COLLOQUIUM MATHEMATICUM

VOL. LXVII 1994 FASC. 2

ON THE INJECTIVITY OF THE GENERALIZED BERS PROJECTION AND ITS FRÉCHET DERIVATIVE

BY

EWA LIGOCKA (WARSZAWA)

1. The statement of the result. Let E be a compact subset of the complex plane $\mathbb{C}$. Denote by D the complement of E in $\widehat{\mathbb{C}}$. We shall consider the plane Lebesgue measure on E. Let $M(E)$ be the open unit ball in $L^\infty(E)$. Denote by $B_2(D)$ the Banach space of holomorphic functions f on D such that $\|f\|_2 = \sup_{z\in D} \varrho(z)^{-2}|f(z)| < \infty$; $\varrho(z)$ denotes here the element of the Poincaré metric on components of D (for nonhyperbolic D we put $B_2(D) = \{0\}$). Let $\mu \in M(E)$. Let $\widetilde{\mu}$ be equal to μ on E and to zero on D. Denote by w^μ the quasiconformal map $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ defined by the Beltrami equation

$$\frac{\partial w^\mu}{\partial \overline{z}} = \widetilde{\mu}\frac{\partial w^\mu}{\partial z}.$$

The mapping w^μ is determined up to composition with Möbius maps. The restriction of w^μ to D is a univalent meromorphic function on D. If f is any meromorphic function then one can define the Schwarzian derivative of f,

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f}\right)^2.$$

1.1. DEFINITION. The *generalized Bers projection* is the mapping Φ : $M(E) \to B_2(D)$ defined by $\Phi(\mu) = S_{w^\mu}$. The Beardon–Gehring theorem implies $S_{w^\mu} \in B_2(D)$ [Be-Ge].

The mapping Φ is holomorphic (see Sugawa [Su], Appendix). We shall denote its Fréchet derivative at $\mu \in M(E)$ by $D\Phi[\mu]$.

T. Sugawa proved in [Su] that if $\operatorname{Int} E = \emptyset$ and $\widehat{\mathbb{C}} \setminus E$ consists of a finite number of hyperbolic components then Φ is an injection. Moreover, if $\widehat{\mathbb{C}}\setminus E$ is connected then for every $\mu \in M(E)$ the Fréchet derivative $D\Phi[\mu]$ is also an injection.

1991 *Mathematics Subject Classification*: Primary 30C62.

The aim of the present note is to extend Sugawa's result to the case of compact sets E for which $\widehat{\mathbb{C}} \setminus E$ has countably many components.

We shall need the following:

1.2. DEFINITION. Let E be a compact set in $\mathbb{C}$. Let A_0 be the component of $\widehat{\mathbb{C}} \setminus E$ containing ∞. Define inductively the set A_{j+1} as the sum of all components D' of $(\mathbb{C} \setminus E) \setminus \bigcup_{k=0}^{j} A_k$ such that $\overline{D'} \cap \bigcup_{k=0}^{j} \overline{A}_k$ contains at least three distinct points.

We shall say that E has a *regular complement* iff $\mathbb{C} \setminus E = \bigcup_{j=0}^{\infty} A_j$.

1.3. Remark. Each compact set E in $\mathbb{C}$ with $\operatorname{Int} E = \emptyset$ such that $\widehat{\mathbb{C}} \setminus E$ consists of a finite number of components has a regular complement.

Each Carathéodory compact set E ($E = \partial A_0$) has a regular complement.

There are also many compact sets E which have irregular complements.

1.4. THEOREM. *Let E be a compact subset of $\mathbb{C}$ with $\operatorname{Int} E = \emptyset$. Assume that E has a regular complement. Then the generalized Bers projection is injective. Moreover, for each $\mu \in M(E)$ the derivative $D\Phi[\mu]$ is a linear injection $L^\infty(E) \to B_2(D)$.*

1.5. Remark. Sugawa [Su] gave an example of a compact set E in $\widehat{\mathbb{C}}$ for which Φ is not injective. For this E the following is true: If D_j, D_i are two distinct components of $\widehat{\mathbb{C}} \setminus E$ then $\overline{D}_i \cap \overline{D}_j = \{\infty\}$. (By using the Möbius transform we can map E into $\mathbb{C}$.)

