \Lambda(\pi/2-\varepsilon)$ and all $g \in G$.

If $K^\dagger$ is the kernel of the semigroup generated by the dual operator $H^\dagger$ then $K_t^\dagger(g) = \overline{K_t(g^{-1})}$ for all $g \in G$.

Proof. Again we use H_0 to denote the principal part of H, and H_1 for the lower order terms. Since H_0 is positive self-adjoint and H_1 is relatively bounded by H_0 with relative bound zero it follows from standard perturbation theory that H generates a continuous semigroup S on L_2. Moreover, it follows from the perturbation theory of holomorphic semigroups that S is holomorphic in the open right half-plane.

Next, we exploit the perturbation arguments developed in the appendix of [BrR2] and subsequently used in [ElR1], [ElR2] and [ElR4] to construct the kernel K of S from the kernel $K^{(0)}$ of the semigroup $S^{(0)}$ generated by the principal part H_0 of H. Since the proof is very similar to the previous applications we only sketch the outline of the argument.

Define $K_t^{(n)}$ by the recursion relation $K_t^{(n)} = -(K^{(n-1)} \,\widehat{*}\, H_1 K^{(0)})_t$ where the convolution product $\widehat{*}$ is defined on $\mathbb{R} \times G$ by

$$(\varphi \,\widehat{*}\, \psi)_t(g) = \int_{\mathbb{R}} ds \int_G dh\, \varphi_s(h)\psi_{t-s}(h^{-1}g) = \int_{\mathbb{R}} ds \int_G dh\, \varphi_{t-s}(h)\psi_s(h^{-1}g).$$

Then one proves that the perturbation series

$$K_t = \sum_{n\geq 0} K_t^{(n)}$$

is L_p-convergent, for all $p \in [1, \infty]$ and all $t > 0$, and identifies K_t as the kernel of S. In fact the series is convergent on exponentially weighted L_p-spaces and this allows one to extend the Gaussian bounds from $K^{(0)}$ to K.

Let $\varrho \geq 0$ and define L_1^ϱ to be the L_1-space with respect to the measure $dg\, e^{\varrho|g|'}$ and denote the norm by

$$\|\varphi\|_1^\varrho = \int_G dg\, e^{\varrho|g|'}|\varphi(g)|.$$

Similarly, let L_∞^ϱ be the space of measurable functions φ for which $g \mapsto e^{\varrho|g|'}|\varphi(g)|$ is essentially bounded with norm

$$\|\varphi\|_\infty^\varrho = \operatorname*{ess\,sup}_{g\in G} e^{\varrho|g|'}|\varphi(g)|.$$

Since $|\cdot|'$ satisfies the triangle inequality it follows from the recursion relation for $K^{(n)}$ that one has the coupled integral inequalities

$$(1)\quad \|K_t^{(n)}\|_\infty^\varrho \le \int_0^t ds\,(\|K_{t-s}^{(n-1)}\|_\infty^\varrho \|H_1 K_s^{(0)}\|_1^\varrho) \wedge (\|K_{t-s}^{(n-1)}\|_1^\varrho \|H_1 K_s^{(0)}\|_\infty^\varrho)$$

and

$$(2)\qquad \|K_t^{(n)}\|_1^\varrho \le \int_0^t ds\,\|K_{t-s}^{(n-1)}\|_1^\varrho \|H_1 K_s^{(0)}\|_1^\varrho.$$

But it follows from the Gaussian bounds of Corollary 2.3 applied to $K^{(0)}$ that with a suitable choice of $a > 0$ and $\omega \ge 0$,

$$\|K_t^{(0)}\|_1^\varrho \le ae^{\omega(1+\varrho^m)t}, \qquad \|H_1 K_t^{(0)}\|_1^\varrho \le at^{-(m-1)/m} e^{\omega(1+\varrho^m)t},$$
$$\|K_t^{(0)}\|_\infty^\varrho \le at^{-D/m} e^{\omega(1+\varrho^m)t}, \qquad \|H_1 K_t^{(0)}\|_\infty^\varrho \le at^{-(D+m-1)/m} e^{\omega(1+\varrho^m)t}$$

for all $\varrho \ge 0$. Now these estimates allow one to solve the integral inequalities (1) and (2). One obtains bounds

$$\|K_t^{(n)}\|_1^\varrho \le a(b^n t^n/n!)^{1/m} e^{\omega(1+\varrho^m)t},$$
$$\|K_t^{(n)}\|_\infty^\varrho \le at^{-D/m}(b^n t^n/n!)^{1/m} e^{\omega(1+\varrho^m)t}.$$

