COLLOQUIUM MATHEMATICUM

VOL. LXVII 1994 FASC. 2

CONVERGENCE OF COMPOUND PROBABILITY MEASURES ON TOPOLOGICAL SPACES

BY

JUN KAWABE (NAGANO)

1. Introduction. Let X and Y be topological spaces. In this paper, we present a sufficient condition imposed on transition probabilities that assures the weak convergence or the relative compactness of compound probability measures $\mu \circ \lambda$ defined by

$$\mu \circ \lambda(D) = \int_X \lambda(x, D_x)\, \mu(dx)$$

for a measure μ on X and a transition probability λ on $X \times Y$.

In information theory, a transition probability λ on $X \times Y$ is called an information channel with input space X and output space Y, and for a probability measure μ on X (which is called the input source) the compound probability measure $\mu \circ \lambda$ plays an important role (see, e.g., Umegaki [11]).

On the other hand, compound probability measures can be viewed as a generalization of convolution measures. In fact, if $X = Y$ is a topological group and a transition probability λ is given by $\lambda(x, B) = \nu(Bx^{-1})$ for all $x \in X$ and all Borel subsets B of Y, where ν is a probability measure on X, then the projection $\mu\lambda$ of $\mu \circ \lambda$ onto Y defined by $\mu\lambda(B) = \mu \circ \lambda(X \times B)$ is the convolution measure $\mu * \nu$. The weak convergence of convolution measures has been looked into in great detail by Csiszár [2, 3].

In Section 2 we recall notations and necessary definitions and results concerning probability measures on topological spaces, and in Section 3 we show that compound probability measures can be defined on the Borel subsets of $X \times Y$ for continuous τ-smooth transition probabilities.

In Section 4 we present a sufficient condition that assures the weak convergence of a net of compound probability measures, and also give a relative compactness criterion for a set of compound probability measures which ex-

1991 *Mathematics Subject Classification*: 60B05, 60B10.

Key words and phrases: continuous transition probabilities, compound probability measures, equicontinuity, Gaussian transition probabilities.

The author is partially supported by Grant-in-Aid for General Scientific Research No. 06640301, the Ministry of Education, Science and Culture, Japan.

tends Prokhorov's compactness criterion for probability measures. In this and the following sections, the equicontinuity of a set of transition probabilities plays an important role.

In Section 5, for Gaussian transition probabilities we express the assumptions of our theorems of Section 4 in terms of the corresponding mean and covariance functions. In Section 6 we give several examples and counterexamples concerning the convergence and uniform tightness of compound probability measures.

Throughout this paper, we suppose that all the topological spaces and all the uniform spaces considered are Hausdorff.

2. Transition probabilities on topological spaces. Let X be a topological space and $\mathcal{B}(X)$ be the σ-algebra of all Borel subsets of X. By a *Borel measure* on X we mean a finite measure defined on $\mathcal{B}(X)$ and we denote by $\mathcal{P}(X)$ the set of all Borel probability measures on X.

In this paper, the following concept of regularity for Borel measures is useful. We say that a Borel measure μ on X is *τ-smooth* if for every increasing net $\{G_\alpha\}$ of open subsets of X, we have $\mu(\bigcup_\alpha G_\alpha) = \sup_\alpha \mu(G_\alpha)$. We denote by $\mathcal{P}_\tau(X)$ the set of all τ-smooth probability measures on X. Every Radon measure is τ-smooth, and if X is regular every τ-smooth measure is regular (see, e.g., Proposition I.3.1 of Vakhania *et al.* [12]). We also know that if X is strongly Lindelöf, that is, every open cover of any open subset of X has a countable subcover (this is satisfied, for instance, if X is a Suslin space), then every Borel measure is τ-smooth (see, e.g., Proposition I.3.1 of [12]). Here we recall that a topological space is called a *Suslin space* if it is a continuous image of some Polish space (see Schwartz [9]).

If X is completely regular, we equip $\mathcal{P}(X)$ with the weakest topology for which the functionals

$$\mathcal{P}(X) \ni \mu \mapsto \int_X f(x)\mu(dx), \qquad f \in C_{\mathrm{b}}(X),$$

are continuous. Here $C_{\mathrm{b}}(X)$ denotes the set of all bounded continuous real-valued functions on X. This topology on $\mathcal{P}(X)$ is called the *weak topology*, and we say that a net $\{\mu_\alpha\}$ in $\mathcal{P}(X)$ *converges weakly* to a Borel probability measure μ, and we write $\mu_\alpha \xrightarrow{w} \mu$, if

$$\lim_\alpha \int_X f(x)\,\mu_\alpha(dx) = \int_X f(x)\,\mu(dx)$$

for every $f \in C_{\mathrm{b}}(X)$, and this is equivalent to the condition that for each open subset G (resp. closed subset F) of X,

$$\liminf_\alpha \mu_\alpha(G) \ge \mu(G) \qquad (\text{resp. } \limsup_\alpha \mu_\alpha(F) \le \mu(F)),$$

provided that $\mu \in \mathcal{P}_\tau(X)$. It is known that the weak topology on $\mathcal{P}_\tau(X)$ is also completely regular, and hence it is uniformizable, that is, it can be derived from a uniformity on $\mathcal{P}_\tau(X)$ (see Topsøe [10, Theorem 11.2]).

Let us recall now Prokhorov's criterion of weak relative compactness of a set of probability measures which is a basic tool in the study of weak convergence. A subset P of $\mathcal{P}(X)$ is said to be *uniformly tight* if for each $\varepsilon > 0$, there exists a compact subset K_ε of X such that $\mu(X - K_\varepsilon) < \varepsilon$ for all $\mu \in P$. It is well known that if X is completely regular, every uniformly tight subset of $\mathcal{P}_\tau(X)$ is relatively compact in $\mathcal{P}_\tau(X)$ (see, e.g., Theorem I.3.6 of [12]). We say that a completely regular topological space X is a *Prokhorov space* if the converse holds, that is, every relatively compact subset of $\mathcal{P}_\tau(X)$ is uniformly tight. According to Prokhorov [8], every Polish space is a Prokhorov space. See Fernique [4] for other examples of Prokhorov spaces.

Let X be a topological space and Y be a completely regular topological space. A (Borel) *transition probability* λ on $X \times Y$ is defined to be a mapping from X into $\mathcal{P}(Y)$ which satisfies

(T1) for every $B \in \mathcal{B}(Y)$, the function $X \ni x \mapsto \lambda(x, B)$ is Borel measurable.

