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AN APPLICATION OF MODULES OF GENERALIZED FRACTIONS TO GRADES OF IDEALS AND GORENSTEIN RINGS

BY

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1. Introduction. Let A be a commutative Noetherian ring. The grade of a proper ideal $\mathfrak{a}$ of A was defined by D. Rees in [5] as the least integer $i \geq 0$ such that $\operatorname{Ext}^i_A(A/\mathfrak{a}, A) \neq 0$. It is well known that the definition of grade of an ideal $\mathfrak{a}$ of A can be generalized by defining the grade of $\mathfrak{a}$ on a finitely generated A-module M such that $\mathfrak{a}M \neq M$ both as the maximum of lengths of M-sequences contained in $\mathfrak{a}$ and the least integer $i \geq 0$ such that $\operatorname{Ext}^i_A(A/\mathfrak{a}, M) \neq 0$.

In this note we shall extend the above definition of grade to non-zero A-modules M which have the property that whenever $\mathfrak{a}$ is an ideal of A and $x_1, \ldots, x_n$ is a poor M-sequence contained in $\mathfrak{a}$ such that $\mathfrak{a} \subseteq Z(M/(\sum_{i=1}^n Ax_i)M)$, then $\mathfrak{a} \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Ass}_A(M/(\sum_{i=1}^n Ax_i)M)$. (In this note, for an A-module N, the set of zero divisors on N is denoted by $Z(N)$.) Our extended definition of grade is based on the theory of modules of generalized fractions which was introduced by Sharp and the author in [7]. Indeed, we shall show that if M is an A-module with the above property, then the set $U_{i+1} = \{(x_1, \ldots, x_i, 1) : x_1, \ldots, x_i \text{ is a poor } M\text{-sequence}\}$ is a triangular subset of A^{i+1} for all $i \geq 0$, and, for an ideal $\mathfrak{a}$ of A with $\mathfrak{a}M \neq M$, the length of all maximal M-sequences contained in $\mathfrak{a}$ is the least integer $i \geq 0$ such that

$$\{x \in U_{i+1}^{-i-1}M : ax = 0\} \neq 0,$$

where $U_{i+1}^{-i-1}M$ is the module of generalized fractions of M with respect to U_{i+1}. Also, in this note, we shall use modules of generalized fractions to obtain characterizations of Gorenstein rings.

Let us recall briefly the main ingredients in the construction of modules of generalized fractions. Throughout this note, A will denote a commutative ring (with identity). For an ideal $\mathfrak{a}$ of A and an A-module M, we shall denote the submodule $\{m \in M : \mathfrak{a}m = 0\}$ by $\operatorname{ann}(\mathfrak{a}, M)$. When discussing modules of generalized fractions, we shall use the notation of [7], except that

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we shall use slightly different notation concerning matrices, in that round brackets will now be used instead of square ones; we shall agree to use n-tuples $(a_1, \ldots, a_n)$ of elements of A and $1 \times n$ row matrices $(a_1, \ldots, a_n)$ over A interchangeably. We still use $^\top$ to denote matrix transpose and, for $n \in \mathbb{N}, D_n(A)$ to denote the set of all $n \times n$ lower triangular matrices over A.

A *triangular subset* of A^n is a non-empty subset U of A^n such that (i) whenever $(x_1, \ldots, x_n) \in U$, then $(x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}) \in U$ for all choices of $\alpha_1, \ldots, \alpha_n \in \mathbb{N}$, and (ii) whenever $x, y \in U$, then there exist $z \in U$ and $H, K \in D_n(A)$ such that $Hx^\top = Ky^\top = z^\top$. Given such a U and an A-module M, one can construct (see Section 2 of [7]) the module of generalized fractions $U^{-n}M = \{\frac{a}{x} : a \in M, x \in U\}$, where $\frac{a}{x}$ denotes the equivalence class of $(a, x) \in M \times U$ under the equivalence relation $\sim$ on $M \times U$ defined as follows. For $a, b \in M$ and $x, y \in U$ we write $(a, x) \sim (b, y)$ if and only if there exist $(z_1, \ldots, z_n) =: z \in U$ and $H, K \in D_n(A)$ such that

$$Hx^\top = z^\top = Ky^\top \quad \text{and} \quad |H|a - |K|b \in \Big(\sum_{i=1}^{n-1} Az_i\Big)M.$$

Now $U^{-n}M$ is an A-module under the operations

$$\frac{a}{x} + \frac{b}{y} = \frac{|H|a + |K|b}{z} \quad \text{and} \quad r\frac{a}{x} = \frac{ra}{x}$$

for $r \in R$, $a, b, \in M$, $x, y \in U$ and any choice of $z \in U$ and $H, K \in D_n(A)$ such that $Hx^\top = z^\top = Ky^\top$.