Then estimating the sum K of the series $K^{(n)}$ with the Hölder inequality yields

$$\|K_t\|_\infty^\varrho \le a' t^{-D/m} e^{\omega(1+\varrho^m)t}.$$

But the latter bound gives

$$e^{\varrho|g|'}|K_t(g)| \le a' t^{-D/m} e^{\omega(1+\varrho^m)t}$$

and minimizing over ϱ one finds

$$|K_t(g)| \le at^{-D/m} e^{\omega t} e^{-b((|g|')^m t^{-1})^{1/(m-1)}}$$

for some $a, b > 0$ and $\omega \ge 0$.

The function K defined by the perturbation expansion is now a function of the Gaussian type and it follows from the standard arguments of "time-dependent" perturbation theory that it is indeed equal to the semigroup kernel of the perturbed semigroup S. But applying similar arguments to the semigroup $S_t^\theta = S_{e^{i\theta}t}$ with $\theta \in \Lambda(\pi/2)$ one establishes that K_z is defined in the open right half-plane and satisfies the desired Gaussian bounds in $\Lambda(\pi/2 - \varepsilon)$. It remains to bound the left derivatives of K.

The estimation of the derivatives $A^\alpha K_t$ is almost identical to the estimation of K_t if $\alpha \in J(d)$ and $\|\alpha\| < m$. The starting point is now the recursion relation

$$(A^\alpha K_t^{(n)})(h) = -\int_0^t ds \int_G dk\,(A^\alpha K_{t-s}^{(n-1)})(k)(H_1 K_s^{(0)})(k^{-1}h),$$

which gives a set of coupled integral inequalities on the weighted spaces which can then be solved with the aid of the above bounds together with the estimates

$$\|A^\alpha K_t^{(0)}\|_1^\varrho \le at^{-\|\alpha\|/m} e^{\omega(1+\varrho^m)t}, \quad \|A^\alpha K_t^{(0)}\|_\infty^\varrho \le at^{-(D+\|\alpha\|)/m} e^{\omega(1+\varrho^m)t},$$

which again follow from the bounds of Corollary 2.3 applied to $K^{(0)}$. Now the derivatives only introduce an extra factor $(t-s)^{-\|\alpha\|/m}$ in the bound on $K_{t-s}^{(n-1)}$ in the integral inequalities and since this factor is integrable at $s = t$ this presents no problem. It merely introduces an additional factor $t^{-\|\alpha\|/m}$ in the overall estimate. If, however, $\|\alpha\| \ge m$ a new problem arises because the additional factor $(t-s)^{-\|\alpha\|/m}$ is not integrable and the recursion relation is ill-defined. But this can be dealt with by the methods of [ElR1].

The $n = 0$ estimates are valid for all α and by splitting the convolution integral into two parts one obtains an alternative recursion relation which is well defined and can be used to estimate the higher derivatives. This second relation has the form

$$\begin{aligned}(A^\alpha K_t^{(n)})(h) = &- \int_0^{t/2} ds \int_G dk\, (A^\alpha K_{t-s}^{(n-1)})(k)(H_1 K_s^{(0)})(k^{-1}h) \\ &- \int_{t/2}^{t} ds \int_G dk\, K_{t-s}^{(n-1)}(k)(A^\alpha L(k) K_s^{(0)})(h).\end{aligned}$$