Let $\mu \in \mathcal{P}(X)$ and λ be a transition probability on $X \times Y$ which satisfies

(T2) for every $D \in \mathcal{B}(X \times Y)$, the function $X \ni x \mapsto \lambda(x, D_x)$ is Borel measurable.

Here for a subset D of $X \times Y$ and $x \in X$, D_x denotes the section determined by x, that is, $D_x = \{y \in Y : (x, y) \in D\}$. Then we can define a Borel probability measure $\mu \circ \lambda$ on $X \times Y$, which is called the *compound probability measure* of μ and λ, by

$$\mu \circ \lambda(D) = \int_X \lambda(x, D_x)\, \mu(dx) \qquad \text{for all } D \in \mathcal{B}(X \times Y).$$

By a standard argument, we can show that Fubini's theorem remains valid for all Borel measurable and $\mu \circ \lambda$-integrable functions f on $X \times Y$:

$$\int_{X \times Y} f(x, y)\, \mu \circ \lambda(dx, dy) = \int_X \int_Y f(x, y)\, \lambda(x, dy)\, \mu(dx).$$

It is obvious that (T2) implies (T1), and (T2) is satisfied, for instance, if the product σ-algebra $\mathcal{B}(X) \times \mathcal{B}(Y)$ coincides with $\mathcal{B}(X \times Y)$ (this is satisfied if X and Y are Suslin spaces; see [9]).

We say that a transition probability λ is *τ-smooth* if the probability measure $\lambda_x \equiv \lambda(x, \cdot)$ is τ-smooth for each $x \in X$, that is, it is a mapping from X into $\mathcal{P}_\tau(Y)$. We also say that λ is *continuous* if it satisfies

(T3) the mapping $X \ni x \mapsto \lambda_x \in \mathcal{P}(Y)$ is continuous.

Let us denote by $C(X, \mathcal{P}_\tau(Y))$ the set of all continuous mappings from X into $\mathcal{P}_\tau(Y)$.

The following proposition clarifies the relation between continuous τ-smooth transition probabilities on $X \times Y$ and continuous mappings from X into $\mathcal{P}_\tau(Y)$, and gives a sufficient condition, imposed on transition probabilities, under which (T2) is satisfied.

PROPOSITION 1. *Let X be a topological space and Y be a completely regular topological space. Let λ be a mapping from X into $\mathcal{P}_\tau(Y)$.*

(1) *λ is continuous if and only if for each open subset U of $X \times Y$, the function $X \ni x \mapsto \lambda(x, U_x)$ is lower semi-continuous on X.*

(2) *If λ is continuous then for each Borel subset D of $X \times Y$, the function $X \ni x \mapsto \lambda(x, D_x)$ is Borel measurable.*

Therefore $C(X, \mathcal{P}_\tau(Y))$ coincides with the set of all continuous τ-smooth transition probabilities on $X \times Y$, and every continuous τ-smooth transition probability on $X \times Y$ satisfies (T2).

Proof. (1) The condition is clearly sufficient for λ to be continuous. Conversely, let U be an open subset of $X \times Y$. Then we can find an increasing net $\{U_\alpha\}$ of finite unions of open rectangles with $U = \bigcup_\alpha U_\alpha$.

Since λ_x is τ-smooth we have $\lambda(x, U_x) = \sup_\alpha \lambda(x, [U_\alpha]_x)$ for all $x \in X$. Since $\lambda : X \to \mathcal{P}_\tau(Y)$ is continuous, it is not too hard to prove that the function $X \ni x \mapsto \lambda(x, [U_\alpha]_x)$ is lower semi-continuous on X for each α. Thus the function $X \ni x \mapsto \lambda(x, U_x)$ is lower semi-continuous on X, because it is the supremum of a family of lower semi-continuous functions on X.

(2) Let $\mathcal{A}$ be the family of sets $D \in \mathcal{B}(X \times Y)$ for which the function $X \ni x \mapsto \lambda(x, D_x)$ is Borel measurable. It is easy to see that $\mathcal{A}$ is a σ-additive class. Since, according to (1), open subsets of $X \times Y$ belong to $\mathcal{A}$, the Borel σ-algebra $\mathcal{B}(X \times Y)$ coincides with $\mathcal{A}$ (see, e.g., Neveu [7]), and the proof of (2) is complete.

If $\lambda \in C(X, \mathcal{P}_\tau(Y))$ then λ clearly satisfies (T3), and, by (2), λ also satisfies (T2) and hence (T1). Therefore $C(X, \mathcal{P}_\tau(Y))$ is contained in the set of all continuous τ-smooth transition probabilities on $X \times Y$, and the reverse inclusion is obvious.

Let us denote by $\mu\lambda$ the projection of $\mu \circ \lambda$ onto Y, that is, $\mu\lambda(B) = \mu \circ \lambda(X \times B)$ for all $B \in \mathcal{B}(Y)$. Now we give typical examples of continuous τ-smooth transition probabilities. Other examples are Gaussian transition probabilities, to be considered in Section 4.

EXAMPLE 1. (1) *For $\nu \in \mathcal{P}_\tau(Y)$, put $\lambda(x, B) = \nu(B)$ for all $x \in X$ and all $B \in \mathcal{B}(Y)$. Then $\lambda \in C(X, \mathcal{P}_\tau(Y))$, and $\mu \circ \lambda = \mu \times \nu$ for each $\mu \in \mathcal{P}(X)$, where $\mu \times \nu$ is the product measure.*

(2) *Let* $X = Y = G$ *be a topological group and* $\nu \in \mathcal{P}_\tau(G)$. *Put* $\lambda(x, B) = \nu(Bx^{-1})$ *for all* $x \in G$ *and all* $B \in \mathcal{B}(G)$. *Then* $\lambda \in C(G, \mathcal{P}_\tau(G))$, *and for each* $\mu \in \mathcal{P}(G)$, $\mu\lambda = \mu * \nu$, *where* $\mu * \nu$ *is the convolution measure.*

(3) *Let* T *be a topological space and* Y *be a Suslin uniform space. Let* $(\Omega, \mathcal{A}, P)$ *be a probability measure space and* $X(t, \omega)$, $t \in T$ *and* $\omega \in \Omega$, *be a* $\mathcal{B}(T) \times \mathcal{A}$-*measurable*, Y-*valued stochastic process which is continuous in probability*, *that is*, *for every* $t_0 \in T$, *every* $\varepsilon > 0$ *and every uniformity* V *on* Y *there exists a neighborhood* U *of* t_0 *such that if* $t \in U$ *then* $P(\{\omega \in \Omega : (X(t, \omega), X(t_0, \omega)) \in V\}) > 1 - \varepsilon$. *Put* $\lambda(t, B) = P(\{\omega \in \Omega : X(t, \omega) \in B\})$ *for all* $t \in T$ *and all* $B \in \mathcal{B}(Y)$. *Then* $\lambda \in C(T, \mathcal{P}_\tau(Y))$.