1.6. Remark. The condition that E has a regular complement is *not* a necessary condition for the validity of our result.

It is possible to formulate far weaker (and far more complicated) conditions on E, which are sufficient for the validity of Theorem 1.4. See Remark 3.4 at the end of this paper for some details.

Following again Sugawa we can state

1.7. COROLLARY. *Let E be as in Theorem 1.4. Then E has Lebesgue measure zero iff the only conformal maps on $\widehat{\mathbb{C}} \setminus E$ which extend to quasiconformal maps $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ are Möbius mappings.*

1.8. COROLLARY. *Assume that there exists a constant c such that each conformal map w on $\widehat{\mathbb{C}} \setminus E$ for which $\|S_w\|_2 < c$ is Möbius. Then E has Lebesgue measure zero.*

Corollary 1.8 is due to Overholt [Ov].

2. Proof of Theorem 1.4. The proof of the injectivity of Φ is the same as in Sugawa's paper [Su]. Assume that $\Phi(\mu_1) = \Phi(\mu_2)$. Note that we can always assume that w^{μ_1} and w^{μ_2} fix 0, 1, ∞ since $S_{m \circ f} = S_f$ for each Möbius map m. Now, $\Phi(\mu_1) - \Phi(\mu_2) = S_{w^{\mu_1}} - S_{w^{\mu_2}}$ implies that

$w^{\mu_1} \circ (w^{\mu_2})^{-1}$ restricted to a component of $D = \mathbb{C} \setminus E$ is a Möbius map. Since E has a regular complement it must be the same Möbius map on every component of D. Thus $\operatorname{Int} E = \emptyset$ implies that $w^{\mu_1} \circ (w^{\mu_2})^{-1}$ is a Möbius map on $\widehat{\mathbb{C}}$ fixing $0, 1, \infty$ and therefore $w^{\mu_1} = w^{\mu_2}$ and $\mu_1 = \mu_2$.

We must now prove the injectivity of $D\Phi[\mu]$. Sugawa [Su] proved that it is sufficient to establish the injectivity of $D\Phi[0]$. It can be proved by differentiating the formula

$$T_\mu \circ \Phi_\mu = \Phi \circ R_\mu - \Phi(\mu)$$

at $0 \in M(E_\mu)$, where $E_\mu = w^\mu(E)$, Φ_μ is the Bers projection $\Phi_\mu : M(E_\mu) \to B_2(D_\mu)$, $D_\mu = \widehat{\mathbb{C}} \setminus E_\mu$,

$$T_\mu f = (f \circ w^\mu) \cdot \left(\frac{dw^\mu}{dz}\right)^2, \qquad T_\mu : B_2(D_\mu) \to B_2(D),$$

and $R_\mu(v)$ is the Beltrami differential of $w^\nu \circ w^\mu$,

$$R_\mu(v) = \frac{\mu \dfrac{\partial w^\mu}{dz} + \nu \circ w^\mu \overline{\dfrac{\partial w^\mu}{\partial z}}}{\dfrac{\partial w^\mu}{\partial z} + \nu \circ w^\mu \overline{\dfrac{\partial w^\mu}{\partial z}} \cdot \overline{\mu}}, \qquad R_\mu : M(E_\mu) \to M(E).$$

We shall use in the sequel the Bers formula (see [Su, Appendix])

$$D\Phi[0](\nu)(z) = -\frac{6}{\pi} \int_E \frac{\nu(t)}{(t-z)^4}\, dV_t.$$

We shall need the following:

2.1. Lemma. *Suppose that $\varphi \in L^\infty(D)$ and $\operatorname{supp}\varphi$ is bounded. The function*

$$F_\varphi(z) = \int_{\mathbb{C}} \frac{\varphi(t)}{t-z}\, dV_t$$

belongs to the Hölder space $\Lambda_\alpha(\mathbb{C})$ for each $\alpha \in (0,1)$.