In the first integral the extra singularity $(t-s)^{-\|\alpha\|/m}$ introduced by the derivatives plays no role because $s \le t/2$. In the second integral the term $A^\alpha L(k) K_s^{(0)}$ will have a nonintegrable singularity at $s = 0$ but this will not cause any difficulty because it is excluded from the region of integration. Nevertheless, it is important for the form of the final bounds that the singularity of this second term is of the correct order. But this follows because $L(k^{-1})A^\alpha L(k)$ is close to A^α in a suitable sense. The appropriate comparison is given by Lemma 4.3 of [ElR4] which is a weighted version of Lemma 4.3 of [ElR1]. If $w \in \langle 0, \infty\rangle$ is such that $w/w_i \in \mathbb{N}$ for all i one has to use the equivalent modulus $|\cdot|''$ defined by

$$\Big|\exp\Big(\sum_{i=1}^d \xi_i a_i\Big)\Big|'' = \Big(\sum_{i=1}^d |\xi_i|^{2w/w_i}\Big)^{1/(2w)}$$

in the proof of Lemma 4.3 in [ElR4]. The detailed estimation of the derivatives is then given by repetition of the arguments of [ElR1]. We omit further details.

The proof of the last statement of the theorem, concerning the dual kernel, is straightforward. ∎

3.2. *General representations.* In the previous subsection we proved that if the principal part P of the form C is a positive Rockland form then the operator $dL(C)$ associated with the left regular representation L of G in L_2 generates a holomorphic semigroup with a kernel which, together with all its derivatives, satisfies Gaussian bounds. Now let $(\mathcal{X}, G, U)$ be a strongly, or weak*, continuous representation of G. Let $H = dU(C)$ and let K be the kernel of the semigroup generated by $dL(C)$. We shall show that $\overline{H}$ generates a continuous semigroup S which is holomorphic in the open right half-plane and which has K as kernel. The idea of the proof is to use the kernel K to define a semigroup on $\mathcal{X}$ and then show that the generator is $\overline{H}$.

First we show that $(K_t)_{t>0}$ is a bounded approximation of the identity.

Lemma 3.3.

$$\lim_{t\to 0} \int_G dg\, K_t(g) = 1.$$

Proof. Let $\varphi \in C_c^\infty(G)$, $\varphi \ge 0$, $\int dg\, \varphi(g) = 1$. Let $s > 0$ be such that $\operatorname{supp} \varphi \subseteq B_s$. Further, let $\psi \in C_c^\infty(G)$, $\psi \ge 0$, be such that $\psi(g) = 1$ for all $g \in B_{s+1}$ and set $\tau = \varphi * \psi$. Then $\tau \in C_c^\infty(G)$ and $\tau(g) = 1$ for all $g \in B_1$. Consider the decomposition

$$\int_G dg\, K_t(g) = \int_G dg\, K_t(g)\tau(g) + \int_G dg\, K_t(g)(1 - \tau(g)).$$

It follows from the Gaussian bounds of Theorem 3.2 that $\lim_{t\to 0} \int dg\, K_t(g) \times (1-\tau(g)) = 0$, and hence it remains to prove that $\lim_{t\to 0} \int dg\, K_t(g)\tau(g) = 1$. Now

$$\begin{aligned}\int_G dg\, K_t(g)\tau(g) &= \int_G dg \int_G dh\, K_t(g)\varphi(h)\psi(h^{-1}g)\\ &= \int_G dg \int_G dh\, K_t(g)\varphi(h)\check{\psi}(g^{-1}h)\\ &= \int_G dh\, (T_t\check{\psi})(h)\varphi(h),\end{aligned}$$

where $\check{\psi}(g) = \psi(g^{-1})$ and T is the semigroup generated by $dL(C)$. Because $\lim_{t\to 0} T_t\check{\psi} = \check{\psi}$ in L_2 we obtain

$$\lim_{t\to 0} \int_G dg\, K_t(g)\tau(g) = \int_G dh \check{\psi}(h)\varphi(h) = \tau(e) = 1$$

and the lemma is proved. ∎

We now prove a result which, together with Corollary 2.3, completes the proof of Theorem 1.1. Moreover, in combination with Theorem 3.2, it extends Theorem 1.1 to nonhomogeneous operators.