Proof. (1) Clearly λ is τ-smooth, and the continuity of λ follows from Lemma I.4.1 of [12] and Proposition 1.

(2) It is easily seen that λ is τ-smooth, and the continuity of λ is an immediate consequence of Lemma I.4.1 of [12] and Proposition 1, together with the fact that for each open subset U of $G \times G$ and each $x \in G$, $\widetilde{U} \equiv \{(x, y) \in G \times G : (x, yx) \in U\}$ is also an open subset of $G \times G$ and $(U_x)x^{-1} = [\widetilde{U}]_x$.

(3) The proof is easy and we omit it.

Proposition 2. *Let* X *be a topological space and* Y *be a completely regular topological space. If* $\mu \in \mathcal{P}_\tau(X)$ *and* $\lambda \in C(X, \mathcal{P}_\tau(Y))$ *then* $\mu \circ \lambda \in \mathcal{P}_\tau(X \times Y)$ *and therefore* $\mu\lambda \in \mathcal{P}_\tau(Y)$.

Proof. Let $\{U_\alpha\}$ be an increasing net of open subsets of $X \times Y$, and set $U = \bigcup_\alpha U_\alpha$. Then $\lambda(x, U_x) = \sup_\alpha \lambda(x, [U_\alpha]_x)$, because λ_x is τ-smooth for each $x \in X$. Since for each α, the bounded positive function $X \ni x \mapsto \lambda(x, [U_\alpha]_x)$ is lower semi-continuous on X by Proposition 1(1), and the net of such functions is increasing, we have

$$\begin{aligned}\mu \circ \lambda(U) &= \int_X \sup_\alpha \lambda(x, [U_\alpha]_x)\, \mu(dx) \\ &= \sup_\alpha \int_X \lambda(x, [U_\alpha]_x)\, \mu(dx) = \sup_\alpha \mu \circ \lambda(U_\alpha)\end{aligned}$$

by the τ-smoothness of μ. Hence $\mu \circ \lambda$ is τ-smooth, and from this the τ-smoothness of $\mu\lambda$ follows.

3. Convergence of compound probability measures. In this section, we study the convergence and relative compactness of compound probability measures.

Let X be a topological space and Y be a uniform space. Denote by $C(X, Y)$ the set of all continuous mappings from X into Y. If $Y = \mathbb{R}$ we write $C(X)$ for $C(X, \mathbb{R})$. We say that a subset H of $C(X, Y)$ is *equicontin-*

uous at $x \in X$ if for each uniformity V, there exists a neighborhood U of x such that $(f(x), f(u)) \in V$ for all $u \in U$ and all $f \in H$. H is *equicontinuous on* X if it is equicontinuous at every $x \in X$. In this paper, we need the following variant of the notion of equicontinuity. We say that a subset H of $C(X,Y)$ is *equicontinuous on a set* $A \subset X$ if the set of all restrictions of functions of H to A is equicontinuous on A.

A topological space X is called a *k-space* if it satisfies the condition that a subset A of X is closed whenever its intersection with every compact subset of X is closed. Every locally compact space and every space which satisfies the first axiom of countability (in particular, a metric space) is a k-space, and a mapping defined on a k-space into a topological space is continuous if it is continuous on each compact set.

In the proofs of theorems below we need the following form of the Ascoli theorem (see, e.g., Kelley [6, p. 234]): If X is a k-space, then $H \subset C(X,Y)$ is relatively compact with respect to the topology of uniform convergence on compact subsets of X if and only if

(a) H is equicontinuous on every compact subset of X, and
(b) $H[x] \equiv \{f(x) : f \in H\}$ is relatively compact in Y for each $x \in X$.

It is not too hard to prove that a subset Q of $C(X, \mathcal{P}_\tau(Y))$ is equicontinuous on every compact subset of X if and only if for each $h \in C_{\mathrm{b}}(Y)$, the set of the functions

$$X \ni x \mapsto \int_Y h(y)\, \lambda(x, dy), \qquad \lambda \in Q,$$

is equicontinuous on every compact subset of X.

In what follows, for $P \subset \mathcal{P}(X)$ and $Q \subset C(X, \mathcal{P}_\tau(Y))$, we set $P \circ Q = \{\mu \circ \lambda : \mu \in P \text{ and } \lambda \in Q\}$ and $PQ = \{\mu\lambda : \mu \in P \text{ and } \lambda \in Q\}$. Now we state our result about the convergence of compound probability measures.

THEOREM 1. *Let X be a completely regular k-space and let Y be a completely regular topological space. Assume that a net $\{\lambda_\alpha\}$ in $C(X, \mathcal{P}_\tau(Y))$ satisfies*

(a) *$\{\lambda_\alpha\}$ is equicontinuous on every compact subset of X,*
(b) *$\{\lambda_\alpha(x, \cdot)\}$ is uniformly tight for each $x \in X$, and*
(c) *there exists $\lambda \in C(X, \mathcal{P}_\tau(Y))$ such that $\lambda_\alpha(x, \cdot) \xrightarrow{w} \lambda(x, \cdot)$ for $x \in X$.*

Then for any uniformly tight net $\{\mu_\alpha\}$ in $\mathcal{P}_\tau(X)$ converging weakly to $\mu \in \mathcal{P}_\tau(X)$, we have $\mu_\alpha \circ \lambda_\alpha \xrightarrow{w} \mu \circ \lambda$.

The following two theorems extend Prokhorov's criterion of relative compactness of a set of probability measures.

THEOREM 2. *Let X and Y be as in Theorem* 1. *Assume that $P \subset \mathcal{P}_\tau(X)$ is uniformly tight and $Q \subset C(X, \mathcal{P}_\tau(Y))$ satisfies*

(a) *Q is equicontinuous on every compact subset of X, and*
(b) *$Q[x] = \{\lambda_x : \lambda \in Q\}$ is uniformly tight for each $x \in X$.*

Then for every net $\{\mu_\alpha \circ \lambda_\alpha\}$ in $P \circ Q$, we can find a subnet $\{\mu_{\alpha'} \circ \lambda_{\alpha'}\}$ of $\{\mu_\alpha \circ \lambda_\alpha\}$, $\mu \in \mathcal{P}_\tau(X)$ and $\lambda \in C(X, \mathcal{P}_\tau(Y))$ which satisfy

(1) $\mu_{\alpha'} \xrightarrow{w} \mu$,
(2) $\lambda_{\alpha'}(x, \cdot) \xrightarrow{w} \lambda(x, \cdot)$ *for each $x \in X$, and*
(3) $\mu_{\alpha'} \circ \lambda_{\alpha'} \xrightarrow{w} \mu \circ \lambda$.