The above concept is indeed a generalization of the familiar concept of ordinary module of fractions: see [7, 3.1].

2. The results. Throughout this section M is an A-module. We say that a sequence $x_1, \ldots, x_n$ of elements of A is a *poor M-sequence* if

$$(Ax_1 + \ldots + Ax_{i-1})M : x_i = (Ax_1 + \ldots + Ax_{i-1})M$$

for all $i = 1, \ldots, n$; it is an *M-sequence* if, in addition, $M \neq (Ax_1 + \ldots + Ax_n)M$.

The following theorem, which is proved in [4, 3.2] by elementary means, plays an important role in this note.

2.1. THEOREM. *Let $x_1, \ldots, x_n$ be a poor M-sequence and let $y_1, \ldots, y_n$ be a sequence of elements of A such that*

$$H(y_1, \ldots, y_n)^\top = (x_1, \ldots, x_n)^\top$$

for some $H \in D_n(A)$. Then the map from $M/(\sum_{i=1}^n Ay_i)M$ to $M/(\sum_{i=1}^n Ax_i)M$ induced by multiplication by $|H|$ is a monomorphism, and $y_1, \ldots, y_n$ is also a poor M-sequence.

2.2. COROLLARY. *Let U be a triangular subset of A^n which consists entirely of poor M-sequences. Let $m \in M$ and $(x_1, \ldots, x_n) \in U$. Then*

$$\frac{m}{(x_1, \ldots, x_n)} = 0$$

in $U^{-n}M$ if and only if $m \in (\sum_{i=1}^{n-1} Ax_i)M$.

Proof. ($\Rightarrow$) Suppose that $\frac{m}{(x_1,\ldots,x_n)} = 0$. Then there exist $(y_1, \ldots, y_n) \in U$ and $H = (h_{ij}) \in D_n(A)$ such that

$$H(x_1, \ldots, x_n)^\top = (y_1, \ldots, y_n)^\top \quad \text{and} \quad |H|m \in \Big(\sum_{i=1}^{n-1} Ay_i\Big)M.$$

Now, $h_{n1}x_1 + \ldots + h_{nn}x_n = y_n$; hence

$$h_{11} \ldots h_{n-1n-1}\Big(y_n - \sum_{i=1}^{n-1} h_{ni}x_i\Big)m \in \Big(\sum_{i=1}^{n-1} Ay_i\Big)M.$$

We may now use [7, 2.2] to see that

$$h_{11} \ldots h_{n-1n-1}m \in \Big(\sum_{i=1}^{n-1} Ay_i\Big)M : y_n = \Big(\sum_{i=1}^{n-1} Ay_i\Big)M.$$

Therefore, by 2.1, $m \in (\sum_{i=1}^{n-1} Ax_i)M$.
($\Leftarrow$) Use [7, 3.3(ii)].

We shall need to use a result of A. M. Riley concerning the saturation of triangular subsets. For an arbitrary triangular subset U of A^n, the set

$$\widetilde{U} = \{x \in A^n : xH^\top \in U \quad \text{for some } H \in D_n(A)\}$$

is called the *saturation* of U; it is easily seen to be a triangular subset of A^n containing U. Riley proves in [6, Chapter II, 2.9] by direct calculation that the natural homomorphism $U^{-n}M \to \widetilde{U}^{-n}M$ is an isomorphism. (The reader is referred to [2, 2.3] for another proof of this.)

The next theorem provides an explicit description, in a certain situation, of the elements of a submodule of a module of generalized fractions which is annihilated by an ideal of A.