THEOREM 3.4. *Let $(\mathcal{X}, G, U)$ be a continuous representation of G and C a form of order m whose principal part is a positive Rockland form. Let $H = dU(C)$ and $H^\dagger = dU_*(C)$ be the dual operator in the dual representation $(\mathcal{F}, G, U_*)$. Then $\overline{H}$ generates a semigroup S which is holomorphic in the right half-plane. Moreover, $\overline{H} = H^{\dagger *}$ and the kernel of the semigroup generated by $dL(C)$ on L_2 is the kernel of S.*

Proof. Let K be the kernel of the semigroup generated by $H_0 = dL(C)$ on L_2. For $t > 0$ define the operator S_t by $S_t x = \int_G dg\, K_t(g) U(g) x$. Then it follows from Lemma 3.3 that S is a continuous semigroup. But the same argument can be applied to the kernels $t \mapsto K_{e^{i\theta} t}$ with $\theta \in \langle -\pi/2, \pi/2\rangle$. Since all constants involved are locally uniform in θ it follows from Kato [Kat], Theorem XI.1.23, that S is a holomorphic semigroup with holomorphy sector the open right half-plane. Let $\widetilde{H}$ be its generator and for $\lambda \in \mathbb{R}$ large enough let $R_\lambda = (\lambda I + \widetilde{H})^{-1}$ be the resolvent. Then R_λ is a continuous operator, so if one replaces $\mathcal{X}$ by $\mathcal{X}_n$ it follows that R_λ maps $\mathcal{X}_n$ into $\mathcal{X}_n$ if $(\mathcal{X}, G, U)$ is a strongly continuous representation. Hence R_λ maps $\mathcal{X}_\infty$ into $\mathcal{X}_\infty$ for any representation.

In order to identify the generator of S, we first assume that U is the left regular representation in $L_1^\varrho = L_1(G; e^{\varrho|g|}dg)$, where $\varrho \ge 0$ is fixed.

Note that $R_\lambda \varphi = r_\lambda * \varphi$, where r_λ is the resolvent kernel associated with the kernel K. Since $R_\lambda \varphi = (\lambda I + H_0)^{-1}\varphi$ for all $\varphi \in C_c^\infty(G)$ and $C_c^\infty(G) \subseteq L^\varrho_{1;\infty}$ one deduces that

$$(\psi, (\lambda I + H) R_\lambda \varphi) = ((\lambda I + H^\dagger)\psi, R_\lambda \varphi) = (\psi, \varphi)$$

for all $\psi, \varphi \in C_c^\infty(G)$. So $(\lambda I + \overline{H}) R_\lambda \varphi = \varphi$ for all $\varphi \in C_c^\infty(G)$. Since R_λ is a continuous operator and $C_c^\infty(G)$ is dense in L_1^ϱ it follows that $\widetilde{H} \subseteq \overline{H}$. Next, using the dual kernel $K^\dagger$ one can define similarly a semigroup $S^\dagger$ on $\mathcal{F}$ with generator $\widetilde{H^\dagger}$. Then it follows from the above that $\widetilde{H^\dagger} \subseteq \overline{H^\dagger}$. But $S^\dagger = S^*$, so $\widetilde{H^\dagger}^{\,*} = \widetilde{H}$. Hence

$$\overline{H} \subseteq H^{\dagger *} = \overline{H^\dagger}^{\,*} \subseteq \widetilde{H^\dagger}^{\,*} = \widetilde{H} \subseteq \overline{H}$$

So $\overline{H} = H^{\dagger *} = \widetilde{H}$ is the generator of S.

Finally, we deduce that the generator $\widetilde{H}$ of the semigroup S is the closure $\overline{H}$ of $dU(C)$ for a general representation U. Let $\varrho > 0$ be so large that $\|U(g)\| \le M e^{\varrho|g|}$ uniformly for all $g \in G$, for some $M > 0$. Let H^L and S^L be the operator and semigroup corresponding to the left regular representation in L_1^ϱ. Let $\varphi \in C_c^\infty(G)$ and $x \in \mathcal{X}$. Then for all $t > 0$ one has

$$S_t U(\varphi) x = U(K_t * \varphi) x = U(S_t^L \varphi) x.$$

Hence by the Duhamel formula

$$S_tU(\varphi)x - U(\varphi)x = U(S_t^L\varphi - \varphi)x$$
$$= -U\Big(\int_0^t ds\, H^L S_s^L\varphi\Big)x = -\int_0^t ds\,(U(S_s^L H^L\varphi)x).$$

