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RADIAL LIMITS OF SUPERHARMONIC FUNCTIONS IN THE PLANE

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1. Introduction. The following result is due to Schneider [10, Theorem 2].

THEOREM A. *If E is a second category subset of* $[0,2\pi)$, *then there is no harmonic function h on $\mathbb{C}$ such that $r^{-\mu}h(re^{i\theta})\to+\infty$ as $r\to+\infty$ for all $\theta\in E$ and all $\mu>0$.*

It is essential that E is second category: Bagemihl and Seidel [5, pp. 187–190] showed that if E is first category and $M:[0,+\infty)\to(0,+\infty)$ is increasing, then there exists a harmonic function h on $\mathbb{C}$ such that $h(re^{i\theta})/M(r)\to+\infty$ as $r\to+\infty$ for all $\theta\in E$. However, using elementary techniques, we shall show that the hypotheses of Theorem A can, in some respects, be relaxed.

If f is an extended real-valued function on $\mathbb{C}$ and μ is a positive number, then we define

$$L(f,\mu)=\{\theta\in[0,2\pi):\lim_{r\to+\infty}r^{-\mu}f(re^{i\theta})=+\infty\}$$

and

$$U(f,\mu)=\{\theta\in[0,2\pi):\limsup_{r\to+\infty}r^{-\mu}f(re^{i\theta})=+\infty\}.$$

THEOREM 1. *Let E be a second category subset of* $[0,2\pi)$. *There is no harmonic function h on $\mathbb{C}$ such that*

$$\liminf_{r\to+\infty}h(re^{i\theta})>-\infty \tag{1}$$

for all $\theta\in E$ and such that $E\subseteq\bigcap_{\mu>0}\overline{U(h,\mu)}$.

Note that in Theorem 1 there is no *a priori* supposition that there is even one value of θ such that

$$\limsup_{r\to+\infty}r^{-\mu}h(re^{i\theta})=+\infty$$

for all positive μ. A similar remark applies to Theorems 2 and 3, below.

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Theorem 1 is false with "superharmonic" in place of "harmonic".

PROPOSITION 1. *There exists a superharmonic function u on $\mathbb{C}$ such that*

$$\liminf_{r\to+\infty} u(re^{i\theta}) > 0$$

for all $\theta \in [0, 2\pi)$ and such that $\bigcap_{\mu>0} U(u,\mu)$ is a residual subset of $[0, 2\pi)$ (and hence $\bigcap_{\mu>0} \overline{U(u,\mu)} = [0, 2\pi]$).

We give two alternative ways of modifying the hypotheses of Theorem 1 so that it becomes valid for superharmonic functions. In the first of these we simply replace $U(u,\mu)$ by $L(u,\mu)$. Recall that a function u on a domain is called *hyperharmonic* if either u is superharmonic or $u \equiv +\infty$.

THEOREM 2. *Let E be a second category subset of $[0, 2\pi)$. If u is hyperharmonic on $\mathbb{C}$,*

$$\liminf_{r\to+\infty} u(re^{i\theta}) > -\infty \tag{2}$$

for all $\theta \in E$, and $E \subseteq \bigcap_{\mu>0} \overline{L(u,\mu)}$, then $u \equiv +\infty$.

Note that Theorem 2 implies that Theorem A holds for superharmonic functions.

Our second superharmonic version of Theorem 1 involves a strengthening of the condition $E \subseteq \bigcap_{\mu>0} \overline{U(u,\mu)}$. This latter condition means that every open interval I which meets E also meets $U(u,\mu)$ for each μ. In the following theorem we require more: each such interval I must meet each set $U(u,\mu)$ in a set which is not too small.

THEOREM 3. *Let E be a second category subset of $[0, 2\pi)$. If u is hyperharmonic on $\mathbb{C}$ and (2) holds for all $\theta \in E$, and if $\{e^{i\theta} : \theta \in I \cap U(u,\mu)\}$ is a non-polar subset of $\mathbb{C}$ for each positive μ and each open interval I such that $E \cap I \neq \emptyset$, then $u \equiv +\infty$.*

For the notion of polar set, we refer to Helms [9, pp. 126–130].

2. An elementary lemma. We shall use the following lemma in the proofs of Theorems 1–3.

LEMMA 1. *Let E be a second category subset of $[0, 2\pi)$. Let $\phi : \mathbb{C} \to (-\infty, +\infty]$ be lower semi-continuous and fine continuous on $\mathbb{C}$. If*

$$\liminf_{r\to+\infty} \phi(re^{i\theta}) > -\infty \tag{3}$$

for all $\theta \in E$, then there exists an open interval J such that $E \cap J \neq \emptyset$ and ϕ is bounded below on the sector $\{re^{i\theta} : r > 0, \theta \in J\}$.

