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ON STRONGLY CLOSED SUBALGEBRAS OF $B(X)$

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Let X be a real or complex Banach space. The strong topology on the algebra $B(X)$ of all bounded linear operators on X is the topology of pointwise convergence of nets of operators. It is given by a basis of neighbourhoods of the origin consisting of sets of the form

$$U(\varepsilon; x_1, \ldots, x_n) = \{T \in B(X) : \|Tx_i\| < \varepsilon,\ i = 1, \ldots, n\}, \tag{1}$$

where $x_1, \ldots, x_n$ are linearly independent elements of X and ε is a positive real number. Closure in the strong topology will be called *strong closure* for short. It is well known that the strong closure of a subalgebra of $B(X)$ is again a subalgebra. In this paper we study strongly closed subalgebras of $B(X)$, in particular, maximal strongly closed subalgebras. Our results are given in Section 1, while in Section 2 we give the motivation for this study and pose several open questions.

1. Maximal strongly closed algebras of operators. Call a proper strongly closed subalgebra A of $B(X)$ a *maximal strongly closed algebra* (m.s.c.a.) if for any subalgebra A_1 satisfying $A \subset A_1 \subset B(X)$ we have either $A_1 = A$, or A_1 is a strongly dense subalgebra of $B(X)$. Let X_0 be a proper closed subspace of X, i.e. $(0) \neq X_0 \neq X$, and put

$$\mathcal{A}(X_0) = \{T \in B(X) : TX_0 \subset X_0\}.$$

One can easily see that $\mathcal{A}(X_0)$ is a proper subalgebra of $B(X)$.

PROPOSITION 1. *For any proper closed subspace X_0 of X, the algebra $\mathcal{A}(X_0)$ is a maximal strongly closed algebra.*

Proof. If $T \notin \mathcal{A}(X_0)$, then there is an x_0 in X_0 such that

$$\inf\{\|Tx_0 - x\| : x \in X_0\} > \varepsilon > 0. \tag{2}$$

Let $U(\varepsilon; x_0)$ be as in (1). Then (2) implies that $T + U(\varepsilon; x_0) \cap \mathcal{A}(X_0) = \emptyset$, which means that $\mathcal{A}(X_0)$ is strongly closed. It remains to be shown that

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$\mathcal{A}(X_0)$ is maximal, i.e. $\mathrm{alg}(\mathcal{A}(X_0), Q)$ is strongly dense in $B(X)$ for every Q in $B(X) \setminus \mathcal{A}(X_0)$. So, let $Q \notin \mathcal{A}(X_0)$. Then there is an $x_0 \in X_0$ such that $y_0 = Qx_0 \notin X_0$. We shall be done if we show that for any linearly independent elements $x_1, \ldots, x_n$ in X and any $y_1, \ldots, y_n \in X$ there is an operator T in $\mathrm{alg}(\mathcal{A}(X_0), Q)$ with $Tx_i = y_i,\ i = 1, \ldots, n$.

Suppose that $x_1, \ldots, x_k \in X \setminus X_0$ and $x_{k+1}, \ldots, x_n \in X_0$ (if $k = 0$ all x_i are in X_0 and if $k = n$ they are in $X \setminus X_0$), and choose $f_i \in X^*$ with $f_i(x_j) = \delta_{ij},\ 1 \le i, j \le n$, where δ_{ij} is the Kronecker symbol. We assume, moreover, that f_i is in $X_0^\perp = \{f \in X^* : X_0 \subset \ker f\}$ for $1 \le i \le k$. Thus for $1 \le i \le k$ the one-dimensional operators $f_i \otimes x_i$ are in $\mathcal{A}(X_0)$. Choose f_0 in X^* so that $f_0 \in X_0^\perp$ and $f_0(y_0) = 1$. Then $f_0 \otimes y_i$ and $f_i \otimes x_0$ are in $\mathcal{A}(X_0)$ for $i = 1, \ldots, n$. Define

$$T_i = \begin{cases} f_i \otimes y_i & \text{for } 1 \le i \le k, \\ (f_0 \otimes y_i)Q(f_i \otimes x_0) & \text{for } k+1 \le i \le n. \end{cases}$$

Clearly all T_i are in $\mathrm{alg}(\mathcal{A}(X_0), Q)$ and $T_i x_j = \delta_{ij} y_j,\ 1 \le i, j \le n$. Thus $T = T_1 + \ldots + T_n$ is as required.

