COLLOQUIUM MATHEMATICUM

VOL. LXVII 1994 FASC. 2

SECTIONAL CURVATURES OF MINIMAL HYPERSURFACES IMMERSED IN S^{2n+1}

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Introduction. Let M be a compact minimal hypersurface in the unit sphere S^{2n+1} $(n > 1)$ with standard Sasakian structure (ϕ, ξ, η, g). We suppose that M is tangent to the structure vector field ξ of S^{2n+1}. We consider the sectional curvature K_{ts} of M spanned by e_t and e_s orthogonal to the structure vector ξ. The purpose of the present paper is to prove that if $K_{ts} + 3g(Je_t, e_s)^2 \geq 1/(2n-1)$, then M is congruent to $S^{2n-1}(r_1) \times S^1(r_2)$, where J is defined by $\phi X = JX + u(X)C$ for any vector X tangent to M, C being the unit normal of M and $u(X) = -g(X, \phi C)$.

The sectional curvature of M spanned by ξ and $-\phi C$ is always zero. Thus we must consider the sectional curvatures K_{ts} on the plane section orthogonal to ξ. Our result is a pinching theorem on a hypersurface M with induced structure from the Sasakian structure on S^{2n+1}.

We would like to thank the referee for his kind advice to complete our result.

1. Preliminaries. Let S^{2n+1} be the $(2n+1)$-dimensional unit sphere. It is well known that S^{2n+1} admits a standard Sasakian structure (ϕ, ξ, η, g). We have

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \quad \eta(\phi X) = 0,$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi)$$

for any vector fields X and Y on S^{2n+1}. We denote by $\overline{\nabla}$ the operator of covariant differentiation with respect to the metric g on S^{2n+1}. We then have

$$\overline{\nabla}_X \xi = \phi X, \quad (\overline{\nabla}_X \phi)Y = -g(X, Y)\xi + \eta(Y)X$$

for any vector fields X and Y on S^{2n+1}.

1991 *Mathematics Subject Classification*: 53C25, 53C40.

Research supported by the basic science research institute program, Korea Ministry of Education, 1992-114.

Let M be an $2n$-dimensional hypersurface in S^{2n+1}. *Throughout this paper, we assume that M is tangent to the structure vector field ξ of S^{2n+1}.*

We denote by the same g the Riemannian metric tensor field induced on M from S^{2n+1}. The operator of covariant differentiation with respect to the induced connection on M will be denoted by ∇. Then the Gauss and Weingarten formulas are, respectively,

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)C \quad \text{and} \quad \overline{\nabla}_X C = -AX$$

for any vector fields X and Y tangent to M, where C denotes the unit normal vector field of M. We call the A appearing here the *second fundamental form* of M. It can be considered as a symmetric $(2n, 2n)$-matrix. If $(\nabla_X A)Y = 0$ for any vector fields X and Y tangent to M, then A is said to be *parallel*.

We put $\phi C = -U$. Then U is a unit field tangent to M. We define a 1-form u by $u(X) = g(U, X)$ for any vector field X tangent to M, and we put

$$\phi X = JX + u(X)C,$$

where JX is the tangential part of ϕX. Then J is an endomorphism on the tangent bundle $T(M)$, satisfying

$$\begin{aligned} &JU = 0, \quad J\xi = 0, \quad u(\xi) = 0, \quad u(U) = 1, \\ &J^2 X = -X + u(X)U + \eta(X)\xi, \quad g(JX, Y) = -g(X, JY). \end{aligned} \tag{1.1}$$

For any vector field X tangent to M, we have

$$\overline{\nabla}_X \xi = \phi X = \nabla_X \xi + g(AX, \xi)C,$$

and so

$$\nabla_X \xi = JX, \quad A\xi = U. \tag{1.2}$$

Moreover, using the Gauss and Weingarten formulas, we obtain (cf. Yano–Kon [3, 4])

$$\nabla_X U = JAX, \tag{1.3}$$

$$(\nabla_X J)Y = u(Y)AX - g(AX, Y)U - g(X, Y)\xi + \eta(Y)X. \tag{1.4}$$

We denote by R the Riemannian curvature tensor of M. Then the Gauss and Codazzi equations of M are, respectively,

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(AY, Z)AX - g(AX, Z)AY, \tag{1.5}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = 0. \tag{1.6}$$

It is well known that a connected complete hypersurface in a sphere with two constant principal curvatures is locally isometric to the product of two spheres (cf. Ryan [2]). We can also prove that a hypersurface in a sphere with parallel second fundamental form has at most two constant principal curvatures (cf. Ryan [3]).