The fine topology is discussed, for example, in [9, Chapter 10].

The first step in our proof of Lemma 1 is to show that the function Φ, defined by

$$\Phi(z) = \inf\{\phi(rz) : r > 0\} \quad (z \in \mathbb{C}),$$

is fine upper semi-continuous on $\mathbb{C}\backslash\{0\}$. It suffices to work at a point z such that $\Phi(z) < +\infty$. If $A > \Phi(z)$, then there exist a positive number r_0 and a fine neighbourhood ω of $r_0 z$ on which $\phi < A$. The set $\{r_0^{-1}\zeta : \zeta \in \omega\} = \Omega$, say, is a fine neighbourhood of z, and if $w \in \Omega$, then $r_0 w \in \omega$, so that $\Phi(w) \le \phi(r_0 w) < A$. Hence Φ is fine upper semi-continuous at z, as required.

Next we show that Φ is upper semi-continuous with respect to the usual topology on $\mathbb{C}\backslash\{0\}$. Suppose that this is not the case. Then there exist $z \in \mathbb{C}\backslash\{0\}$, a number $A > \Phi(z)$, and a sequence (z_n) such that $z_n \to z$ and $\Phi(z_n) > A$ for all n. For each n, let $L_n = \{rz_n : r > 0\}$. Then $\Phi = \Phi(z_n)$ on L_n, and hence $\Phi > A$ on $\bigcup_{n=1}^{\infty} L_n = L$, say. Since Φ is fine upper semi-continuous at z, there exists a fine neighbourhood ω_0 of z on which $\Phi < A$. Now $\mathbb{C}\backslash\omega_0$ is thin at z (see, for example, Brelot [6, p. 90]), but L is not thin at z, since every circle of centre z clearly meets L (see, for example [9, p. 216]). These conclusions are contradictory, since $\Phi > A$ on L and hence $L \subseteq \mathbb{C}\backslash\omega_0$.

It now follows that the function $\theta \to \Phi(e^{i\theta})$ is upper semi-continuous on $[0, 2\pi]$, so that if

$$B_n = \{\theta \in [0, 2\pi] : \Phi(e^{i\theta}) \ge -n\} \quad (n = 1, 2, \ldots),$$

then each B_n is closed, and hence ∂B_n is nowhere dense. Since, for each θ, the function $r \to \phi(re^{i\theta})$ is lower semi-continuous on $[0, +\infty)$, it follows since (3) holds for all $\theta \in E$, that $\Phi(e^{i\theta}) > -\infty$ for all $\theta \in E$, and hence $E \subseteq \bigcup_{n=1}^{\infty} B_n$. It now follows that $E \cap B_m^\circ \ne \emptyset$ for some m; otherwise E would be a subset of the first category set $\bigcup_{n=1}^{\infty} \partial B_n$. Since $\phi(re^{i\theta}) \ge \Phi(e^{i\theta}) \ge -m$ for each $r > 0$ and each $\theta \in B_m$, the conclusion of the lemma will hold if we take J to be an open interval such that $J \subseteq B_m^\circ$ and $E \cap J \ne \emptyset$.

3. Proof of Theorems 1–3. We shall use the following form of a classical theorem of Ahlfors and Heins [1].

LEMMA 2. *Let u be positive and superharmonic in the sector $S = \{re^{i\theta} : r > 0,\ \theta \in J\}$, where J is an open interval of length $l \in (0, 2\pi]$.*

(i) *For all $\theta \in J$,*

$$\liminf_{r \to +\infty} r^{-\pi/l} u(re^{i\theta}) < +\infty. \tag{4}$$

(ii) *For all $\theta \in J\backslash P$, where $\{e^{i\theta} : \theta \in P\}$ is a polar subset of $\mathbb{C}$,*

$$u(re^{i\theta}) = O(r^{\pi/l}) \quad (r \to +\infty). \tag{5}$$

(iii) *If, further, u is harmonic on S, then* (5) *holds for all $\theta \in J$.*

In the case where $l = \pi$, parts (i) and (ii) are weak versions of the results (B) and (A) respectively, given in [1, p. 341], and (iii) is an easy consequence of the half-plane Poisson integral representation of a positive harmonic function (see, e.g., Tsuji [11, pp. 149–151]). Lemma 2 for arbitrary l can be obtained from the special case where $l = \pi$ by application of a conformal mapping.