Therefore $P \circ Q$ is a relatively compact subset of $\mathcal{P}_\tau(X \times Y)$.

Remark 1. If we set $X = \{x_0\}$ (x_0 is a fixed element of X) and $P = \{\delta_{x_0}\}$ (δ_{x_0} is the Dirac measure on X which is concentrated at x_0) in Theorem 2, then we have Prokhorov's criterion for relative compactness of a set of probability measures (see Section 2).

Theorem 3. *Let X and Y be as in Theorem 1 and assume that Y is a Prokhorov space. Assume that $P \subset \mathcal{P}_\tau(X)$ is uniformly tight and $Q \subset C(X, \mathcal{P}_\tau(Y))$ satisfies*

(a) *Q is equicontinuous on every compact subset of X, and*
(b) *$Q[x]$ is relatively compact in $\mathcal{P}_\tau(Y)$ for each $x \in X$.*

Then $P \circ Q \subset \mathcal{P}_\tau(X \times Y)$ is uniformly tight.

We have more direct proofs of Theorems 1 and 2 than our proofs below when Y is a Prokhorov space (see the proof of Theorem 3 and Remark 2). However, in the case when Y is an arbitrary completely regular topological space, we first prove the lemmas below.

Lemma 1. *Let X be a k-space and Y be a compact space.*

(1) *For each $f \in C_{\mathrm{b}}(X \times Y)$ and each $\lambda \in C(X, \mathcal{P}_\tau(Y))$, the function*

$$X \ni x \mapsto \int_Y f(x, y)\, \lambda(x, dy)$$

is continuous on X.

(2) *Assume that $Q \subset C(X, \mathcal{P}_\tau(Y))$ is equicontinuous on every compact subset of X. Then for every net $\{\lambda_\alpha\}$ in Q, we can find a subnet $\{\lambda_{\alpha'}\}$ of $\{\lambda_\alpha\}$ and $\lambda \in C(X, \mathcal{P}_\tau(Y))$ such that*

$$\lim_{\alpha'} \sup_{x \in K} \Big| \int_Y f(x, y)\, \lambda_{\alpha'}(x, dy) - \int_Y f(x, y)\, \lambda(x, dy) \Big| = 0 \tag{3.1}$$

for each $f \in C_{\mathrm{b}}(X \times Y)$ and each compact subset K of X.

Proof. (1) Let K be a compact subset of X. By the Stone–Weierstrass theorem, f can be uniformly approximated on $K \times Y$ by functions of the form $\sum_{i=1}^n g_i h_i$, where the g_i's are in $C(K)$ and the h_i's are in $C(Y)$. Then the function $X \ni x \mapsto \int_Y f(x,y)\,\lambda(x,dy)$ is continuous on K, and since X is a k-space, it is continuous on X, because it is uniformly approximated on K by the functions $X \ni x \mapsto \sum_{i=1}^{n} g_i(x) \int_Y h_i(y)\,\lambda(x,dy)$ which are continuous on K by the continuity of λ.

(2) Since Y is compact, $Q[x]$ is clearly uniformly tight for each $x \in X$. Thus by the Ascoli theorem, Q is relatively compact in $C(X,\mathcal{P}_\tau(Y))$ with respect to the topology of uniform convergence on compact subsets of X. Therefore for every net $\{\lambda_\alpha\}$ in Q, we can find a subnet $\{\lambda_{\alpha'}\}$ of $\{\lambda_\alpha\}$ and $\lambda \in C(X,\mathcal{P}_\tau(Y))$ such that

$$\lim_{\alpha'} \sup_{x\in K} \Big| \int_Y h(y)\,\lambda_{\alpha'}(x,dy) - \int_Y h(y)\lambda(x,dy) \Big| = 0 \tag{3.2}$$

for each $h \in C(Y)$ and each compact subset K of X. Then it is easily verified that (3.2) remains valid for every $f \in C_{\mathrm{b}}(X \times Y)$ by the Stone–Weierstrass theorem and a standard argument.

If X is a uniform space, we denote by $U_{\mathrm{b}}(X)$ the set of all bounded uniformly continuous real-valued functions on X. Let us remark that a net $\{\mu_\alpha\}$ of probability measures on a uniform space X converges weakly to a τ-smooth probability measure μ on X if and only if

$$\lim_{\alpha} \int_X f(x)\,\mu_\alpha(dx) = \int_X f(x)\,\mu(dx)$$

for every $f \in U_{\mathrm{b}}(X)$ (for a proof see, e.g., Lemma 3 of [2]). In the following lemma, we have the advantage of considering uniform spaces and $U_{\mathrm{b}}(X\times Y)$ instead of topological spaces and $C_{\mathrm{b}}(X \times Y)$.

LEMMA 2. *Let X be a uniform k-space and Y be a totally bounded uniform space.*

(1) *For each $f \in U_{\mathrm{b}}(X \times Y)$ and each $\lambda \in C(X,\mathcal{P}_\tau(Y))$, the function*

$$X \ni x \mapsto \int_Y f(x,y)\,\lambda(x,dy)$$

is continuous on X.

(2) *Assume that $Q \subset C(X,\mathcal{P}_\tau(Y))$ satisfies* (a) *and* (b) *of Theorem* 2. *Then for every net $\{\lambda_\alpha\}$ in Q, we can find a subnet $\{\lambda_{\alpha'}\}$ of $\{\lambda_\alpha\}$ and $\lambda \in C(X,\mathcal{P}_\tau(Y))$ such that* (3.1) *is valid for each $f \in U_{\mathrm{b}}(X \times Y)$ and each compact subset K of X.*

(3) *Assume that $Q \subset C(X,\mathcal{P}_\tau(Y))$ satisfies* (a) *and* (b) *of Theorem* 2. *Then for each $f \in U_{\mathrm{b}}(X \times Y)$, the set of the functions*

$$X \ni x \mapsto \int_Y f(x,y)\,\lambda(x,dy), \qquad \lambda \in Q,$$

is equicontinuous on every compact subset of X.