PROPOSITION 2. *If* $\dim X_0 < \infty$ *or* $\operatorname{codim} X_0 < \infty$, *then* $\mathcal{A}(X_0)$ *is maximal in* $B(X)$ *in the sense that if* A_1 *is a subalgebra of* $B(X)$ *satisfying* $\mathcal{A}(X_0) \subset A_1 \subset B(X)$, *then either* $A_1 = \mathcal{A}(X_0)$, *or* $A_1 = B(X)$.

Proof. Assume first that $\dim X_0 < \infty$, and let $X_0 = \mathrm{span}\{x_1, \ldots, x_n\}$, where $x_1, \ldots, x_n$ are linearly independent. We have to show that $\mathrm{alg}(\mathcal{A}(X_0), Q) = B(X)$ for every Q in $B(X) \setminus \mathcal{A}(X_0)$. Observe first that $Qx_i \notin X_0$ for some i, since otherwise $Q \in \mathcal{A}(X_0)$. We can assume that $y_1 = Qx_1 \notin X_0$.

Let $T \in B(X)$ and put $u_i = Tx_i,\ i = 1, \ldots, n$. Choose P_i, V_i in $\mathcal{A}(X_0)$ so that $P_i y_1 = u_i$, $V_i x_i = x_1$ and $V_j x_i = 0$ for $i \ne j$ (one can easily construct such operators on $\mathrm{span}\{y_1, x_1, \ldots, x_n\}$ and extend them to the whole of X using the fact that they are finite-dimensional). We have $P_i Q V_i x_i = u_i$ and $P_j Q V_j x_i = 0$ for $i \ne j$. Thus

$$\Big(\sum_{j=1}^n P_j Q V_j\Big) x_i = u_i, \quad \text{or} \quad \Big(T - \sum_{j=1}^n P_j Q V_j\Big) x_i = 0 \quad \text{for } i = 1, \ldots, n.$$

This means that $R = T - \sum_{j=1}^n P_j Q V_j$ is in $\mathcal{A}(X_0)$, and so $T = R - \sum_{j=1}^n P_j Q V_j$ is in $\mathrm{alg}(\mathcal{A}(X_0), Q)$. Hence, in this case $\mathrm{alg}(\mathcal{A}(X_0), Q) = B(X)$.

Similarly we treat the case $\operatorname{codim} X_0 < \infty$. We have $\dim X_0^\perp = \operatorname{codim} X_0$, so that there are linearly independent functionals $f_1, \ldots, f_n$ such that $X_0^\perp = \mathrm{span}\{f_1, \ldots, f_n\}$. Choose x_i in X so that $f_i(x_j) = \delta_{ij},\ 1 \le i, j \le n$. Let $Q \in B(X) \setminus \mathcal{A}(X_0)$. We have to show that $\mathrm{alg}(\mathcal{A}(X_0), Q) = B(X)$. Observe first that there is an i such that $Q^* f_i$ is not in X_0. Otherwise $Q^* X_0^\perp \subset X_0^\perp$, which means that $f(Qx) = 0$ for every f in $X_0^\perp$ and every x in X_0. But this implies that $Qx \in X_0$ whenever $x \in X_0$, or $Q \in \mathcal{A}(X_0)$—a contradiction. Therefore we can assume that $q_1 = Q^* f_1 \notin X_0$.

