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COMPACTNESS IN APPROXIMATION SPACES

BY

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In this paper we give a characterization of the relatively compact subsets of the so-called approximation spaces. We treat some applications: (1) we obtain some convergence results in such spaces, and (2) we establish a condition for relative compactness of a set lying in a Besov space.

0. Introduction. In the following, all definitions concerning approximation spaces are adopted from [2].

A *quasi-norm* is a non-negative function $\|\cdot\|_X$ defined on a (real or complex) linear space X for which the following conditions are satisfied:

(1) If $\|f\|_X = 0$ for some $f \in X$, then $f = 0$.

(2) $\|\lambda f\|_X = |\lambda|\ \|f\|_X$ for $f \in X$ and all scalars λ.

(3) There exists a constant $c_X \geq 1$ such that

$$\|f+g\|_X \leq c_X[\|f\|_X + \|g\|_X] \quad \text{for } f, g \in X.$$

The quasi-norms $\|\cdot\|_X^{(1)}$ and $\|\cdot\|_X^{(2)}$ are said to be *equivalent* if

$$\|f\|_X^{(2)} \leq a\|f\|_X^{(1)} \quad \text{and} \quad \|f\|_X^{(1)} \leq b\|f\|_X^{(2)} \quad \text{for all } f \in X,$$

where a and b are suitable constants.

A quasi-norm $\|\cdot\|_X$ is called a *p-norm* $(0 < p \leq 1)$ if

$$\|f+g\|_X^p \leq \|f\|_X^p + \|g\|_X^p \quad \text{for } f, g \in X.$$

The condition (3) is satisfied with $c_X := 2^{1/p-1}$.

A *quasi-Banach space* is a linear space X equipped with a quasi-norm $\|\cdot\|_X$ such that every Cauchy sequence is convergent.

An *approximation scheme* (X, A_n) is a quasi-Banach space X together with a sequence of subsets A_n such that the following conditions are satisfied:

(1) $A_1 \subseteq A_2 \subseteq \ldots \subseteq X$.

(2) $\lambda A_n \subseteq A_n$ for all scalars λ and $n = 1, 2, \ldots$

(3) $A_m + A_n \subseteq A_{m+n}$ for $m, n = 1, 2, \ldots$

We put $A_0 := \{0\}$.

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Let (X, A_n) be an approximation scheme. For $f \in X$ and $n = 1, 2, \ldots,$ the nth *approximation number* is defined by

$$\alpha_n(f, X) := \inf\{\|f - a\|_X : a \in A_{n-1}\}.$$

Let $0 < \varrho < \infty$ and $0 < u < \infty$. Then the *approximation space* X_u^ϱ, or more precisely $(X, A_n)_u^\varrho$, consists of all elements $f \in X$ such that $(n^{\varrho - 1/u}\alpha_n(f, X)) \in l_u$, where $n = 1, 2, \ldots$ We put

$$\|f\|_{X_u^\varrho} := \|(n^{\varrho-1/u}\alpha_n(f, X))\|_{l_u} \quad \text{for } f \in X_u^\varrho.$$

Then X_u^ϱ is a quasi-Banach space.

We mention (see [2]) that an element $f \in X$ belongs to X_u^ϱ if and only if

$$(2^{k\varrho}\alpha_{2^k}(f, X)) \in l_u, \quad \text{where } k = 0, 1, \ldots$$

Moreover,

$$\|f\|^*_{X_u^\varrho} := \|(2^{k\varrho}\alpha_{2^k}(f, X))\|_{l_u}$$

defines an equivalent quasi-norm on X_u^ϱ.

In the sequel $c_1, c_2, \ldots$ are positive constants depending on certain exponents, but not on natural numbers.

1. Relatively compact sets in X_u^ϱ. The main result of our work is

THEOREM 1. *Let (X, A_n) be an approximation scheme. Let A be a subset of X_u^ϱ. Then A is relatively compact in X_u^ϱ if and only if the following two conditions are satisfied*:

(1) *A is relatively compact in X.*
(2) $\lim_n \sum_{k=n}^\infty [2^{k\varrho}\alpha_{2^k}(f, X)]^u = 0$ *uniformly on A.*

Proof. If A is a relatively compact set in X_u^ϱ then, from the inequality $\|f\|_X \le \|f\|^*_{X_u^\varrho}$ for $f \in X_u^\varrho$, it is obvious that A is relatively compact in X. Since A is a precompact set in X_u^ϱ, given $\varepsilon > 0$, we can find $f_1, \ldots, f_m \in A$ such that, for every $f \in A$,

$$\|f - f_j\|^*_{X_u^\varrho} \le \varepsilon \quad \text{for some } j \in \{1, \ldots, m\}.$$