P r o o f. Denote by Z the completion of Y. Since Y is totally bounded, Z is compact and there exists a uniform isomorphism θ of Y onto a dense subset of Z. Let us first note that the following (i) and (ii) are valid:

(i) If $f \in U_{\rm b}(X\times Y)$ then it has a unique uniformly continuous extension $f^Z \in U_{\rm b}(X \times Z)$ which satisfies $f^Z(x,\theta(y)) = f(x,y)$ for all $x \in X$ and all $y \in Y$.

(ii) For each $\lambda \in C(X,\mathcal{P}_\tau(Y))$, if we define

$$\lambda^Z(x,D) = \lambda(x,\theta^{-1}(D)) \quad \text{for all } x \in X \text{ and all } D \in \mathcal{B}(Z),$$

then $\lambda^Z \in C(X,\mathcal{P}_\tau(Z))$ and

$$\int\limits_Z f^Z(x,z)\,\lambda^Z(x,dz) = \int\limits_Y f(x,y)\,\lambda(x,dy)$$

for every $f \in U_{\rm b}(X \times Y)$ and every $x \in X$.

(1) From (i), (ii) and Lemma 1(1), (1) follows.

(2) Set $Q^Z = \{\lambda^Z : \lambda \in Q\}$. Then it is easily verified that $Q^Z \subset C(X,\mathcal{P}_\tau(Z))$ is equicontinuous on every compact subset of X. Therefore by Lemma 1, for every net $\{\lambda_\alpha\}$ in Q, we can find a subnet $\{\lambda_{\alpha'}\}$ of $\{\lambda_\alpha\}$ and $\gamma \in C(X,\mathcal{P}_\tau(Z))$ such that

$$\lim_{\alpha'}\sup_{x\in K}\Big|\int\limits_Z f(x,z)\,\lambda^Z_{\alpha'}(x,dz) - \int\limits_Z f(x,z)\,\gamma(x,dz)\Big| = 0 \tag{3.3}$$

for every $f \in U_{\rm b}(X \times Z)$ and every compact subset K of X. From (i), (ii) and (3.3), it is sufficient to prove that there exists $\lambda \in C(X,\mathcal{P}_\tau(Y))$ with $\lambda^Z = \gamma$.

To prove this we first show that $\theta(Y)$ is a γ_x-thick subset of Z for all $x \in X$, that is,

$$(\gamma_x)_*(Z-\theta(Y)) \equiv \sup\{\gamma_x(D) : D \in \mathcal{B}(Z) \text{ and } D \subset Z-\theta(Y)\} = 0.$$

Fix $x \in X$. Since $Q[x]$ is uniformly tight, we can find a sequence $\{K_n\}$ of compact subsets of Y such that $\lambda(x,Y-K_n) < 1/n$ for all $\lambda \in Q$. On the other hand, it follows from (3.3) that $\lambda^Z_{\alpha'}(x,\cdot)$ converges weakly to the τ-smooth probability measure $\gamma(x,\cdot)$. Let $D \in \mathcal{B}(Z)$ with $D \subset Z-\theta(Y)$. Then for all $n \ge 1$, D is contained in $Z-\theta(K_n)$, which are open subsets of Z. Hence

$$\begin{aligned}\gamma(x,D) \le \gamma(x,Z-\theta(K_n)) &\le \liminf_{\alpha'} \lambda^Z_{\alpha'}(x,Z-\theta(K_n))\\ &= \liminf_{\alpha'} \lambda_{\alpha'}(x,Y-K_n) \le 1/n.\end{aligned}$$

Letting $n \to \infty$ we obtain $\gamma(x,D) = 0$, and so $\theta(Y)$ is γ_x-thick for all $x \in X$.

According to Halmos [5; Theorem A, p. 75], if we define for each $x \in X$ and each $B \in \mathcal{B}(Y)$,

$$\lambda(x,B) = \gamma(x,D),$$

where $D \in \mathcal{B}(Z)$ with $\theta(B) = \theta(Y) \cap D$, it is easily verified that λ is well defined and $\lambda \in C(X, \mathcal{P}_\tau(Y))$ with $\lambda^Z = \gamma$. So the proof of (2) is complete.

(3) Fix $f \in U_{\mathrm{b}}(X \times Y)$. Note first that for each $\lambda \in C(X, \mathcal{P}_\tau(Y))$, by (1), we can define a bounded continuous function φ_λ on X by

$$\varphi_\lambda(x) = \int_Y f(x,y)\,\lambda(x,dy), \qquad x \in X.$$

By the Ascoli theorem, we only have to show that $\Phi_Q = \{\varphi_\lambda : \lambda \in Q\}$ is relatively compact in $C(X)$ with respect to the topology of uniform convergence on compact subsets of X.

Let $\{\varphi_{\lambda_\alpha}\}$ be a net in Φ_Q and let K be a compact subset of X. For simplicity, we write φ_α for φ_{λ_α}. Then by (2), we can find a subnet $\{\varphi_{\alpha'}\}$ of $\{\varphi_\alpha\}$ and $\lambda \in C(X, \mathcal{P}_\tau(Y))$ such that

$$\lim_{\alpha'} \sup_{x \in K} \Big|\varphi_{\alpha'}(x) - \int_Y f(x,y)\lambda(x,dy)\Big| = 0.$$

Put $\varphi(x) = \int_Y f(x,y)\,\lambda(x,dy)$ for $x \in X$. Then $\varphi \in C_{\mathrm{b}}(X)$ by (1), and $\lim_{\alpha'} \sup_{x\in K} |\varphi_{\alpha'}(x) - \varphi(x)| = 0$. Hence Φ_Q is relatively compact in $C(X)$.

LEMMA 3. *Let X be a completely regular topological space and let $\{\mu_\alpha\}$ be a net in $\mathcal{P}(X)$ which is uniformly tight. Assume that a net $\{\varphi_\alpha\}$ in $C_{\mathrm{b}}(X)$ satisfies*

(a) *$\{\varphi_\alpha\}$ is uniformly bounded*, *and*

(b) *$\{\varphi_\alpha\}$ is equicontinuous on every compact subset of X.*

If $\mu \in \mathcal{P}_\tau(X)$ and $\mu_\alpha \xrightarrow{w} \mu$, and if $\varphi \in C_{\mathrm{b}}(X)$ and $\varphi_\alpha(x) \to \varphi(x)$ for each $x \in X$, then

$$\lim_\alpha \int_X \varphi_\alpha(x)\mu_\alpha(dx) = \int_X \varphi(x)\mu(dx).$$

Proof. We omit the proof, which is an easy modification of the proof of Theorem 2 in [2].

Now we prove the theorems of this section.

Proof of Theorem 1. First note that any completely regular topological space is uniformizable and it can be totally bounded for uniformities yielding its topology. Thus we assume that X is a uniform k-space and Y is a totally bounded uniform space.