Therefore

$$\|t^{-1}(S_tU(\varphi)x - U(\varphi)x) - U(H^L\varphi)x\| = \Big\|t^{-1}\int_0^t ds\, U(S_s^L H^L\varphi - H^L\varphi)x\Big\|$$
$$\le \sup_{0<s\le t}\|S_s^L H^L\varphi - H^L\varphi\|_1^\varrho\|x\|.$$

Since S^L is a continuous semigroup, it follows that $U(\varphi)x$ is in the domain of the generator $\widetilde{H}$ of S and

$$\widetilde{H}U(\varphi)x = U(H^L\varphi)x = HU(\varphi)x.$$

Let $\mathcal{D} = \operatorname{span}\{U(\varphi)x : \varphi \in C_c^\infty(G), x \in \mathcal{X}\}$ be the Gårding space. Then $\widetilde{H} \supseteq H|_{\mathcal{D}}$. But $\mathcal{D}$ is dense in $\mathcal{X}'_m$. So $\widetilde{H} \supseteq H$ and hence $\widetilde{H} \supseteq \overline{H}$. Since $R_\lambda x \in \mathcal{X}_\infty$ for all $x \in \mathcal{X}_\infty$ it follows that $(\lambda I + H)R_\lambda x = x$ for all $x \in \mathcal{X}_\infty$. One can then deduce as above for the left regular representation in L_1^ϱ by using duality that $\widetilde{H} = \overline{H} = H^{\dagger *}$. This proves the theorem. ■

COROLLARY 3.5. *The kernel $z \mapsto K_z(g)$ is analytic in the open right half-plane uniformly for all $g \in G$.*

Proof. If S is the holomorphic semigroup generated by $dU(C)$ on L_∞, where U is the left regular representation, then $S_{z_1}K_{z_2} = K_{z_1+z_2}$ for all z_1, z_2 in the right half-plane. Hence the map $z \mapsto K_z$ from $\Lambda(\pi/2)$ into L_∞ is holomorphic. Fix $z_0 \in \Lambda(\pi/2)$. Then there exists $K'_{z_0} \in L_\infty$ such that

$$\lim_{z\to z_0}\Big\|\frac{K_z - K_{z_0}}{z - z_0} - K'_{z_0}\Big\|_\infty = 0.$$

But $g \mapsto (K_z - K_{z_0})(g)/(z - z_0)$ is a continuous function, so

$$\lim_{z\to z_0}\frac{K_z(g) - K_{z_0}(g)}{z - z_0} = K'_{z_0}(g)$$

uniformly for all $g \in G$. Therefore $z \mapsto K_z(g)$ is analytic. ■

3.3. *Regularity.* Finally, we consider unitary representations and prove optimal regularity results.

THEOREM 3.6. *Let C be a form of order m whose principal part P is a positive Rockland form. Suppose $(\mathcal{X}, G, U)$ is a unitary representation and let $H = dU(C)$. Then*

I. *The operator H is closed.*
II. *For all $n \in \mathbb{N}$ one has $D(H^n) = \mathcal{X}'_{nm}$ with equivalent norms.*

P r o o f. We only have to prove Statement I since the principal part of C^n is a positive Rockland form. The equivalence of the norms follows from the closed graph theorem.

Consider the form C_1 defined so that

$$dV(C_1) = \sum_{\substack{\alpha\in J(d)\\ \|\alpha\|\le m}} (-1)^{|\alpha|} A^{\langle\alpha_*,\alpha\rangle}$$

in any representation $(\mathcal{Y}, G, V)$, where $\langle\alpha_*,\alpha\rangle$ denotes the multi-index formed by composition of α and the index α_* obtained by reversing the order of α. Then C_1 is a form of order $2m$ whose principal part P_1 is a positive Rockland form. Indeed, if V is a nontrivial irreducible unitary representation, $x \in \mathcal{Y}_\infty(V)$ and $dV(P_1)x = 0$, then $(x, dV(P_1)x) = 0$, so $A^\alpha x = 0$ in the representation V for all $\alpha \in J(d)$ with $\|\alpha\| = m$. Hence $dV(P)x = 0$ and $x = 0$ since P is a Rockland form.