Moreover, given $\varepsilon > 0$, there exists a natural number n_1 such that for $n \ge n_1$ and $i \in \{1, \ldots, m\}$ we have

$$\sum_{k=n}^\infty [2^{k\varrho}\alpha_{2^k}(f_i, X)]^u \le \varepsilon^u,$$

and then

$$\sum_{k=n}^\infty [2^{k\varrho}\alpha_{2^k}(f, X)]^u = 2^{u\varrho}\sum_{k=n-1}^\infty [2^{k\varrho}\alpha_{2\cdot 2^k - 1}(f - f_j + f_j, X)]^u$$

$$\le c_1 2^{u\varrho} \sum_{k=n-1}^{\infty} [2^{k\varrho}\alpha_{2^k}(f-f_j,X) + 2^{k\varrho}\alpha_{2^k}(f_j,X)]^u$$

$$\le c_2 2^{u\varrho}\Big(\sum_{k=n-1}^{\infty} [2^{k\varrho}\alpha_{2^k}(f-f_j,X)]^u + \sum_{k=n-1}^{\infty} [2^{k\varrho}\alpha_{2^k}(f_j,X)]^u\Big)$$

$$\le c_2 2^{u\varrho+1}\varepsilon^u \quad \text{for } n \ge n_1+1.$$

Conversely, if (f_n) is a sequence of points of A we will prove that (f_n) contains a subsequence (f_{n_k}) which is a Cauchy sequence in X_u^ϱ. Then (f_{n_k}) is convergent in X_u^ϱ, and therefore A is relatively compact in X_u^ϱ.

Let (β_n) be a sequence of real numbers such that $0 \le \beta_n \le 1$ for $n = 1,2,\ldots$ We have

$$\begin{aligned}
&\alpha_{2\cdot 2^k-1}(f_n-f_m,X)\\
&\qquad = (1-\beta_k)\alpha_{2\cdot 2^k-1}(f_n-f_m,X) + \beta_k\alpha_{2\cdot 2^k-1}(f_n-f_m,X)\\
&\qquad \le (1-\beta_k)\|f_n-f_m\|_X + c_X\beta_k(\alpha_{2^k}(f_n,X)+\alpha_{2^k}(f_m,X)).
\end{aligned}$$

Hence

$$\begin{aligned}
&(\|f_n-f_m\|^*_{X_u^\varrho})^u\\
&\quad = \|f_n-f_m\|_X^u + \sum_{k=1}^{\infty}[2^{k\varrho}\alpha_{2^k}(f_n-f_m,X)]^u\\
&\quad \le \|f_n-f_m\|_X^u + \sum_{k=0}^{\infty}[2^{(k+1)\varrho}\alpha_{2^{k+1}-1}(f_n-f_m,X)]^u\\
&\quad = \|f_n-f_m\|_X^u + 2^{\varrho u}\sum_{k=0}^{\infty}[2^{k\varrho}\alpha_{2^{k+1}-1}(f_n-f_m,X)]^u\\
&\quad \le \|f_n-f_m\|_X^u + c_1 2^{\varrho u}\|f_n-f_m\|_X^u\sum_{k=0}^{\infty}[(1-\beta_k)2^{k\varrho}]^u\\
&\qquad + c_2 2^{\varrho u}\sum_{k=0}^{\infty}[\beta_k 2^{k\varrho}\alpha_{2^k}(f_n,X)]^u + c_2 2^{\varrho u}\sum_{k=0}^{\infty}[\beta_k 2^{k\varrho}\alpha_{2^k}(f_m,X)]^u.
\end{aligned}$$

By condition (2), given $\varepsilon > 0$, there exists a natural number n_0 such that, for all $f \in A$,

$$\sum_{k=n_0}^{\infty} [2^{k\varrho}\alpha_{2^k}(f,X)]^u \le \varepsilon.$$

Since, by condition (1), A is relatively compact in X, the sequence (f_n) contains a subsequence (f_{n_k}) which converges in X and therefore (f_{n_k}) is a