To prove the theorem, we only have to show that for each $f \in U_{\mathrm{b}}(X \times Y)$, we have

$$\lim_\alpha \int_{X\times Y} f(x,y)\,\lambda_\alpha(x,dy)\,\mu_\alpha(dx) = \int_{X\times Y} f(x,y)\,\lambda(x,dy)\,\mu(dx).$$

Fix $f \in U_{\mathrm{b}}(X \times Y)$. By Lemma 2(3), if we put for each $x \in X$,

$$\varphi_\alpha(x) = \int_Y f(x,y)\,\lambda_\alpha(x,dy) \quad \text{and} \quad \varphi(x) = \int_Y f(x,y)\,\lambda(x,dy),$$

then $\{\varphi_\alpha\}$ satisfies (a) and (b) of Lemma 3, and $\varphi_\alpha(x) \to \varphi(x)$ for each $x \in X$, since $\lambda_\alpha(x,\cdot) \xrightarrow{w} \lambda(x,\cdot)$ for each $x \in X$. Consequently, by Lemma 3, we have

$$\lim_\alpha \int_X \varphi_\alpha(x)\,\mu_\alpha(dx) = \int_X \varphi(x)\,\mu(dx),$$

and this implies $\mu_\alpha \circ \lambda_\alpha \xrightarrow{w} \mu \circ \lambda$.

Proof of Theorem 2. As noticed at the beginning of the proof of Theorem 1, we assume that X is a uniform k-space and Y is a totally bounded uniform space.

Since P is uniformly tight, it is relatively compact in $\mathcal{P}_\tau(X)$. Consequently, by Lemma 2(2), for every net $\{\mu_\alpha \circ \lambda_\alpha\}$ in $P \circ Q$ we can find a subnet $\{\mu_{\alpha'} \circ \lambda_{\alpha'}\}$ of $\{\mu_\alpha \circ \lambda_\alpha\}$, $\mu \in \mathcal{P}_\tau(X)$ and $\lambda \in C(X, \mathcal{P}_\tau(Y))$ such that for every $f \in U_{\mathrm{b}}(X \times Y)$ and every compact subset K of X,

$$\mu_{\alpha'} \xrightarrow{w} \mu \tag{3.6}$$

and

$$\lim_{\alpha'} \sup_{x \in K} \Big| \int_Y f(x,y)\,\lambda_{\alpha'}(x,dy) - \int_Y f(x,y)\,\lambda(x,dy) \Big| = 0. \tag{3.7}$$

From (3.7) it follows that $\lambda_{\alpha'}(x,\cdot) \xrightarrow{w} \lambda(x,\cdot)$ for each $x \in X$. So, by Theorem 1 we have $\mu_{\alpha'} \circ \lambda_{\alpha'} \xrightarrow{w} \mu \circ \lambda$, and the proof is complete.

Proof of Theorem 3. By (a), (b) and the Ascoli theorem, Q is relatively compact in $C(X, \mathcal{P}_\tau(Y))$ with respect to the topology of uniform convergence on compact subsets of X, while P is relatively compact in $\mathcal{P}_\tau(X)$ since it is uniformly tight. Hence for every net $\{\mu_\alpha \lambda_\alpha\}$ in PQ, we can find a subnet $\{\mu_{\alpha'} \lambda_{\alpha'}\}$ of $\{\mu_\alpha \lambda_\alpha\}$, $\mu \in \mathcal{P}_\tau(X)$ and $\lambda \in C(X, \mathcal{P}_\tau(Y))$ such that for every $h \in C_{\mathrm{b}}(Y)$ and every compact subset K of X,

$$\mu_{\alpha'} \xrightarrow{w} \mu \tag{3.8}$$

and

$$\lim_{\alpha'} \sup_{x \in K} \Big| \int_Y h(y)\,\lambda_{\alpha'}(x,dy) - \int_Y h(y)\,\lambda(x,dy) \Big| = 0. \tag{3.9}$$

Define bounded continuous functions $\varphi_{\alpha'}$ and φ on X by

$$\varphi_{\alpha'}(x) = \int_Y h(y)\,\lambda_{\alpha'}(x,dy) \quad \text{and} \quad \varphi(x) = \int_Y h(y)\,\lambda(x,dy).$$

Then $\{\varphi_{\alpha'}\}$ satisfies (a) and (b) of Lemma 3, while from (3.9) it follows that $\varphi_{\alpha'}(x) \to \varphi(x)$ for each $x \in X$. Therefore by Lemma 3 we have

$$\lim_{\alpha'} \int_X \varphi_{\alpha'}(x)\,\mu_{\alpha'}(dx) = \int_X \varphi(x)\,\mu(dx),$$

which implies $\mu_{\alpha'}\lambda_{\alpha'} \xrightarrow{w} \mu\lambda$, and hence PQ is relatively compact.

Since Y is a Prokhorov space by assumption, PQ is uniformly tight. Noting that P and PQ are the projections of $P \circ Q \subset \mathcal{P}_\tau(X \times Y)$ onto X and Y respectively, $P \circ Q$ is uniformly tight by an easily verified result that $\Sigma \subset \mathcal{P}(X \times Y)$ is uniformly tight if and only if so are $\Sigma_X = \{\sigma_X : \sigma \in \Sigma\} \subset \mathcal{P}(X)$ and $\Sigma_Y = \{\sigma_Y : \sigma \in \Sigma\} \subset \mathcal{P}(Y)$, where σ_X and σ_Y denote the projections of σ onto X and Y, respectively.

Remark 2. The author does not know whether PQ is uniformly tight or not under the assumptions of Theorem 2. However, we know that even for subsets of probability measures on Suslin spaces, relative compactness does not imply uniform tightness in general (see Example I.6.4 of Fernique [4]).

4. Gaussian transition probabilities. In this section we treat Gaussian transition probabilities which are important examples of equicontinuous sets of transition probabilities. Let us first give necessary information about Gaussian measures needed in the sequel.

Let H be a real separable Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. A Borel probability measure μ on H is said to be *Gaussian* if for each $u \in H$, the function $H \ni w \mapsto \langle w, u\rangle$ is a real (possibly degenerate) Gaussian random variable on the probability measure space $(H, \mathcal{B}(H), \mu)$. Since every Gaussian measure μ on H satisfies

$$\int_H \|w\|^2\,\mu(dw) < \infty,$$

there are a *mean vector* $m\in H$ and a *covariance operator* $S\in\mathcal{S}(H)$ such that

$$\langle m, u\rangle = \int_H \langle w, u\rangle\,\mu(dw) \quad \text{and} \quad \langle Su, v\rangle = \int_H \langle w - m, u\rangle\langle w - m, v\rangle\,\mu(dw)$$

for all $u, v \in H$. Here $\mathcal{S}(H)$ denotes the set of all non-negative symmetric trace class operators on H; it is endowed with the metric topology derived from the trace norm $\|\cdot\|_{\mathrm{tr}}$.