Choose x_0 in X_0 so that $q_1(x_0) = 1$. Let $T \in B(X)$ and put $h_i = T^* f_i$, $i = 1, \ldots, n$. Set $P_i = h_i \otimes x_0$, so that $P_i \in \mathcal{A}(X_0)$, $i = 1, \ldots, n$. Put $V_i = f_1 \otimes x_i$, $i = 1, \ldots, n$. Then $V_i \in \mathcal{A}(X_0)$ and

$$V_j^* f_i = (x_j \otimes f_1) f_i = f_i(x_j) f_1 = \delta_{ij} f_1, \qquad 1 \le i, j \le n.$$

Thus

$$P_j^* Q^* V_j^* f_i = \delta_{ij} P_j^* Q^* f_1 = \delta_{ij} P_j^* q_1 = \delta_{ij} h_j.$$

This implies

$$\Big(\sum_{j=1}^{n} P_j^* Q^* V_j^*\Big) f_i = h_i = T^* f_i, \qquad i = 1, \ldots, n,$$

or

$$\Big(T^* - \sum_{j=1}^{n} P_j^* Q^* V_j^*\Big) f = 0 \qquad \text{for all } f \text{ in } X_0^{\perp}.$$

This means that

$$f\Big(\Big[T - \sum_{j=1}^{n} V_j Q P_j\Big] x\Big) = 0$$

for all f in $X_0^{\perp}$ and all x in X, hence $\operatorname{im}[T - \sum_{j=1}^{n} V_j Q P_j] \subset X_0$. Thus $R = T - \sum_{j=1}^{n} V_j Q P_j \in \mathcal{A}(X_0)$, so that we have $T = R - \sum_{j=1}^{n} V_j Q P_j \in \operatorname{alg}(\mathcal{A}(X_0), Q)$. The conclusion follows.

The following result is a partial converse to Proposition 2.

Proposition 3. *Let H be a separable Hilbert space, and let H_0 be a closed subspace of H with $\dim H_0 = \operatorname{codim} H_0 = \infty$. Then there is a norm closed subalgebra A of $B(H)$ such that*

$$\mathcal{A}(H_0) \subsetneq A \subsetneq B(H). \tag{3}$$

Proof. Choose an orthonormal basis $(e_i)_{i=1}^{\infty}$ in H so that H_0 is the closure of $\operatorname{span}\{e_{2i-1}\}_{i=1}^{\infty}$. Put $Qx = (x, e_1) e_2$, so that $Q \notin \mathcal{A}(H_0)$. Let S be the unilateral shift given by $Se_i = e_{i+1}$. We shall show that S does not belong to the norm closure A of the algebra $A_0 = \operatorname{alg}(\mathcal{A}(H_0), Q)$, so that A is the desired algebra.

It is sufficient to show that

$$\lim_{i} (W e_{2i-1}, e_{2i}) = 0 \tag{4}$$

for all W in A_0. Since $I \in A_0$ and $Q^2 = 0$, each W in A_0 can be written as

$$W = \sum_{i=1}^{k} V_i + R_0, \tag{5}$$

where $R_0 \in \mathcal{A}(H_0)$ and each V_i is of the form

$$V = R_m Q R_{m-1} Q \ldots R_2 Q R_1 \tag{6}$$

with $R_i \in \mathcal{A}(H_0)$. Since $(R_0 e_{2i-1}, e_{2i}) = 0$ for all i, in view of (5) it is sufficient to prove (4) for $W = V$ of the form (6). We have

$$V e_{2i-1} = c_i R_m e_2,$$

where $c_i = (R_1 e_{2i-1}, e_1)(R_2 e_2, e_1) \ldots (R_{m-1} e_2, e_1)$, so that the sequence (c_i) is bounded by $\|R_1\| \ldots \|R_{m-1}\|$. Now (4) follows from $\lim_i (R_m e_2, e_{2i}) = 0$.

2. Motivation and open problems. As the motivation for the study of strongly closed algebras of operators we mention two well-known problems.