Cauchy sequence in X. We put

$$K := 1 + c_1 2^{u\varrho} \sum_{k=0}^{n_0-1} 2^{k\varrho u}.$$

Then there exists a natural number n_1 such that $p, q \geq n_1$ implies

$$\|f_{n_p} - f_{n_q}\|_X \leq (\varepsilon/K)^{1/u}.$$

If we take (β_n) with $\beta_n = 0$ for $1 \leq n < n_0$ and $\beta_n = 1$ for $n \geq n_0$, from the above inequalities we arrive at

$$(\|f_{n_p} - f_{n_q}\|^*_{X^\varrho_u})^u$$
$$\leq \|f_{n_p} - f_{n_q}\|^u_X \Big[1 + c_1 2^{\varrho u} \sum_{k=0}^{n_0-1} 2^{k\varrho u}\Big] + \varepsilon c_2 2^{\varrho u+1} \leq \varepsilon[1 + c_2 2^{\varrho u+1}].$$

This completes the proof.

We also give a compactness criterion in a particular case. For standard notions of bases in Banach spaces we refer to [5].

THEOREM 2. *Let X be a Banach space with a basis $\{f_n\}$. Let (X, A_n) be the approximation scheme built from the sequence of subsets*

$$A_n := [f_1, \ldots, f_n] \quad \textit{for } n = 1, 2, \ldots$$

Let A be a subset of X^ϱ_u. Then A is relatively compact in X^ϱ_u if and only if the following two conditions are satisfied:

(1) *A is bounded in X.*
(2) $\lim_n \sum_{k=n}^\infty [2^{k\varrho} \alpha_{2^k}(f, X)]^u = 0$ *uniformly on A.*

Proof. The necessity follows from Theorem 1.

To prove the sufficiency, we define the operator $P_n : X \to X$ by

$$P_n(f) := \sum_{i=1}^n f_i^*(f) f_i \quad \text{for } f \in X,$$

where $\{f_n^*\}$ is the sequence of coefficient functionals associated with the basis $\{f_n\}$. The approximation scheme (X, A_n) is *linear* in the sense of [2], and it follows that

$$\|f - P_{n-1}(f)\|_X \leq c\alpha_n(f, X)$$

for all $f \in X$ and $n = 1, 2, \ldots$, where $c := 1 + \sup \|P_n\|$. From condition (2) we obtain

$$\lim_n \sum_{k=n}^\infty [2^{k\varrho} \|f - P_{2^k-1}(f)\|_X]^u = 0 \quad \text{uniformly on } A.$$

Hence, given $\varepsilon > 0$, there exists a natural number k such that, for all $f \in A$,

$$\|f - P_{2^k-1}(f)\|_X \le \varepsilon/2.$$

Since A is bounded in X, $P_{2^k-1}(A)$ is precompact in X, and then there exists a set $\{g_1, \ldots, g_m\}$ such that for every $f \in A$ there exists $j \in \{1, \ldots, m\}$ with

$$\|P_{2^k-1}(f) - g_j\|_X \le \varepsilon/2,$$

and therefore

$$\|f - g_j\|_X \le \|f - P_{2^k-1}(f)\|_X + \|P_{2^k-1}(f) - g_j\|_X \le \varepsilon.$$

Hence A is precompact in X, and then A is relatively compact in X. The result now follows from Theorem 1.

2. Some applications. Now we obtain some consequences of the preceding results. First, we establish various convergence theorems.

THEOREM 3. *Let (X, A_n) be an approximation scheme. Suppose that $f_n \to f$ in X and that*

$$\lim_n \sum_{k=n}^{\infty} [2^{k\varrho} \alpha_{2^k}(f_m, X)]^u = 0 \quad \text{uniformly on } A,$$

where $A := \{f_m : m \in \mathbb{N}\}$. Then $f_n \to f$ in X_u^ϱ.

Proof. Since $f_n \to f$ in X, the set $A \cup \{f\}$ is compact, hence A is relatively compact in X. From the uniform convergence assumption, we have $A \subset X_u^\varrho$. Applying Theorem 1 we conclude that A is relatively compact in X_u^ϱ. Then f is the only adherent value of the sequence (f_n) and therefore $f_n \to f$ in X_u^ϱ.

The following dominated convergence theorem (see [4, p. 39] for operators in the Schatten classes) is an immediate consequence of Theorem 3.

THEOREM 4. *Let (X, A_n) be an approximation scheme. Suppose that $f_n \to f$ in X, with $f \in X_u^\varrho$, and that*

$$\alpha_k(f_n) \le \alpha_k(f) \quad \text{for } k, n = 1, 2, \ldots$$

Then $f_n \to f$ in X_u^ϱ.