The following lemma is well known for Gaussian measures on a Hilbert space, and acutually shown in Chevet [1], using the method of tensor products, for Gaussian measures on a real separable Banach space which is of type 2 and has the approximation property.

LEMMA 4. (1) *Let A be a set of Gaussian measures on H with mean vectors $\{m_\mu : \mu \in A\}$ and covariance operators $\{S_\mu : \mu \in A\}$. Then A is uniformly tight if and only if $\{m_\mu : \mu \in A\}$ and $\{S_\mu : \mu \in A\}$ are relatively compact in H and $\mathcal{S}(H)$, respectively.*

(2) *Let μ_n $(n \geq 1)$ and μ be Gaussian measures on H with mean vectors m_n and m and with covariance operators S_n and S, respectively. Then $\mu_n \xrightarrow{w} \mu$ if and only if*

$$\lim_{n\to\infty} \|m_n - m\| = 0 \quad \textit{and} \quad \lim_{n\to\infty} \|S_n - S\|_{\mathrm{tr}} = 0.$$

Let X be a topological space. A transition probability λ on $X \times H$ is said to be *Gaussian* if for each $x \in X$, λ_x is a Gaussian measure on H. For each $x \in X$, we denote by $m(x)$ and $S(x)$ the mean vector and the covariance operator of λ_x respectively, and we say that the functions $m : X \to H$ and $S : X \to \mathcal{S}(H)$ are the *mean function* and the *covariance function* of λ, respectively. Since a Gaussian measure is uniquely determined by its mean vector and covariance operator, it is easily verified that a Gaussian transition probability λ is also uniquely determined by its mean function m and covariance function S, and hence we write $\lambda = \mathcal{TN}[m, S]$.

In what follows, let X be a completely regular k-space and assume that every compact subset of X satisfies the first axiom of countability. These conditions are satisfied, for instance, when X is either (1) a first countable locally compact space, or (2) a metric space, or (3) a regular Suslin k-space. From Lemma 4, it follows that a Gaussian transition probability $\lambda = \mathcal{TN}[m, S]$ on $X \times H$ is continuous if and only if $m \in C(X, H)$ and $S \in C(X, \mathcal{S}(H))$, and λ is τ-smooth since every Borel measure on H is τ-smooth.

The following theorem states that for Gaussian transition probabilities, we can express the conditions (a) and (b) of the theorems of Section 3 in terms of the corresponding mean and covariance functions.

THEOREM 4. *Let $\lambda_n = \mathcal{TN}[m_n, S_n]$ $(n \geq 1)$ and $\lambda = \mathcal{TN}[m, S]$ be continuous Gaussian transition probabilities on $X \times H$. Assume that*

(a) *$\{m_n\} \subset C(X, H)$ and $\{S_n\} \subset C(X, \mathcal{S}(H))$ are equicontinuous on every compact subset of X, and*

(b) *$\lim_{n\to\infty} \|m_n(x) - m(x)\| = 0$ and $\lim_{n\to\infty} \|S_n(x) - S(x)\|_{\mathrm{tr}} = 0$ for each $x \in X$.*

Then $\{\lambda_n\}$ satisfies (a) *and* (b) *of Theorems* 1–3*. Therefore for any uniformly tight sequence $\{\mu_n\}$ in $\mathcal{P}_\tau(X)$ converging weakly to $\mu \in \mathcal{P}_\tau(X)$, we have $\mu_n \circ \lambda_n \xrightarrow{w} \mu \circ \lambda$ and $\{\mu_n \circ \lambda_n\}$ is uniformly tight.*

In order to prove the theorem above, we need the following lemma, which is known (see, e.g., Kelley [6; Problem M, p. 241]).

LEMMA 7. *Let $\{\varphi_n\}$ be a sequence of real-valued functions defined on a topological space X which satisfies the first axiom of countability, and let $x \in X$. Assume that*

(a) φ_n *is continuous for all* $n \geq 1$, *and*

(b) φ_n *converges continuously to a function* φ *at* x, *i.e.*, $\lim_{n\to\infty} \varphi_n(x_n) = \varphi(x)$ *for any sequence* $\{x_n\}$ *converging to* x.

Then $\{\varphi_n\}$ *is equicontinuous at* x.

Proof of Theorem 4. By (b) and Lemma 4, $\lambda_n(x,\cdot) \xrightarrow{w} \lambda(x,\cdot)$ for each $x \in X$, and this implies that $\{\lambda_n(x,\cdot)\}$ is uniformly tight for each $x \in X$, since H is a Prokhorov space. Hence the conditions (b) of Theorems 1–3 are valid.

Fix $h \in C_{\mathrm{b}}(X)$ and put for each $x \in X$,

$$\varphi_n(x) = \int_H h(u)\,\lambda_n(x,du) \quad \text{and} \quad \varphi(x) = \int_H h(u)\,\lambda(x,du).$$

Now we show that $\{\varphi_n\} \subset C(H)$ is equicontinous on every compact subset K of X in order to prove that the conditions (a) of Theorems 1–3 are valid. From (a) and (b) of the present theorem, it is easily proved that m_n and S_n converge continuously on K to m and S, respectively. Hence if a sequence $\{x_n\}$ in K converges to $x \in K$, then $\lambda_n(x_n,\cdot) \xrightarrow{w} \lambda(x,\cdot)$ by Lemma 4. From this we have $\varphi_n(x_n) \to \varphi(x)$, and hence φ_n converges continuously to φ on K. Since K is first countable by assumption, $\{\varphi_n\}$ is equicontinuous on K by Lemma 7. Consequently, by Theorem 1 we have $\mu_n \circ \lambda_n \xrightarrow{w} \mu \circ \lambda$, and from Theorem 3 it follows that $\{\mu_n \circ \lambda_n\}$ is uniformly tight since H is a Prokhorov space.

Remark 3. In Theorem 4, the condition that $\{\mu_n\}$ is uniformly tight is automatically satisfied if X is a Prokhorov space.

Remark 4. We can obtain a similar result to Theorem 4 for Gaussian transition probabilities on nuclear spaces such as, for instance, the space of slowly increasing functions and the space of distributions.