2.3. THEOREM. *Let $\mathfrak{a}$ be an ideal of A and let U be a triangular subset of A^n which consists entirely of poor M-sequences. Suppose that there exists $x = (x_1, \ldots, x_n) \in U$ such that $x_1, \ldots, x_n \in \mathfrak{a}$. Then each element of* $\mathrm{ann}(\mathfrak{a}, (U \times \{1\})^{-n-1}M)$ *can be written in the form $\frac{m}{(x_1,\ldots,x_n,1)}$ for some $m \in M$.*

Proof. By the above result of Riley and 2.1, we can suppose that U is saturated. Let

$$X = \frac{b}{(y_1, \dots, y_n, 1)} \in \operatorname{ann}(\mathfrak{a}, (U \times \{1\})^{-n-1} M),$$

where $b \in M$ and $(y_1, \dots, y_n) \in U$. We may assume that there exists $H \in D_n(A)$ such that

$$H(x_1, \dots, x_n)^\top = (y_1, \dots, y_n)^\top.$$

Then, by [2, 3.2], $(x_1, \dots, x_i, x_{i+1}y_{i+1}, y_{i+2}, \dots, y_n) \in U$ for each $i = 0, \dots, n$; hence $(x_1, \dots, x_i, y_{i+1}, \dots, y_n) \in U$ since U is saturated and

$$(x_1, \dots, x_i, x_{i+1}y_{i+1}, y_{i+2}, \dots, y_n)^\top = D(x_1, \dots, x_i, y_{i+1}, \dots, y_n)^\top,$$

where D is the diagonal matrix $\operatorname{diag}(1, \dots, 1, x_{i+1}, 1, \dots, 1)$.

Suppose, inductively, that i is an integer with $0 \le i < n$ and it has been proved that

$$X = \frac{m_i}{(x_1, \dots, x_i, y_{i+1}, \dots, y_n, 1)}$$

for some $m_i \in M$. This is certainly the case when $i = 0$. We have $x_{i+1}X = 0$. Hence by 2.2,

$$x_{i+1}m_i = x_1m'_1 + \dots + x_im'_i + y_{i+1}m'_{i+1} + \dots + y_nm'_n$$

for some $m'_1, \dots, m'_n \in M$. Therefore, again by 2.2,

$$\begin{aligned} X &= \frac{x_{i+1}m_{i+1}}{(x_1, \dots, x_i, x_{i+1}y_{i+1}, y_{i+2}, \dots, y_n, 1)} \\ &= \frac{y_{i+1}m'_{i+1}}{(x_1, \dots, x_i, x_{i+1}y_{i+1}, y_{i+2}, \dots, y_n, 1)} \\ &= \frac{m'_{i+1}}{(x_1, \dots, x_{i+1}, y_{i+2}, \dots, y_n, 1)}. \end{aligned}$$

This completes the inductive step and the result is proved by induction.

Theorem 2.3 has some immediate consequences which we record here.

2.4. Consequences. (1) Let the situation be as in 2.3. Then, in view of 2.2 and 2.3, the A-homomorphism

$$\phi : \Big(\sum_{i=1}^n Ax_i\Big)M : \mathfrak{a} \to \operatorname{ann}(\mathfrak{a}, (U \times \{1\})^{-n-1}M)$$

given by $\phi(m) = \frac{m}{(x_1, \dots, x_n, 1)}$ is surjective and $\ker \phi = (\sum_{i=1}^n Ax_i)M$. Hence

$$\Big(\sum_{i=1}^n Ax_i\Big)M : \mathfrak{a} \Big/ \Big(\sum_{i=1}^n Ax_i\Big)M \cong \operatorname{ann}(\mathfrak{a}, (U \times \{1\})^{-n-1}M).$$

(2) For $z = (z_1, \dots, z_n) \in A^n$, let $U_z = \{(z_1^{\alpha_1}, \dots, z_n^{\alpha_n}) : \alpha_1, \dots, \alpha_n \in \mathbb{N}\}$ (a triangular subset of A^n). Let the situation be as in 2.1, and let V be the saturation of U_x. (Note that, in view of [7, 3.9] and 2.1, each element of