2. Pinching theorem. Let M be a minimal hypersurface in S^{2n+1}. We use the convention that the ranges of indices are

$$i, j, k = 0, 1, \dots, 2n-1; \quad r, s, t = 1, \dots, 2n-1.$$

From (1.2) we can choose an orthonormal basis $e_0 = \xi, e_1, \dots, e_{2n-1}$ of $T_x(M)$ such that

$$Ae_t = \lambda_t e_t + u(e_t)\xi, \quad t = 1, \dots, 2n-1.$$

Generally, we obtain

$$\begin{aligned} g(\nabla^2 A, A) &= \sum g((R(e_i, e_j)A)e_i, Ae_j) \\ &= \sum g(R(e_i, e_j)Ae_i, Ae_j) - \sum g(AR(e_i, e_j)e_i, Ae_j). \end{aligned} \tag{2.1}$$

We now compute the right hand side of (2.1). First of all, we have

$$\begin{aligned} &\sum g(R(e_i, e_j)Ae_i, Ae_j) \\ &\quad = 2\sum g(R(\xi, e_t)A\xi, Ae_t) + \sum g(R(e_t, e_s)Ae_t, Ae_s) \\ &\quad = -2\sum \lambda_t^2 g(Je_t, Je_t) - \sum \lambda_t\lambda_s K_{ts} - 2\sum \lambda_t^2 g(Je_t, Je_t) \\ &\quad = -4\sum \lambda_t^2 g(Je_t, Je_t) - \sum \lambda_t\lambda_s K_{ts}, \end{aligned}$$

where K_{ts} denotes the sectional curvature spanned by e_t and e_s, and

$$\begin{aligned} &-\sum g(AR(e_i, e_j)e_i, Ae_j) \\ &\quad = -\sum g(R(\xi, e_s)\xi, A^2 e_s) - \sum g(AR(e_t, e_s)e_t, Ae_s) \\ &\qquad - \sum g(R(e_t, \xi)e_t, AU) \\ &\quad = \sum \lambda_t^2 g(Je_t, Je_t) + \sum \lambda_t^2 K_{ts} + 2n \\ &\qquad - 2g(AU, AU) + (2n-1) - g(AU, AU) \\ &\quad = \sum \lambda_t^2 K_{ts} + \sum \lambda_t^2 g(Je_t, Je_t) - 3g(AU, AU) + (4n-1). \end{aligned}$$

Substituting these equations into (2.1), we find

$$\begin{aligned} g(\nabla^2 A, A) &= \sum \lambda_t^2 K_{ts} - \sum \lambda_t\lambda_s K_{ts} \\ &\quad - 3\sum \lambda_t^2 g(Je_t, Je_t) - 3g(AU, AU) + (4n-1). \end{aligned} \tag{2.2}$$

On the other hand,

$$\begin{aligned} \sum \lambda_t^2 g(Je_t, Je_t) + g(AU, AU) &= \sum g(JAe_t, JAe_t) + g(AU, AU) \\ &= \sum g(Ae_t, Ae_t) = \operatorname{Tr} A^2 - 1. \end{aligned}$$

Thus (2.2) becomes

$$g(\nabla^2 A, A) = \tfrac{1}{2}\sum(\lambda_t - \lambda_s)^2 K_{ts} - 3(\operatorname{Tr} A^2 - 1) + (4n-1), \tag{2.3}$$

and hence

$$
\begin{aligned}
(2.4)\qquad & -\tfrac{1}{2}\Delta \operatorname{Tr} A^2 + g(\nabla A, \nabla A) \\
&= -\tfrac{1}{2}\sum(\lambda_t-\lambda_s)^2 K_{ts} + 3\operatorname{Tr} A^2 - (4n+2) \\
&= -\tfrac{1}{2}\sum(\lambda_t-\lambda_s)^2 (K_{ts} + 3g(Je_t,e_s)^2) \\
&\quad + \tfrac{3}{2}\sum(\lambda_t-\lambda_s)^2 g(Je_t,e_s)^2 + 3\operatorname{Tr} A^2 - (4n+2) \\
&= -\tfrac{1}{2}\sum(\lambda_t-\lambda_s)^2 (K_{ts} + 3g(Je_t,e_s)^2) \\
&\quad + \tfrac{3}{2}|[J,A]|^2 + 3\operatorname{Tr} A^2 - (4n+2).
\end{aligned}
$$