Now suppose that u is a hyperharmonic function on $\mathbb{C}$ satisfying (2) for all θ belonging to a second category subset E of $[0, 2\pi)$. By Lemma 1, there exists an open interval J such that $E \cap J \neq \emptyset$ and u is bounded below on the sector $S = \{re^{i\theta} : r > 0,\ \theta \in J\}$. We may suppose that J is of length l, where $0 < l < 2\pi$, and by adding a constant to u we may suppose that $u > 0$ on S.

To complete the proof of Theorem 1, suppose further that u is harmonic on $\mathbb{C}$ and $E \subseteq \overline{U(u, \pi/l)}$. Since $E \cap J \neq \emptyset$, we have $J \cap U(u, \pi/l) \neq \emptyset$, which says that there exists $\theta \in J$ such that (5) fails, contrary to Lemma 2(iii).

If the hypotheses of Theorem 2 are satisfied, then $J \cap L(u, \pi/l) \neq \emptyset$, so that (4) fails for some $\theta \in J$. By Lemma 2(i), u cannot be superharmonic on $\mathbb{C}$; hence $u \equiv +\infty$.

If the hypotheses of Theorem 3 are satisfied, then $\{e^{i\theta} : \theta \in J \cap U(u, \pi/l)\}$ is a non-polar set, and by Lemma 2(ii), this is impossible if u is superharmonic on $\mathbb{C}$. Hence $u \equiv +\infty$.

4. Proof of Proposition 1. We start by constructing an example in the half-plane $D = \{x + iy : y > 0\}$. Let F be a countable dense subset of $(0, \pi)$. The set $Q = \{ne^{i\theta} : n = 1, 2, \ldots; \theta \in F\}$ is a countable, hence polar, subset of D. Let v be a Green potential on D such that $v = +\infty$ on Q, and let

$$F_n = \{\theta \in (0, \pi) : v(ne^{i\theta}) = +\infty\} \qquad (n = 1, 2, \ldots).$$

For all positive integers m and n, the set $\{\theta \in (0, \pi) : v(ne^{i\theta}) \leq m\}$ is relatively closed and nowhere dense in $(0, \pi)$, so that each set F_n is residual in $(0, \pi)$, and hence so also is $\bigcap_{n=1}^{\infty} F_n = F^*$, say. Note that $v(ne^{i\theta}) = +\infty$ for all positive integers n and all $\theta \in F^*$.

Now let

$$\Omega_1 = \{x + iy : |y| \in [0, 2) \cup (3, +\infty)\}$$

and

$$\Omega_2 = \{x + iy : |y| \in [0, 1) \cup (4, +\infty)\}.$$

Define w on Ω_1 by

$$w(x + iy) = \begin{cases} v(x + iy) & (y > 3), \\ 0 & (|y| < 2), \\ v(x - iy) & (y < -3). \end{cases}$$

Clearly w is superharmonic on Ω_1. Also, there exists a superharmonic function u on $\mathbb{C}$ such that $u = w + 1$ on Ω_2 (see, for example, [2, Theorem 2]). It is easy to verify that u has the required properties.

5. Higher dimensions. Lemma 2 has a generalization in which the sector S is replaced by a cone in $\mathbb{R}^N$ and the exponent π/l is replaced by a positive constant depending on the angle of the cone (see Azarin [4]), but Lemma 1 has no straightforward generalization to $\mathbb{R}^N$ ($N \geq 3$). Hence our proofs of Theorems 1–3 do not generalize to higher dimensions, and we shall show that natural analogues of Theorems 2 and 3 are indeed false in $\mathbb{R}^N$ when $N \geq 3$. However, since harmonic functions are continuous, the use of Lemma 1 is not essential to the proof of Theorem 1, and Theorem 1 is easily generalized. We note that Armitage and Goldstein [3, Theorem 2] have generalized Theorem A to $\mathbb{R}^N$.

Let Σ denote the unit sphere in $\mathbb{R}^N$.

THEOREM 1′. *Let E be a second category subset of Σ. There is no harmonic function h on $\mathbb{R}^N$ such that*

$$\liminf_{r\to+\infty} h(r\zeta) > -\infty \tag{6}$$

for all $\zeta \in E$ and such that for each positive number μ the closure of the set

$$\{\zeta \in \Sigma : \limsup_{r\to+\infty} r^{-\mu} h(r\zeta) = +\infty\}$$

contains E.

We indicate the proof. Suppose that there exists a harmonic function h with the properties described. The continuity of h and the hypothesis that (6) holds for all $\zeta \in E$ imply that h is bounded below on some open cone K, with the origin as vertex, such that $E \cap K \neq \emptyset$ (cf. [3, Lemma 1]). By hypothesis, it follows that for each $\mu > 0$ there exists $\zeta_\mu \in K$ such that

$$\limsup_{r\to+\infty} r^{-\mu} h(r\zeta_\mu) = +\infty.$$

For values of μ larger than some critical value, depending on the angle of K, this contradicts the N-dimensional version of Lemma 2(iii).