THEOREM 5. *Let X be a quasi-Banach space equipped with a p-norm $\|\cdot\|_X$ $(0 < p \le 1)$. Let (X, A_n) be an approximation scheme. Suppose that $f_n \to f$ in X and that*

$$\|f_n\|^*_{X_u^\varrho} \to \|f\|^*_{X_u^\varrho}.$$

Then $f_n \to f$ in X_u^ϱ.

Proof. It follows from

$$|\alpha_k(f_n, X)^p - \alpha_k(f, X)^p| \le \|f_n - f\|_X^p$$

and $f_n \to f$ in X that $\lim_n \alpha_k(f_n) = \alpha_k(f)$ for $k = 1, 2, \ldots$ Obviously, the corresponding approximation numbers are defined from the p-norm $\|\cdot\|_X$.

Since $\|f_n\|^*_{X^\varrho_u} \to \|f\|^*_{X^\varrho_u}$, given $\varepsilon > 0$, there exists a natural number n_1 such that for $n \geq n_1$ we have

$$\sum_{k=0}^{\infty}[2^{k\varrho}\alpha_{2^k}(f_n, X)]^u \leq \varepsilon + \sum_{k=0}^{\infty}[2^{k\varrho}\alpha_{2^k}(f, X)]^u.$$

Also $f \in X^\varrho_u$, and then there exists a natural number n_0 such that

$$\sum_{k=0}^{\infty}[2^{k\varrho}\alpha_{2^k}(f, X)]^u \leq \varepsilon + \sum_{k=0}^{n_0}[2^{k\varrho}\alpha_{2^k}(f, X)]^u.$$

Combining the above inequalities we obtain

$$(*) \qquad \sum_{k=0}^{\infty}[2^{k\varrho}\alpha_{2^k}(f_n, X)]^u \leq 2\varepsilon + \sum_{k=0}^{n_0}[2^{k\varrho}\alpha_{2^k}(f, X)]^u \quad \text{for } n \geq n_1.$$

Using

$$\lim_n \sum_{k=0}^{n_0}[2^{k\varrho}\alpha_{2^k}(f_n, X)]^u = \sum_{k=0}^{n_0}[2^{k\varrho}\alpha_{2^k}(f, X)]^u,$$

we get a natural number n_2 such that for $n \geq n_2$ we have

$$(**) \qquad \sum_{k=0}^{n_0}[2^{k\varrho}\alpha_{2^k}(f, X)]^u \leq \varepsilon + \sum_{k=0}^{n_0}[2^{k\varrho}\alpha_{2^k}(f_n, X)]^u.$$

Hence $(*)$ and $(**)$ for $n \geq \max(n_1, n_2)$ yield

$$\sum_{k=0}^{\infty}[2^{k\varrho}\alpha_{2^k}(f_n, X)]^u \leq 3\varepsilon + \sum_{k=0}^{n_0}[2^{k\varrho}\alpha_{2^k}(f_n, X)]^u,$$

and then

$$\sum_{k=n_0+1}^{\infty}[2^{k\varrho}\alpha_{2^k}(f_n, X)]^u \leq 3\varepsilon.$$

We take $m_0 := \max(n_1, n_2, 2)$. Since $f_1, \ldots, f_{m_0-1} \in X^\varrho_u$, given $\varepsilon > 0$, we obtain a natural number n_3 such that $n \geq n_3$ and $k \in \{1, \ldots, m_0 - 1\}$ imply

$$\sum_{i=n}^{\infty}[2^{i\varrho}\alpha_{2^i}(f_k, X)]^u \leq 3\varepsilon.$$

Therefore, from the two preceding inequalities we see that for $m \geq \max(n_0 + 1, n_3)$ and $n = 1, 2, \ldots,$

$$\sum_{k=m}^{\infty}[2^{k\varrho}\alpha_{2^k}(f_n, X)]^u \leq 3\varepsilon.$$

Thus

$$\lim_n \sum_{k=n}^{\infty} [2^{k\varrho} \alpha_{2^k}(f_m, X)]^u = 0$$

uniformly on $\{f_m : m \in \mathbb{N}\}$, and the result follows from Theorem 3.

To prove a compactness criterion in Besov spaces, we start with some notation. Let I be the interval [0, 1] and let m be an integer, $m \geq -1$. We consider the orthonormal systems $\{f_n^{(m)} : n \geq -m\}$ of spline functions of order m defined on I (for definition and properties see e.g. [1]). The system $\{f_n^{(m)} : n \geq -m\}$ is a basis in $C(I)$ and $L_p(I)$ for $1 \leq p < \infty$.