Proof. F_φ is a solution of the differential equation $\partial u/\partial \overline{z} = \varphi$. Take R so large that $\overline{\operatorname{supp}\varphi} \subset B(0,R)$. Let $v = \frac{\partial}{\partial z} G_R\, \varphi$, where G_R is the operator solving the Dirichlet problem

$$\frac{\partial^2 v}{\partial z \partial \overline{z}} = \frac{1}{4}\Delta v = \varphi \quad \text{on } B(0,R), \qquad v \equiv 0 \quad \text{on } \partial B(0,R).$$

By the classical L^p estimates of the solution of the Dirichlet problem and the Sobolev imbedding theorem, $v \in \Lambda_\alpha(B(0,R))$, $0 < \alpha < 1$. By the ellipticity of $d/d\overline{z}$, also $F_\varphi \in \Lambda_\alpha(B(0,R))$. Since F_φ is holomorphic on $\mathbb{C} \setminus \overline{\operatorname{supp}\varphi}$, $F_\varphi \in \Lambda_\alpha(\mathbb{C})$.

LEMMA 2.2. *Let E be a compact set in $\mathbb{C}$. Assume that* $\operatorname{Int} E = \emptyset$ *and E has a regular complement. Let $\varphi \in L^\infty(E)$. If*

$$H_\varphi(z) = \int_E \frac{\varphi(t)}{(t-z)^4}\, dV_t = 0 \quad \text{for each } z \in D = \mathbb{C} \setminus E$$

then

$$F_\varphi(z) = \int_E \frac{\varphi(t)}{t-z}\, dV_t = 0 \quad \text{for each } z \in D.$$

Proof. We have

$$H_\varphi(z) = c\frac{d^3}{dz^3} F_\varphi(z).$$

This implies that on each component D_i of D, $F_\varphi(z) = a_i z^2 + b_i z + c_i$. Since F_φ is Hölder on $\mathbb{C}$ (by Lemma 2.1) and E has a regular complement, $a_i = a_j$, $b_i = b_j$, $c_i = c_j$ for all i, j and $F_\varphi(z) = az^2 + bz + c$ on $\mathbb{C}$. However, $F_\varphi(z)$ vanishes at infinity and therefore $F_\varphi(z) = 0$.

End of the proof of Theorem 1.4. Let $D\Phi[0](\nu) = 0$, $\nu \in M(E)$. Then $H_\nu(z) = \iint_E (\nu(t)/(t-z)^4)\, dV_t$ vanishes on D, and so does $F_\nu(z) = \iint_E (\nu(t)/(t-z))\, dV_t$ by Lemma 2.2.

Take $a \in D$ and consider the expansion of F_ν at a,

$$F_\nu(z) = \sum_{n=0}^{\infty} (z-a)^n \int_E \frac{\nu(t)}{(t-a)^{n+1}}\, dV_t.$$

We have $\iint_E (\nu(t)/(t-a)^k)\, dV_t = 0$ for $k \ge 1$.

Putting $a = \infty$ we obtain the expansion

$$F_\nu(z) = \sum_{n=0}^{\infty} z^{-n-1} \int_E t^n \nu(t)\, dV_t.$$

Hence $\iint_E r(t)\nu(t)\, dV_t = 0$ for every rational function $\nu(t)$ with poles outside E.

The Brennan theorem (see [Br1] and [Me-Si, Th. 7.4 and the proof of Th. 1.7]) implies that for every compact set E in $\mathbb{C}$ and every p with $1 \le p < 2$ the space $R(E)$ of rational functions with poles outside E is dense in $L^p(E) \cap \operatorname{Hol}(\operatorname{Int} E)$, the space of those functions from $L^p(E)$ which are holomorphic on $\operatorname{Int} E$. Thus we have $L^1(E) = \overline{R(E)}$ if $\operatorname{Int} E = \emptyset$. Hence $\iint_E f(t)\nu(t)\, dV_t = 0$ for every $f \in L^1(E)$ and $\nu = 0$ a.e. on E.