I. *The problem of Fell and Doran*. Let X be a topological vector space and $L(X)$ the algebra of all continuous linear operators on X. Let A be an algebra over the same field of scalars as X (i.e. $\mathbb{R}$ or $\mathbb{C}$). A *representation* T of A on X is a homomorphism $a \to T_a$ of A into $L(X)$, with $T_e = I$ if A has unit e. A representation T is said to be *irreducible* if no proper closed subspace X_0 of X is invariant with respect to all T_a, or, equivalently, if every orbit

$$O(T; x_0) = \{T_a x_0 : a \in A\},$$

where $x_0 \neq 0$, is dense in X. Similarly, T is *n-fold irreducible* ($n \in \mathbb{N}$) if for any n-tuple $x_1, \ldots, x_n$ of linearly independent elements of X the orbit

$$O(T; x_1, \ldots, x_n) = \{(T_a x_1, \ldots, T_a x_n) \in X^n : a \in A\}$$

is dense in X^n equipped with the Cartesian product topology. Finally, T is *totally irreducible* if it is n-fold irreducible for every n in $\mathbb{N}$. The problem posed in [1] by Fell and Doran (Problem II, p. 321, see also [5]) is as follows.

Let X be a complete locally convex space and suppose that T is an irreducible representation on X of a complex algebra A such that the commutant $T' = \{S \in L(X) : T_a S = S T_a \text{ for all } a \text{ in } A\}$ consists only of scalar multiples of the identity operator. Does it follow that T is totally irreducible?

The problem makes sense for an arbitrary topological vector space X and also for real spaces and algebras. The problem is open for Banach spaces and even for Hilbert spaces, of course if the dimension of X is infinite.

If X is a Banach space, then a representation T on X is totally irreducible if and only if the algebra $\{T_a \in B(X) : a \in A\}$ is strongly dense in $B(X)$. Thus, if we are looking for a counterexample to the Fell and Doran Problem, we must construct a proper strongly closed subalgebra of $B(X)$ with trivial commutant and with no proper invariant subspace. Our algebras $\mathcal{A}(X_0)$ have trivial commutants, but X_0 is an invariant subspace.

II. *The Transitive Algebra Problem* (see [4], Chapter VIII). Let H be a complex Hilbert space. An algebra $A \subset B(H)$ is said to be *transitive* if it is strongly closed, contains the identity operator, and has no proper invariant subspace. The *Transitive Algebra Problem* is the question whether a transitive algebra on H must be equal to $B(H)$.

Again this leads to the study of strongly closed subalgebras of $B(H)$.

The following remarks can be useful in studying strongly closed subalgebras of $B(X)$.

Let M be a proper closed linear subspace of X^n, $n \geq 1$. Assume that M contains a point $\widetilde{x} = (\widetilde{x}_1, \ldots, \widetilde{x}_n)$ with linearly independent coordinates. Put

$$\mathcal{A}(M) = \{T \in B(X) : (x_1, \ldots, x_n) \in M \text{ implies } (Tx_1, \ldots, Tx_n) \in M\}.$$

It is easy to see that $\mathcal{A}(M)$ is a strongly closed subalgebra of $B(X)$ and our assumption on M implies that $\mathcal{A}(M) \neq B(X)$, since otherwise M would be dense in X^n. On the other hand, if $\mathcal{A}$ is a proper strongly closed subalgebra of $B(X)$, then there are linearly independent elements $x_1, \ldots, x_n$ in X such that the orbit

$$O(\mathcal{A}; x_1, \ldots, x_n) = \{(Tx_1, \ldots, Tx_n) \in X^n : T \in \mathcal{A}\}$$

is not dense in X^n. If M is the closure of this orbit in X^n, then we easily see that $\mathcal{A} \subset \mathcal{A}(M)$. Thus we have