We also have

$$
\begin{aligned}
g(\nabla A, \nabla A) &= \sum g((\nabla_t A)e_s, e_r)^2 + 3\sum g((\nabla_t A)\xi, (\nabla_t A)\xi) \\
&= \sum g((\nabla_t A)e_s, e_r)^2 + 3|[J,A]|^2,
\end{aligned}
$$

where ∇_t denotes covariant differentiation in the direction of e_t. Thus (2.4) reduces to

$$
\begin{aligned}
(2.5)\qquad & -\tfrac{1}{2}\Delta \operatorname{Tr} A^2 + \sum g((\nabla_t A)e_s, e_r)^2 \\
&= -\tfrac{1}{2}\sum(\lambda_t-\lambda_s)^2 (K_{ts} + 3g(Je_t,e_s)^2) \\
&\quad - \tfrac{3}{2}|[J,A]|^2 + 3\operatorname{Tr} A^2 - (4n+2).
\end{aligned}
$$

Since

$$\operatorname{Tr} A^2 = \sum g(AJe_i, AJe_i) + g(AU, AU) + g(A\xi, A\xi),$$

(1.2) and (1.4) imply

$$\operatorname{div}(\nabla_U U) = 2n - \operatorname{Tr} A^2 + \tfrac{1}{2}|[J,A]|^2.$$

Hence we have

$$
\begin{aligned}
(2.6)\qquad & -\tfrac{1}{2}\Delta \operatorname{Tr} A^2 + \sum g((\nabla_t A)e_s, e_r)^2 \\
&= -\tfrac{1}{2}\sum(\lambda_t-\lambda_s)^2 (K_{ts} + 3g(Je_t,e_s)^2) + (2n-2) - 3\operatorname{div}(\nabla_U U).
\end{aligned}
$$

Suppose that $K_{ts} + 3g(Je_t,e_s)^2 \geq 1/(2n-1)$. Then, using $\operatorname{Tr} A^2 = \sum \lambda_t^2 + 2$, we have

$$
\begin{aligned}
-\tfrac{1}{2}\Delta \operatorname{Tr} A^2 + \sum g((\nabla_t A)e_s, e_r)^2 &\leq -\sum \lambda_t^2 + (2n-2) - 3\operatorname{div}(\nabla_U U) \\
&= -\operatorname{Tr} A^2 + 2n - 3\operatorname{div}(\nabla_U U) \\
&= -\tfrac{1}{2}|[J,A]|^2 - 2\operatorname{div}(\nabla_U U).
\end{aligned}
$$

If M is compact, we have $g((\nabla_t A)e_s, e_r) = 0$ for all t, s and r, that is, A is η-parallel and $JA = AJ$. Then $g((\nabla_\xi A)X, Y) = g([J,A]X, Y) = 0$ for any vector fields X and Y tangent to M. Hence, by (1.6), the second fundamental form A of M is parallel. Thus M has two constant principal curvatures. Since $JA = AJ$ we may set

$$AU = aU + \xi, \qquad a = g(AU, U).$$

Then we can prove

LEMMA 2.1. *Let M be a hypersurface in S^{2n+1}. If $AU = aU + \xi$, then a is a constant.*

Proof. From the assumption we have

$$(\nabla_X A)U + AJAX = (Xa)U + JAX + JX.$$

Using the Codazzi equation, we find

$$\begin{aligned} g((\nabla_X A)U, Y) &- g((\nabla_Y A)U, X) \\ &= (Xa)u(Y) + ag(JAX, Y) + g(JX, Y) - g(JAAX, Y) \\ &\quad - (Ya)u(X) - ag(JAY, X) - g(JY, X) + g(AJAY, X) = 0. \end{aligned}$$

Hence

$$\begin{aligned} (Xa)u(Y) - (Ya)u(X) + ag((JA + AJ)X, Y) \\ + 2g(JX, Y) - 2g(AJAX, Y) = 0. \end{aligned}$$

Putting $X = U$, we obtain $Ya = (Ua)u(Y)$. Therefore

$$ag((JA + AJ)X, Y) + 2g(JX, Y) - 2g(AJAX, Y) = 0.$$

We put $\beta = Ua$. Then $Xa = \beta u(X)$ and $Ya = \beta u(Y)$. Thus

$$\nabla_X \nabla_Y a = (X\beta)u(Y) + \beta g(Y, JAX) + \beta g(U, \nabla_X Y),$$

which yields

$$R(X, Y)a = (X\beta)u(Y) - (Y\beta)u(X) + \beta g((JA + AJ)X, Y) = 0.$$

Putting $X = U$ or $Y = U$, we find $(U\beta)u(Y) = Y\beta$ and $(U\beta)u(X) = X\beta$. Consequently, $\beta g((JA + AJ)X, Y) = 0$. If we assume that $AJ + JA = 0$, then $g(JX, Y) = g(AJAX, Y)$, which implies

$$g(JX, JX) = g(JAX, AJX) = -g(JAX, JAX).$$

Hence $JX = 0$. This is a contradiction. Consequently, $\beta = 0$, that is, $Ua = 0$ and then $Xa = (Ua)u(X) = 0$ for any vector field X tangent to M. This shows that a is a constant.