5. Examples and counter-examples. In this section we give the following examples and counter-examples concerning the uniform tightness and the convergence of compound probability measures:

- P is *not* uniformly tight and Q is *not* equicontinuous, but PQ is uniformly tight (see Example 2(2)).
- P is uniformly tight, but PQ and $P \circ Q$ are *not* (see Example 2(3)).
- Q is *not* equicontinuous, but $P \circ Q$ and PQ are uniformly tight (see Example 3(2)).
- $\mu_\alpha \xrightarrow{w} \mu$, $\lambda_\alpha(x,\cdot) \xrightarrow{w} \lambda(x,\cdot)$ for each $x \in X$ and $\mu_\alpha \circ \lambda_\alpha$ converges weakly, but $\mu \circ \lambda$ is *not* a limit point of $\mu_\alpha \circ \lambda_\alpha$ (see Example 3(3)).

In what follows, δ_x denotes the Dirac measure concentrated at x, that is, $\delta_x(B) = 1$ if $x \in B$, and $\delta_x(B) = 0$ if $x \notin B$.

EXAMPLE 2. *Let* $X = Y = \mathbb{R}$. *For each* $n \geq 1$, *put*

$$s_n^2(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ n^2 x & \text{for } 0 < x \leq 1/n, \\ 2n - n^2 x & \text{for } 1/n < x \leq 2/n, \\ 0 & \text{for } 2/n < x. \end{cases}$$

Define a continuous τ*-smooth transition probability* λ_n *on* $\mathbb{R} \times \mathbb{R}$ *by* $\lambda_n = \mathcal{TN}[0, s_n^2]$ *and put* $Q = \{\lambda_n\}$.

(1) *For each* $x \in \mathbb{R}$, $Q[x] = \{\lambda_n(x, \cdot)\}$ *is uniformly tight*, *but* Q *is not equicontinuous at* $x = 0$.

(2) *Put* $P = \{\delta_n\}$. *Then* P *and* $P \circ Q$ *are not uniformly tight*, *while* PQ *is uniformly tight.*

(3) *Put* $P = \{\delta_{1/n}\}$. *Then* P *is uniformly tight*, *but* PQ *and* $P \circ Q$ *are not uniformly tight.*

Proof. Put

$$\varphi_n(x) = \int_{\mathbb{R}} e^{iv}\, \lambda_n(x, dv) = \exp\{-\tfrac{1}{2} s_n^2(x)\}$$

for each $n \geq 1$ and each $x \in \mathbb{R}$. Then $\varphi_n(1/n) = e^{-n/2}$ and $\varphi_n(0) = 1$. Therefore $\{\varphi_n\}$ is not equicontinuous at $x = 0$, and neither is Q. On the other hand, since $\sup_{n \geq 1} s_n^2(x) < \infty$ for each $x \in \mathbb{R}$, $Q[x]$ is uniformly tight. Consequently, (1) is proved.

Before starting to prove (2) and (3), note that a subset A of $\mathcal{P}(\mathbb{R}^n)$ is uniformly tight if and only if the set $\{\widehat{\mu} : \mu \in A\}$ of characteristic functions defined by

$$\widehat{\mu}(p_1, \ldots, p_n) = \int_{\mathbb{R}^n} e^{i\Sigma_{j=1}^n p_j u_j}\, \mu(du_1, \ldots, du_n)$$

for $p = (p_1, \ldots, p_n) \in \mathbb{R}^n$ is equicontinuous at $p = 0 \in \mathbb{R}^n$.

Since $\widehat{\delta}_n(p) = e^{inp}$ and $(\delta_n \circ \lambda_n)^\wedge(p, q) = e^{inp}$ for all $p, q \in \mathbb{R}$, $\{\widehat{\delta}_n\}$ and $\{(\delta_n \circ \lambda_n)^\wedge\}$ are not equicontinuous at $p = 0$ and at $(p, q) = (0, 0)$, respectively. Therefore P and $P \circ Q$ are not uniformly tight. On the other hand, PQ is uniformly tight, since $(\delta_n \lambda_n)^\wedge(q) = 1$ for all $q \in \mathbb{R}$ and hence $\{(\delta_n \lambda_n)^\wedge\}$ is equicontinuous at $q = 0$. Consequently, (2) is proved, and (3) is proved similarly.

EXAMPLE 3. Let $X = Y = \mathbb{R}$. For each $n \geq 1$, put

$$s_n^2(x) = \frac{n|x|}{1 + n^2 x^2}, \qquad x \in \mathbb{R}.$$

Define a continuous τ-smooth transition probability λ_n on $\mathbb{R} \times \mathbb{R}$ by $\lambda_n = \mathcal{TN}[0, s_n^2]$ and put $Q = \{\lambda_n\}$.

(1) For each $x \in \mathbb{R}$, $Q[x]$ is uniformly tight, but Q is not equicontinuous at $x = 0$.

(2) Put $P = \{\delta_{1/n}\}$. Then $P \circ Q$ and PQ are uniformly tight.

(3) Put $\mu_n = \delta_{1/n}$, $\mu = \delta_0$ and $\lambda(x, \cdot) = \delta_0$ for each $x \in \mathbb{R}$. Then, though $\mu_n \xrightarrow{w} \mu$, $\lambda_n(x, \cdot) \xrightarrow{w} \lambda(x, \cdot)$ for every $x \in \mathbb{R}$ and $\mu_n \circ \lambda_n$ converges weakly, $\mu \circ \lambda$ is not a limit point of $\mu_n \circ \lambda_n$ (actually, $\mu_n \lambda_n \xrightarrow{w} \mu\gamma$ and $\mu_n \circ \lambda_n \xrightarrow{w} \mu \circ \gamma$, where $\gamma = \mathcal{TN}[0, 1/2]$).

Proof. (1) and (2) are proved in the same way as Example 2. By Levy's continuity theorem, (3) is proved by noticing that

$$(\mu_n \circ \lambda_n)^\wedge(p, q) \to \exp\{-\tfrac{1}{4}q^2\} = (\mu \circ \gamma)^\wedge(p, q) \quad \text{as } n \to \infty,$$

while $(\mu \circ \lambda)^\wedge(p, q) = 1$ for all $(p, q) \in \mathbb{R} \times \mathbb{R}$.

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DEPARTMENT OF MATHEMATICS
FACULTY OF ENGINEERING
SHINSHU UNIVERSITY
WAKASATO, NAGANO 380, JAPAN
E-mail: JKAWABE@GIPWC.SHINSHU-U.AC.JP

Reçu par la Rédaction le 24.5.1993