U_x, V and U_y is a poor M-sequence.) Let $\mathfrak{a}$ be an ideal of A and suppose that $y_1, \ldots, y_n \in \mathfrak{a}$. Then, by (1)

$$\begin{aligned}
&\operatorname{ann}(\mathfrak{a}, (U_y \times \{1\})^{-n-1}M) \\
&\quad\cong \Big(\sum_{i=1}^{n} Ay_i\Big)M : \mathfrak{a} \Big/ \Big(\sum_{i=1}^{n} Ay_i\Big)M \cong \operatorname{ann}(\mathfrak{a}, (V \times \{1\})^{-n-1}M) \\
&\quad\cong \Big(\sum_{i=1}^{n} Ax_i\Big)M : \mathfrak{a} \Big/ \Big(\sum_{i=1}^{n} Ax_i\Big)M \cong \operatorname{ann}(\mathfrak{a}, (U_x \times \{1\})^{-n-1}M).
\end{aligned}$$

For the remaining part of this section, we shall assume that A is Noetherian and that, whenever $\mathfrak{a}$ is an ideal of A and $x_1, \ldots, x_n$ is a poor M-sequence contained in $\mathfrak{a}$ such that $\mathfrak{a} \subseteq Z(M/(\sum_{i=1}^{n} Ax_i)M)$, then $\mathfrak{a} \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Ass}_A(M/(\sum_{i=1}^{n} Ax_i)M)$. At this stage, under the above assumptions, we can establish the following.

2.5. THEOREM. *For each positive integer n, let V_n be the set of all poor M-sequences of length n. Then*

(i) *V_n is a triangular subset of A^n for all $n \in \mathbb{N}$;*

(ii) *if $\mathfrak{a}$ is an ideal of A such that $\mathfrak{a}M \neq M$, then the length of a maximal M-sequence in $\mathfrak{a}$ is the least integer i (≥ 0) such that* $\operatorname{ann}(\mathfrak{a}, U_{i+1}^{-i-1}M) \neq 0$, *where $U_{n+1} = V_n \times \{1\}$ for all $n \in \mathbb{N}$ and $U_1 = \{1\}$.*

Proof. (i) First note that, for all $n \in \mathbb{N}, (1, \ldots, 1) \in V_n$ and that, in view of [7, 3.9], $(x_1^{t_1}, \ldots, x_n^{t_n}) \in V_n$ whenever $(x_1, \ldots, x_n) \in V_n$ and $t_1, \ldots, t_n \in \mathbb{N}$. Thus we have to check the second condition of the definition of triangular subsets. We shall do this by induction on n. It is clear that V_1 is a triangular subset of A^1.

Suppose, inductively, that $n > 1$ and we have proved that V_{n-1} is a triangular subset. Let $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in V_n$. Then, by induction, there exists $(z_1, \ldots, z_{n-1}) \in V_{n-1}$ such that

$$z_i \in \Big(\sum_{j=1}^{i} Ax_j\Big) \cap \Big(\sum_{j=1}^{i} Ay_j\Big)$$

for $i = 1, \ldots, n-1$. Since

$$\sum_{i=1}^{n-1}(Ax_i)M : \sum_{i=1}^{n} Ax_i = \Big(\sum_{i=1}^{n-1} Ax_i\Big)M,$$

it follows from 2.4(2) that

$$\sum_{i=1}^{n-1}(Az_i)M : \sum_{i=1}^{n} Ax_i = \Big(\sum_{i=1}^{n-1} Az_i\Big)M;$$

hence $\sum_{i=1}^{n} Ax_i \not\subseteq \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}_A(E)$, where $E = M/(\sum_{i=1}^{n-1} Az_i)M$. Similarly, $\sum_{i=1}^{n} Ay_i \not\subseteq \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}_A(E)$. Therefore, in view of the above assumption which we imposed on M, there exists $z_n \in (\sum_{i=1}^{n} Ax_i) \cap (\sum_{i=1}^{n} Ay_i)$ such that $z_n \notin Z(E)$. This completes the inductive step.

(ii) Let $\mathfrak{a}$ be an ideal of A such that $\mathfrak{a}M \neq M$. Let $x_1, \ldots, x_n$ be a maximal M-sequence in $\mathfrak{a}$. Then $(Ax_1 + \ldots + Ax_i)M : \mathfrak{a} = (Ax_1 + \ldots + Ax_i)M$ for all $i = 0, \ldots, n-1$, and, by our assumption on M, $(Ax_1 + \ldots + Ax_n)M : \mathfrak{a} \neq (Ax_1 + \ldots + Ax_n)M$. Hence, by 2.4(1), n is the least integer i such that $\mathrm{ann}(\mathfrak{a}, U_{i+1}^{-i-1}M) \neq 0$.