PROPOSITION 4. *Every proper strongly closed subalgebra of $B(X)$ is contained in a proper strongly closed subalgebra of the form $\mathcal{A}(M)$, for some proper closed subspace M of X^n. In particular, every maximal strongly closed algebra must be of the form $\mathcal{A}(M)$.*

Let $\mathcal{A}$ be a strongly closed subalgebra of $B(X)$. We say that it is of *order* n if it is contained in a proper algebra of the form $\mathcal{A}(M)$ with $M \subset X^n$, and is not contained in any proper algebra $\mathcal{A}(M)$ with $M \subset X^k$, $k < n$. By Proposition 4 every proper strongly closed subalgebra of $B(X)$ has some positive order. It is clear that every strongly closed algebra of operators which has a proper invariant subspace is of order 1, and by Proposition 1 it is contained in an m.s.c.a. of order 1. We do not know whether this is true for algebras of order higher than one. In fact, we do not know any example of an infinite-dimensional m.s.c.a. of order higher than one. Both questions could be answered in the affirmative if we had an affirmative answer to the following

PROBLEM 1. Is every proper strongly closed subalgebra of $B(X)$ contained in some maximal strongly closed algebra?

(The usual technique of the Kuratowski–Zorn Lemma fails here, because we can have a chain $A_1 \subset A_2 \subset \ldots$ of proper strongly closed algebras such that the union $\bigcup_{i=1}^{\infty} A_i$ is a strongly dense subalgebra of $B(X)$.)

There exist algebras of order 2. For example, if T is an operator in $B(l_1)$ without a proper closed invariant subspace (see [5]), then the commutant T' is a proper strongly closed subalgebra of $B(l_1)$ which is not of order 1, and it is of order 2 because it coincides with $\mathcal{A}(M)$, where M is the graph of T.

PROBLEM 2. Does there exist a Banach space X such that $B(X)$ has proper strongly closed subalgebras of arbitrarily high orders?

Or a weaker question:

PROBLEM 3. Does there exist, for every natural n, a Banach space X such that $B(X)$ has a proper strongly closed subalgebra of order n?

We do not even know the answer to the following question.

PROBLEM 4. Does there exist a Banach space X such that $B(X)$ has a subalgebra of order 3?

An affirmative answer to Problem 4 would give a negative solution of the Problem of Fell and Doran: the representation given by the identity map $\mathcal{A} \to \mathcal{A}$ is the desired counterexample. It is irreducible since $\mathcal{A}$ is not of order 1 and it has trivial commutant since $\mathcal{A}$ is not of order 2. The Transitive Algebra Problem would be solved in the negative by showing that for a complex Hilbert space H there is a subalgebra of $B(H)$ of order higher than one.

If X is a complex finite-dimensional space, then every proper subalgebra of $B(X)$ is of order 1 (the Burnside Theorem, see [2], p. 276). For a finite-dimensional real space it is possible to have an m.s.c.a. of order 2 (this is the only example of an m.s.c.a. of order greater than one known to the author). Take the space $\mathbb{R}^4$ interpreted as real quaternions $x = a + bi + cj + dk$. The basis of $B(\mathbb{R}^4)$ is given by the operators T_v^u, $u, v \in (1, i, j, k)$, given by $x \to uxv$. The commutant

$$(T_1^i)' = \operatorname{span}\{T_1^i, T_i^i, T_j^i, T_k^1, T_1^1, T_i^1, T_j^1, T_k^1\}$$

is an m.s.c.a. of order two. This example shows that for a real finite-dimensional Hilbert space the Transitive Algebra Problem is answered in the negative. We do not know whether such an example is possible in infinite dimensions.

PROBLEM 5. Let X be an infinite-dimensional Banach space. Is it possible to have a maximal strongly closed algebra on X which is a commutant of some element in $B(X)$?

Added in proof. Proposition 1 is essentially known. In case of a Hilbert space it follows immediately from Corollary 2 of [3], or from Theorem 8.12 of [4]. The proofs can be adjusted so that they cover the case of a Banach space.

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