The best approximation in $L_p(I)$ for $1 \leq p < \infty$ and in $C(I)$ for $p = \infty$ is defined by

$$E_{n,p}^{(m)}(f) := \inf_{\{a_{-m},\ldots,a_n\}} \Big\| f - \sum_{j=-m}^{n} a_j f_j^{(m)} \Big\|_p .$$

The modulus of smoothness of order $r \geq 1$ of the function $f \in L_p(I)$ is defined for finite p and $\delta r \leq 1$ by

$$\omega_r^{(p)}(f, \delta) := \sup_{0<h\leq\delta} \Big(\int_0^{1-rh} |\Delta_h^r f(t)|^p \, dt \Big)^{1/p}$$

and for $p = \infty$ by

$$\omega_r^{(\infty)}(f, \delta) := \sup\{|\Delta_h^r f(t)| : 0 \leq t < t + rh \leq 1,\ h \leq \delta\},$$

where Δ_h^r denotes the forward progressive difference of order r with increment h.

Let $0 < \alpha < m + 1 + 1/p$, $1 \leq \vartheta < \infty$. The space $B_{p,\vartheta}^{\alpha,m}(I)$ is defined as the set of functions which belong to $L_p(I)$ for $1 \leq p < \infty$ and to $C(I)$ for $p = \infty$, and for which

$$|f|_{p,\vartheta}^{\alpha,m} := \Big(\int_0^1 [t^{-\alpha} \omega_{m+2}^{(p)}(f, t)]^\vartheta \, \frac{dt}{t} \Big)^{1/\vartheta}$$

is finite. It is a Banach space with respect to the norm

$$\|f\|_{B_{p,\vartheta}^{\alpha,m}(I)} := \|f\|_p + |f|_{p,\vartheta}^{\alpha,m}.$$

For $f \in B_{p,\vartheta}^{\alpha,m}(I)$ we put

$$\|f\|'_{B_{p,\vartheta}^{\alpha,m}(I)} := \|f\|_p + \Big(\sum_{n=0}^{\infty} [2^{n\alpha} E_{2^n,p}^{(m)}(f)]^\vartheta \Big)^{1/\vartheta} .$$

It was proved in [3] that $\|\cdot\|_{B_{p,\vartheta}^{\alpha,m}(I)}$ and $\|\cdot\|'_{B_{p,\vartheta}^{\alpha,m}(I)}$ are equivalent norms.

THEOREM 6. *Let* $m \ge -1$, $1 \le p \le \infty$, $1 \le \vartheta < \infty$ *and*

$$0 < \alpha < m + 1 + 1/p.$$

Let A *be a subset of* $B^{\alpha,m}_{p,\vartheta}(I)$. *Then* A *is relatively compact in* $B^{\alpha,m}_{p,\vartheta}(I)$ *if and only if the following two conditions are satisfied*:

(1) A *is bounded in* $L_p(I)$ *for* $1 \le p < \infty$ *and in* $C(I)$ *for* $p = \infty$.

(2) *For every* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that for every measurable set* $E \subset I$ *of measure* $m(E) < \delta$ *and for all* $f \in A$,

$$\int_E [t^{-\alpha}\omega^{(p)}_{m+2}(f,t)]^{\vartheta}\,\frac{dt}{t} \le \varepsilon.$$

Proof. Suppose that A is relatively compact in $B^{\alpha,m}_{p,\vartheta}(I)$. Then, given $\varepsilon > 0$, there exists a finite set $\{f_1,\ldots,f_q\} \subset A$ such that for every $f \in A$, there exists $i \in \{1,\ldots,q\}$ with

$$\|f - f_i\|_{B^{\alpha,m}_{p,\vartheta}(I)} \le \varepsilon.$$

Hence

$$\int_0^1 [t^{-\alpha}\omega^{(p)}_{m+2}(f-f_i,t)]^{\vartheta}\,\frac{dt}{t} \le \varepsilon^{\vartheta}.$$

Since $\{f_1,\ldots,f_q\} \subset A$, given $\varepsilon > 0$, there exists a $\delta > 0$ such that for every measurable set $E \subset I$ of measure $m(E) < \delta$ and for every $j \in \{1,\ldots,q\}$,

$$\int_E [t^{-\alpha}\omega^{(p)}_{m+2}(f_j,t)]^{\vartheta}\,\frac{dt}{t} \le \varepsilon^{\vartheta}.$$