Proof of the corollaries. If the only conformal maps as in the statement of Corollary 1.7 are Möbius then $\Phi \equiv 0$. The injectivity of Φ implies that $M(E) = \{0\}$ and E has measure zero. If each w for which $\|S_w\|_2 < c$ is Möbius then $\{0\}$ is an isolated point of $\Phi(M(E))$. By the identity theorem, $\Phi \equiv 0$ and E has measure zero.

3. Remarks

3.1. Remark. If we drop the assumption that $\operatorname{Int} E = \emptyset$ we can put our Theorem 1.4 in a (seemingly) more general form.

THEOREM 1.4′. *Let E be a compact set in $\mathbb{C}$ with a regular complement and let Φ be the Bers projection.*

1) *Φ is injective iff* $\operatorname{Int} E = \emptyset$.
2) $\operatorname{Ker} D\Phi[0] = \{\mu \in M(E) : \iint_E f(t)\mu(t)\, dV_t = 0,\ \forall f \in L^1(E) \cap \operatorname{Hol}(\operatorname{Int} E)\}$.
3) $\operatorname{Ker} D\Phi[0] = \{0\} \Leftrightarrow \exists \mu \in M(E)\ \operatorname{Ker} D\Phi[\mu] = \{0\}$
$\Leftrightarrow \forall \mu \in M(E)\ \operatorname{Ker} D\Phi[\mu] = \{0\} \Leftrightarrow \operatorname{Int} E = \emptyset$.

The proof remains almost the same. Note that if $\operatorname{Int} E \neq \emptyset$ then the fiber $\Phi^{-1}(0)$ is very large. It contains in particular all C^1 diffeomorphisms of $\widehat{\mathbb{C}}$ equal to the identity on $\widehat{\mathbb{C}} \setminus \operatorname{Int} E$. If $E = \overline{B(0,1)} = \overline{\Delta}$ then $\Phi^{-1}(0)$ is the known class F of q.c. homeomorphisms of the unit disc equal to the identity on the circle.

3.2. Remark. Theorem 1.4′ can be valid for some compact sets E with irregular complement and $\operatorname{Int} E \neq \emptyset$. It suffices that polynomials are dense in $L^1(E) \cap \operatorname{Hol}(\operatorname{Int} E)$. An interesting class of such domains was described by Brennan [Br2]:

Let E_1 be a compact set in $\mathbb{C}$ with connected complement and let D_1 be a Jordan domain with C^2-smooth boundary such that $\overline{D}_1 \subset E_1$. Take $E = E_1 \setminus D_1$. The polynomials are dense in $L^1(E) \cap \operatorname{Hol}(E)$ iff $\int_{\partial D_1} \ln \delta(z)\, |dz| = -\infty$, where $\delta(z) = \operatorname{dist}(z, \mathbb{C} \setminus E_1)$.

Note that if $\operatorname{Int} E = \emptyset$ then E has a regular complement.

3.3. Remark. The formula $T_\mu \circ \Phi_\mu = \Phi \circ R_\mu - \Phi(\mu)$, $\mu \in M(E)$, used in the proof of Theorem 1.4 yields

$$D\Phi[\mu] = T_\mu \circ D\Phi_\mu[0] \circ (DR_\mu[0])^{-1}.$$

Thus by the Bers formula

$$D\Phi[\mu](\nu)(z) = -\frac{6}{\pi} \int_E \frac{\left(\frac{\partial w^\mu}{\partial t}(t)\right)^2 \cdot \nu(t) \left(\frac{\partial w^\mu}{\partial z}(z)\right)^2}{(w^\mu(t) - w^\mu(z))^4}\, dV_t$$

for $\nu \in L^\infty(E)$.