By Proposition 2.1 there exists $\mu_1 > 0$ such that $\|dL(P)\varphi\|_2 \ge \mu_1 N'_{2;m}(\varphi)$ for all $\varphi \in L'_{2;m}$. Moreover, by Theorem 2.1 in Chapter I of [HeN2], and the remark immediately after it, there exists $\mu_2 > 0$ such that $\|dV(P)x\| \ge \mu_2 N'_m(x)$ uniformly for all unitary irreducible representations $(\mathcal{Y}, G, V)$ and $x \in \mathcal{Y}'_m(V)$. Let $\mu = 2^{-1}\min(\mu_1,\mu_2)$. Then

$$(x, dV(P^2 - \mu P_1)x) \ge \mu(x, dV(P_1)x)$$

if V is an irreducible unitary representation or if V is the left regular representation on L_2 and x is a C^∞-vector. Therefore $P^2 - \mu P_1$ is a positive Rockland form. So $C^\dagger C - \mu C_1$ is a form of order $2m$ whose principal part is a positive Rockland form.

Then the closure of $dU(C^\dagger C - \mu C_1)$ is a generator of a semigroup, so it is self-adjoint and by spectral theory it is bounded below. Let $-\varrho$ be a lower bound. Then

$$\|dU(C)x\|^2 = \mu(dU(C_1)x, x) + (dU(C^\dagger C - \mu C_1)x, x) \ge \mu(\|x\|'_m)^2 - \varrho\|x\|^2$$

for all $x \in \mathcal{X}'_{2m}(U)$. Since $\mathcal{X}_\infty(U)$ is a core for $\overline{dU(C)}$, by [BrR1], Corollary 3.1.7, and $\mathcal{X}'_m$ is complete, it follows that $D(\overline{dU(C)}) \subseteq \mathcal{X}'_m$ and hence $dU(C)$ is closed. ∎

It is also possible to obtain regularity results for the left regular representation on the L_p-spaces with respect to left Haar measure if $p \in \langle 1, \infty\rangle$. These are basically a result of the good kernel bounds and the regularity on L_2.

THEOREM 3.7. *Let U be the left regular representation on L_p, where $p \in \langle 1,\infty\rangle$, and let $H = dU(C)$. Then*

I. *The operator H is closed.*
II. *For all $n \in \mathbb{N}$ one has $D(H^n) = L'_{p;nm}$ with equivalent norms.*

Proof. The proof is precisely the same as that for subcoercive operators in [BER]. It is based on a Lie group version of the usual weak L_1-estimates of singular integration theory combined with the regularity properties already obtained for L_2 together with duality and interpolation arguments. ■

THEOREM 3.8. *Let U be the left regular representation on L_p, where $p \in \langle 1, \infty \rangle$, and let $H = dU(C)$. If $\theta \in \langle 0, \pi/2 \rangle$ then there is a $\nu_0 \geq 0$, independent of p, such that the operators $\nu I + H$, $\nu > \nu_0$, have a bounded functional analysis over the functions which are bounded and holomorphic in the sector $\Lambda(\varphi)$ with $\varphi \in \langle \pi/2 - \theta, \pi]$.*

Proof. The proof is precisely the same as in [ElR3]. It is again based on a Lie group version of arguments of singular integration theory. ■

Acknowledgements. This work was carried out whilst the first two authors were visiting the School of Mathematical Sciences at ANU. They wish to thank the ANU for financial support. The first author also wishes to thank Macquarie University for financial support. All authors are grateful to Alan McIntosh for helpful discussions.

We thank the referee for indicating to us a recent paper [DHZ] where similar kernel estimates have been obtained for real t by different methods. The referee has also pointed out that the better decay properties for K_1 as in [DHZ], Section 5, can also be deduced from our method.

REFERENCES

[Agm] S. Agmon, *Lectures on Elliptic Boundary Value Problems*, Van Nostrand Math. Stud. 2, van Nostrand, Princeton, 1965.

[AMT] P. Auscher, A. McIntosh and Ph. Tchamitchian, *Noyau de la chaleur d'opérateurs elliptiques complexes*, Math. Research Letters 1 (1994), 37–45.

[BrR1] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics*, Vol. 1, 2nd ed., Springer, New York, 1987.