THEOREM 2.1. *Let M be a compact minimal hypersurface in S^{2n+1} $(n > 1)$. If the sectional curvature K of M satisfies*

$$K_{ts} + 3g(Je_t, e_s)^2 \geq 1/(2n - 1),$$

then M is congruent to $S^{2n-1}(r_1) \times S^1(r_2)$, where $r_1 = ((2n - 1)/(2n))^{1/2}$ and $r_2 = (1/(2n))^{1/2}$.

Proof. Since $AJ = JA$, we can choose an orthonormal basis $e_0 = \xi$, $e_1 = U$, $e_2, \ldots, e_{2n-2}$ such that $e_{n-1+p} = Je_p$ $(p = 1, \ldots, n - 1)$ and

$$Ae_p = \lambda_p e_p, \quad AJe_p = \lambda_p Je_p, \quad p = 2, \ldots, n - 1.$$

We consider the matrix

$$\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$$

where $a = g(AU, U)$, $g(AU, \xi) = 1$ and $g(A\xi, \xi) = 0$. Its eigenvalues λ and μ satisfy $t^2 - at - 1 = 0$, and hence $\lambda + \mu = a$ and $\lambda\mu = -1$. Moreover, for any p, q $(= 1, \ldots, n-1)$, $p \neq q$,

$$\begin{aligned} 0 &= g((R(e_p, e_q)A)e_p, Ae_q) \\ &= g(R(e_p, e_q)Ae_p, Ae_q) - g(AR(e_p, e_q)e_p, Ae_q) = -\tfrac{1}{2}(\lambda_p - \lambda_q)^2 K_{pq}. \end{aligned}$$

From the assumption we see that $K_{pq} + 3g(Je_p, e_q)^2 = K_{pq} > 0$. Hence $\lambda_p = \lambda_q$ for all p and q. Consequently, we can put

$$\lambda_p = \lambda, \qquad p = 2, \ldots, n-1.$$

Then we may set, from the minimality of M, $\lambda = 1/(2n-1)^{1/2}$ and $\mu = -(2n-1)^{1/2}$. Therefore, M has two constant principal curvatures with multiplicities $2n-2$ and 1. From this and a well known theorem (cf. Ryan [2]) we have our result (see also Theorem 7.1 in [4]).

Remark. Let $\mathbb{C}P^n$ denote the complex n-dimensional projective space equipped with the Fubini–Study metric normalized so that the maximum sectional curvature is 4. We suppose that the following diagram is commutative:

$$\begin{array}{ccc} M & \longrightarrow & S^{2n+1} \\ \pi \downarrow & & \downarrow \pi \\ N & \longrightarrow & \mathbb{C}P^n \end{array}$$

where M is a hypersurface in S^{2n+1} tangent to the structure vector field ξ of S^{2n+1}, N is a real hypersurface in $\mathbb{C}P^n$ and the vertical arrows are Riemannian fiber bundles (cf. [5; Chapter V]). Then the sectional curvatures K of M and K' of N satisfy

$$K'(X, Y) = K(X^*, Y^*) + 3g(X^*, JY^*)^2$$

for any vectors X and Y tangent to N, where * denotes the horizontal lift with respect to the connection η (see [5; p. 144, Lemma 1.2]).

On the other hand, Kon [1] proved the following theorem: Let N be a compact real minimal hypersurface in $\mathbb{C}P^n$. If the sectional curvature K' of N satisfies $K' \geq 1/(2n-1)$, then N is the geodesic hypersphere $\pi(S^{2n-1}(r_1) \times S^1(r_2))$, where $r_1 = ((2n-1)/(2n))^{1/2}$ and $r_2 = (1/(2n))^{1/2}$. Our main theorem corresponds to the theorem above in case of hypersurfaces in an odd-dimensional sphere with contact structure.

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Reçu par la Rédaction le 9.6.1993;
en version modifiée le 22.3.1994