Moreover, if E has a regular complement then by Theorem 1.4′,

$$\operatorname{Ker} D\varphi[\mu] = \left\{\nu \in L^\infty(E) : \int_E \nu \cdot \left(\frac{\partial w^\mu}{\partial z}\right)^2 \cdot f \circ w^\mu = 0\ \forall f \in L^1(E_\mu) \cap \operatorname{Hol}(\operatorname{Int} E_\mu)\right\}.$$

3.4. Remark. As was mentioned before, the condition that E has a regular complement can be weakened in the following way: Put $A_{00}^0 =$ component of $\mathbb{C} \setminus E$ containing ∞. Define inductively $A_{0j}^0 = A_j$ as in Definition 1.2. Put $A_0^0 = \bigcup_{j=0}^{\infty} A_{0j}^0$. Let A_{10}^0 be any component of $(\mathbb{C} \setminus E) \setminus \overline{A_0^0}$. Construct the sets A_{1j}^0 in the same way as before. Take $A_1^0 = \bigcup_{j=1}^{\infty} A_j^0$. Let A_{20}^0 be a component of $(\mathbb{C} \setminus E) \setminus \overline{A_0^0 \cup A_1^1}$. Construct the set $A_2^0 = \bigcup_{j=0}^{\infty} A_{2j}^0$, and so on. After constructing A_k^0, $k = 1, 2, \ldots$, put $A_{00}^1 = A_0^0$ and repeat the previous construction taking A_k^0 instead of components of $\mathbb{C} \setminus E$ to obtain a sequence of sets A_{0k}^1. Put $A_0^1 = \bigcup_{j=0}^{\infty} A_{0j}^1$ and proceed to define A_k^1. Take $A_{00}^2 = A_0^1$ and repeat the construction with A_k^1 instead of A_k^0. As a result we get a sequence of sets A_0^n. We shall say that E has a w_1-*regular complement* if $\mathbb{C} \setminus E = \bigcup_{n=0}^{\infty} A_0^n$. Since in the construction in Definition 1.2 the closure of $\bigcup_{j=0}^{k} A_k$ always had three distinct points in common with $\overline{A}_{k+1}$ and we repeated the same construction again and again, all our results remain true for compact sets with w_1-regular complement. Moreover, we can take $(\mathbb{C} \setminus E) \setminus \bigcup_{n=0}^{\infty} A_0^n$, choose some component of it and repeat this construction to formulate a weaker condition of having w_2-regular complement. In this way one can in principle define a sequence of weaker and weaker conditions of w_n-regularity of the complement of E. Each of those conditions will be sufficient for the validity of Theorem 1.4 and of the rest of our results, but none will be necessary.

REFERENCES

[Be-Ge] A. F. Beardon and F. W. Gehring, *Schwarzian derivatives, the Poincaré metric and the kernel function*, Comment. Math. Helv. 55 (1980), 50–64.

[Br1] J. Brennan, *Point evaluation and invariant subspaces*, Indiana Univ. Math. J. 20 (1971), 879–881.

[Br2] —, *Invariant subspaces and weighted polynomial approximation*, Ark. Mat. 11 (1973), 167–189.

[Me-Si] M. S. Melnikov and S. O. Sinanyan, *Problems in approximation theory of functions of one complex variable*, in: Itogi Nauki i Tekhniki, Sovrem. Problemy Mat. 4, Moscow, 1975, 143–250 (in Russian).

[Ov] M. Overholt, *The area of the complement of a conformally rigid domain*, Proc. Amer. Math. Soc. 103 (1988), 448–450.

[Su] T. Sugawa, *The Bers projection and the λ-lemma*, J. Math. Kyoto Univ. 32 (1992), 701–713.

INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
P.O. BOX 137
00-950 WARSZAWA, POLAND

Reçu par la Rédaction le 5.7.1993