[BrR2] —, —, *Subelliptic operators on Lie groups: variable coefficients*, Acta Appl. Math. (1994), to appear.

[BER] R. J. Burns, A. F. M. ter Elst and D. W. Robinson, *L_p-regularity of subelliptic operators on Lie groups*, J. Operator Theory 30 (1993), to appear.

[Dzi] J. Dziubański, *On semigroups generated by subelliptic operators on homogeneous groups*, Colloq. Math. 64 (1993), 215–231.

[DHZ] J. Dziubański, W. Hebisch and J. Zienkiewicz, *Note on semigroups generated by positive Rockland operators on graded homogeneous groups*, Studia Math. 110 (1994), 115–126.

[ElR1] A. F. M. ter Elst and D. W. Robinson, *Subcoercivity and subelliptic operators on Lie groups II: The general case*, Potential Anal. (1994), to appear.

[ElR2] —, —, *Subcoercive and subelliptic operators on Lie groups: variable coefficients*, Publ. RIMS Kyoto Univ. 29 (1993), 745–801.

[ElR3] A. F. M. ter Elst and D. W. Robinson, *Functional analysis of subelliptic operators on Lie groups*, J. Operator Theory 30 (1993), to appear.

[ElR4] —, —, *Weighted strongly elliptic operators on Lie groups*, J. Funct. Anal. (1994), to appear.

[FoS] G. B. Folland and Stein, *Hardy Spaces on Homogeneous Groups*, Math. Notes 28, Princeton University Press, Princeton, 1982.

[Heb] W. Hebisch, *Sharp pointwise estimate for the kernels of the semigroup generated by sums of even powers of vector fields on homogeneous groups*, Studia Math. 95 (1989), 93–106.

[HeS] W. Hebisch and A. Sikora, *A smooth subadditive homogeneous norm on a homogeneous group*, ibid. 96 (1990), 231–236.

[HeN1] B. Helffer et J. Nourrigat, *Caractérisation des opérateurs hypoelliptiques homogènes invariants à gauche sur un groupe de Lie nilpotent gradué*, Comm. Partial Differential Equations 4 (1979), 899–958.

[HeN2] —, —, *Hypoellipticité maximale pour des opérateurs polynômes de champs de vecteurs*, Progr. Math. 58, Birkhäuser, Boston, 1985.

[Kat] T. Kato, *Perturbation Theory for Linear Operators*, 2nd ed., Grundlehren Math. Wiss. 132, Springer, Berlin, 1984.

[Mil] K. G. Miller, *Parametrices for hypoelliptic operators on step two nilpotent Lie groups*, Comm. Partial Differential Equations 5 (1980), 1153–1184.

[NeS] E. Nelson and W. F. Stinespring, *Representation of elliptic operators in an enveloping algebra*, Amer. J. Math. 81 (1959), 547–560.

[Nir] L. Nirenberg, *Remarks on strongly elliptic partial differential operators*, Comm. Pure Appl. Math. 8 (1955), 649–675.

[Rob] D. W. Robinson, *Elliptic Operators and Lie Groups*, Oxford Math. Monographs, Oxford University Press, Oxford, 1991.

[Roc] C. Rockland, *Hypoellipticity for the Heisenberg group*, Trans. Amer. Math. Soc. 240 (1978), 1–52.

[VSC] N. T. Varopoulos, L. Saloff-Coste and T. Coulhon, *Analysis and Geometry on Groups*, Cambridge Tracts in Math. 100, Cambridge University Press, Cambridge, 1992.

IRMAR
UNIVERSITÉ DE RENNES I
CAMPUS DE BEAULIEU
35042 RENNES CEDEX
FRANCE

DEPARTMENT OF MATHEMATICS
AND COMPUTING SCIENCE
EINDHOVEN UNIVERSITY OF TECHNOLOGY
P.O. BOX 513
5600 MB EINDHOVEN
THE NETHERLANDS

CENTRE FOR MATHEMATICS AND ITS APPLICATIONS
SCHOOL OF MATHEMATICAL SCIENCES
AUSTRALIAN NATIONAL UNIVERSITY
CANBERRA, ACT 0200
AUSTRALIA

Reçu par la Rédaction le 14.9.1993