Now that 2.5 has been proved, we can give the following definition.

2.6. Definition. For all $n \geq 0$, let U_{n+1} be the set of all $(x_1, \ldots, x_n, 1) \in A^{n+1}$ such that $x_1, \ldots, x_n$ is a poor M-sequence. Let $\mathfrak{a}$ be an ideal of A such that $aM \neq M$. Then the *grade* of $\mathfrak{a}$ on M, $\mathrm{grade}(\mathfrak{a}, M)$, is the length of any maximal M-sequence contained in $\mathfrak{a}$ (they have the same length by 2.5(ii)) and also the least integer n (≥ 0) such that $\mathrm{ann}(\mathfrak{a}, U_{n+1}^{-n-1}M) \neq 0$.

It is easy to see that the above definition of grade has the following property.

2.7. Proposition. *Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ be ideals of A such that $\mathfrak{a}_i M \neq M$ for all $i = 1, \ldots, r$. Then*

$$\mathrm{grade}\Big(\bigcap_{i=1}^{r} \mathfrak{a}_i, M\Big) = \mathrm{grade}\Big(\prod_{i=1}^{r} \mathfrak{a}_i, M\Big) = \min\{\mathrm{grade}(\mathfrak{a}_i, M) : 1 \leq i \leq r\}.$$

The next propositions, for which we shall assume that A is local with maximal ideal $\mathfrak{m}$ and $\dim A = n$, provide characterizations of Gorenstein local rings. The reader is referred to [1] and [3] for several equivalent definitions of Gorenstein rings. In particular, recall that A is Gorenstein if and only if, equivalently, either the injective dimension $\mathrm{id}_A(A)$ of A is n or A is Cohen–Macaulay and $\dim_{A/m}((\mathfrak{a} : m)/\mathfrak{a}) = 1$ for some (respectively, any) parametric ideal $\mathfrak{a}$ of A.

2.8. Proposition. *Let $U_{n+1} = \{(x_1, \ldots, x_n, 1) : x_1, \ldots, x_n$ is a poor A-sequence$\}$. Then the following statements are equivalent*:

(i) *A is a Gorenstein ring*;
(ii) $\dim_{A/\mathfrak{m}}(\mathrm{ann}(\mathfrak{m}, U_{n+1}^{-n-1}A)) = 1$;
(iii) *$U_{n+1}^{-n-1}A$ is a non-zero injective A-module.*

Proof. The case $n = 0$ is clear. So, suppose that $n > 0$. By 2.4(1) and the above observations we see that (i)$\Leftrightarrow$(ii). Also, if (iii) holds, then A is Cohen–Macaulay and, for a parametric ideal $\mathfrak{q}$ of A, the ring $A/\mathfrak{q}$ is self-injective, since $A/\mathfrak{q} \cong \mathrm{ann}(\mathfrak{q}, U_{n+1}^{-n-1}A)$ by 2.4(1) and the latter $A/\mathfrak{q}$-module is injective. Hence (i) follows. Therefore, in order to complete the proof, we show that (i)⇒(iii).

Let $\mathfrak{b}$ be an ideal of A and let $f:\mathfrak{b}\to U_{n+1}^{-n-1}A$ be an A-homomorphism. Since $\mathfrak{b}$ is finitely generated, there exists an A-sequence $x_1,\dots,x_n$ such that

$$f(\mathfrak{b})\subseteq A\frac{1}{(x_1,\dots,x_n,1)}.$$

Also, by the Artin–Rees Theorem,

$$\mathfrak{b}\cap\Big(\sum_{i=1}^n Ax_i\Big)^{t+1}\subseteq\Big(\sum_{i=1}^n Ax_i\Big)\mathfrak{b}$$

for some $t\geq 0$. Therefore, in view of 2.2, we have

$$\operatorname{im} f\subseteq \operatorname{ann}(\mathfrak{q},U_{n+1}^{-n-1}A)\quad\text{and}\quad \mathfrak{b}\cap\mathfrak{q}\subseteq\ker f,$$

where $\mathfrak{q}=\sum_{i=1}^n Ax_i^{t+1}$. Hence f induces an $A/\mathfrak{q}$-homomorphism

$$f^*:(\mathfrak{b}+\mathfrak{q})/\mathfrak{q}\to\operatorname{ann}(\mathfrak{q},U_{n+1}^{-n-1}A).$$