Consequently,

$$\begin{aligned}
&\int_E [t^{-\alpha}\omega^{(p)}_{m+2}(f,t)]^{\vartheta}\,\frac{dt}{t}\\
&\le \int_E [t^{-\alpha}\omega^{(p)}_{m+2}(f-f_i,t) + t^{-\alpha}\omega^{(p)}_{m+2}(f_i,t)]^{\vartheta}\,\frac{dt}{t}\\
&\le \bigg(\bigg(\int_E [t^{-\alpha}\omega^{(p)}_{m+2}(f-f_i,t)]^{\vartheta}\,\frac{dt}{t}\bigg)^{1/\vartheta} + \bigg(\int_E [t^{-\alpha}\omega^{(p)}_{m+2}(f_i,t)]^{\vartheta}\,\frac{dt}{t}\bigg)^{1/\vartheta}\bigg)^{\vartheta}\\
&\le (2\varepsilon)^{\vartheta}.
\end{aligned}$$

The set A is bounded in $L_p(I)$ or in $C(I)$ since

$$\|f\|_p \le \|f\|_{B^{\alpha,m}_{p,\vartheta}(I)} \quad \text{for } f \in B^{\alpha,m}_{p,\vartheta}(I).$$

Conversely, assume that (1) and (2) are satisfied. There exists a natural number n_0 such that for $n \geq n_0$ we have $1/2^n < \delta$ and $2^n \geq m+2$. If

$$F := \bigcup_{k \geq n} [1/2^{k+1}, 1/2^k],$$

then $m(F) < \delta$. For $k \geq n$ we have $2^{k+1} \geq 2^k \geq 2^n \geq m+2$ and from [1] we obtain

$$E^{(m)}_{2^{k+1},p}(f) \leq M_m \omega^{(p)}_{m+2}(f, 1/2^{k+1}),$$

hence for $q > 1$ and for all $f \in A$ we have

$$\frac{2^{-\alpha\vartheta}}{2M_m^\vartheta} \sum_{k=n}^{n+q} 2^{(k+1)\alpha\vartheta} [E^{(m)}_{2^{k+1},p}(f)]^\vartheta \leq \sum_{k=n}^{n+q} \int\limits_{1/2^{k+1}}^{1/2^k} [t^{-\alpha} \omega^{(p)}_{m+2}(f,t)]^\vartheta \frac{dt}{t}$$
$$\leq \int\limits_F [t^{-\alpha} \omega^{(p)}_{m+2}(f,t)]^\vartheta \frac{dt}{t} \leq \varepsilon.$$

Therefore, given $\varepsilon > 0$, there exists a natural number n_1 such that for $n \geq n_1$ we have

$$(*) \qquad \sup_{f \in A} \sum_{k=n+1}^{\infty} (2^k)^{\alpha\vartheta} [E^{(m)}_{2^k,p}(f)]^\vartheta \leq \varepsilon.$$

Define $A^{(m)}_n := [f^{(m)}_{-m}, \ldots, f^{(m)}_{-m+n-1}]$ and consider the approximation scheme $(L_p(I), A^{(m)}_n)$ for $1 \leq p < \infty$ and $(C(I), A^{(m)}_n)$ for $p = \infty$. By $(*)$, given $\varepsilon > 0$, there exists a natural number n_2 such that for $n \geq n_2$ we have

$$\sup_{f \in A} \sum_{k=n}^{\infty} [2^{k\alpha} \alpha_{2^k}(f, L_p(I))]^\vartheta \leq \varepsilon,$$

with $1 \leq p < \infty$, and the same holds for $p = \infty$. Applying Theorem 2 we conclude that A is relatively compact in $L_p(I)^\alpha_\vartheta$ for $1 \leq p < \infty$ and in $C(I)^\alpha_\vartheta$ for $p = \infty$. Finally, using the norm $\|\cdot\|'_{B^{\alpha,m}_{p,\vartheta}(I)}$ we obtain the embeddings

$$L_p(I)^\alpha_\vartheta \subseteq B^{\alpha,m}_{p,\vartheta}(I) \quad \text{for } 1 \leq p < \infty, \quad C(I)^\alpha_\vartheta \subseteq B^{\alpha,m}_{\infty,\vartheta}(I)$$

(in fact, in both cases there are equalities), and then A is relatively compact in $B^{\alpha,m}_{p,\vartheta}(I)$.

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