On the other hand, it follows from the hypothesis and 2.4(1) that $\operatorname{ann}(\mathfrak{q},U_{n+1}^{-n-1}A)$ is an injective $A/\mathfrak{q}$-module. Thus, there exists $X\in\operatorname{ann}(\mathfrak{q},U_{n+1}^{-n-1}A)$ such that $f(y)=f^*(\mathfrak{q}+y)=yX$ for all $y\in\mathfrak{b}$. It therefore follows that $U_{n+1}^{-n-1}A$ is an injective A-module.

It is a well known change of rings theorem that if $x\in\mathfrak{m}\setminus Z(A),\overline{A}=A/Ax$, and M is an A-module such that $x\notin Z(M)$, then $\operatorname{id}_{\overline{A}}(M/xM)=\operatorname{id}_A(M)-1$. Next, we shall apply this theorem to give a characterization of Gorenstein local rings in terms of the injective dimension of a certain module of generalized fractions. To do this, we need to introduce some further notation. For a sequence $x_1,\dots,x_i$ ($i\geq 0$) of elements of a commutative ring R, we shall use $U(x)_{i+1}$ to denote the triangular subset

$$\{(x_1^{\alpha_1},\dots,x_i^{\alpha_i},1):\alpha_1,\dots,\alpha_i\in\mathbb{N}\}$$

of R^{i+1}. Note that if $i=0$, then $U(x)_{i+1}=\{1\}$ and $U(x)_{i+1}^{-i-1}R\cong R$.

2.9. Proposition. *The following statements are equivalent*:

(i) *A is a Gorenstein ring*;

(ii) *if $x_1,\dots,x_i$ is a subset of a system of parameters for A, then* $\operatorname{id}_A(U(x)_{i+1}^{-i-1}A)=n-i$.

Proof. (i)⇒(ii). We prove this by induction on n. The case $n=0$ is clear. Let $n\geq 1$, and suppose that the result is true for $n-1$. Let $x_1,\dots,x_i$ be a subset of a system of parameters for A. Let U_{n+1} be as in 2.8. If $i=n$, then, with the aid of [8, 3.6] and 2.2, it is easy to see that the natural A-homomorphism $U(x)_{n+1}^{-n-1}A\to U_{n+1}^{-n-1}A$ is an isomorphism; hence, by 2.8 and the hypothesis, $U(x)_{n+1}^{-n-1}A$ is an injective A-module. So, we may assume that $i < n$. Let $x_{i+1} \in A$ be such that $x_1, \ldots, x_i, x_{i+1}$ forms a subset of a system of parameters for A, and let $\psi : A \to A/Ax_{i+1}$ be the natural homomorphism. Then $\psi(A)$ is a Gorenstein ring of dimension $n-1$ and $\psi(x_1), \ldots, \psi(x_i)$ is a subset of a system of parameters for $\psi(A)$. Hence, by induction,

$$\mathrm{id}_{\psi(A)}(U(\psi(x))_{i+1}^{-i-1}\psi(A)) = n - 1 - i.$$

Next, using 2.2, it is easy to see that $x_{i+1} \notin Z(U(x)_{i+1}^{-i-1}A)$ and that the map $\phi : U(x)_{i+1}^{-i-1}A \to U(\psi(x))_{i+1}^{-i-1}\psi(A)$ given by

$$\phi\left(\frac{a}{(x_1^{\alpha_1}, \ldots, x_i^{\alpha_i}, 1)}\right) = \frac{\psi(a)}{(\psi(x_1)^{\alpha_1}, \ldots, \psi(x_i)^{\alpha_i}, \psi(1))}$$

is an A-epimorphism and $\ker \phi = x_{i+1}U(x)_{i+1}^{-i-1}A$. Therefore, by the above-mentioned change of rings theorem, $\mathrm{id}_A(U(x)_{i+1}^{-i-1}A) = n - i$. The result now follows by induction.

(ii)⇒(i). This is clear by Bass' fundamental theorem in [1]. Note that it follows from the hypothesis that $\mathrm{id}_A(A) = n$.

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