Now we justify the remark that Theorems 2 and 3 fail in $\mathbb{R}^N$ when $N \geq 3$.

PROPOSITION 2. *Let $M : [0, +\infty) \to (0, +\infty)$ be an increasing function. There exist a subset E of the unit sphere Σ of $\mathbb{R}^N$, where $N \geq 3$, and a superharmonic function u on $\mathbb{R}^N$ such that*

(i) *E is a residual subset of Σ,*
(ii) *E has full surface area measure,*
(iii) *$u(r\zeta)/M(r) \to +\infty$ as $r \to +\infty$ for each $\zeta \in E$.*

Proposition 2 clearly shows that the straightforward generalization of Theorem 2 to $\mathbb{R}^N$ is false. The same remark applies also to Theorem 3, for the condition that E has full measure implies that $G \cap E$ is a non-polar subset of $\mathbb{R}^N$ for each non-empty relatively open subset G of Σ.

To prove Proposition 2 we use the following lemmas.

LEMMA 3. *Let $M : [0, +\infty) \to (0, +\infty)$ be an increasing function. There exist a subset E_1 of Σ and a harmonic function h on $\mathbb{R}^N$ such that E_1 has full surface area measure and*

$$h(r\zeta)/M(r) \to +\infty \qquad (r \to +\infty)$$

for each $\zeta \in E_1$.

In the case where $N = 2$ this lemma is a special case of the result of Bagemihl and Seidel [5] cited in §1; for the case where $N \geq 3$, which we require here, we refer to [3, Example 6].

LEMMA 4. *Suppose that $N \geq 3$. There exist a residual subset E_2 of Σ and a positive superharmonic function v on $\mathbb{R}^N$ such that $v(r\zeta) = +\infty$ whenever $r > 0$ and $\zeta \in E_2$.*

To prove Lemma 4, we show first that if $\zeta \in \Sigma$, then there exists a positive homogeneous superharmonic function w on $\mathbb{R}^N$ such that $w(r\zeta) = +\infty$ for all $r > 0$. It suffices to deal with the case where $\zeta = (1, 0, \ldots, 0)$. In the case where $N \geq 4$ we may take w to be the potential given by

$$w(x_1, \ldots, x_N) = (x_2^2 + \ldots + x_N^2)^{(3-N)/2},$$

which is homogeneous of degree $3 - N$. In the case $N = 3$ we take w to be the potential given by

$$w(x_1, x_2, x_3) = \int_0^\infty t^{-1/2}\{(x_1 - t)^2 + x_2^2 + x_3^2\}^{-1/2}\, dt.$$

It is easy to verify that $w = +\infty$ on the positive x_1-axis, that $w \not\equiv +\infty$, and that w is homogeneous of degree $-1/2$.

Now let $\{\zeta_1, \zeta_2, \ldots\}$ be a countable dense subset of Σ. For each j let w_j be a positive superharmonic function on $\mathbb{R}^N$ such that $w_j(r\zeta_j) = +\infty$ for all $r > 0$ and w_j is homogeneous of degree $-1/2$ or $3 - N$, according as $N = 3$ or $N \geq 4$. Let $y \in \mathbb{R}^N$ be such that $w_j(y) < +\infty$ for each j, and define v on $\mathbb{R}^N$ by

$$v = \sum_{j=1}^\infty 2^{-j} w_j / w_j(y).$$

Then $v \not\equiv +\infty$, since $v(y) = 1$. Hence v is superharmonic on $\mathbb{R}^N$. From the homogeneity of the functions w_j it follows that v is homogeneous. Let $E_2 = \{\zeta \in \Sigma : v(\zeta) = +\infty\}$. Then E_2 contains the dense set $\{\zeta_1, \zeta_2, \ldots\}$, and hence for each positive integer n the closed set $\{\zeta \in \Sigma : v(\zeta) \le n\}$ is nowhere dense in Σ, so that E_2 is a residual subset of Σ. By the homogeneity of v, we have $v(r\zeta) = +\infty$ for each $r > 0$ and each $\zeta \in E_2$.

To complete the proof of Proposition 2, we take h and E_1 as in Lemma 3 and v and E_2 as in Lemma 4 and define $u = h + v$ and $E = E_1 \cup E_2$. It is clear that u and E have the properties described.

It would be interesting to determine whether or not it is possible to have $E = \Sigma$ in Proposition 2.

6. Theorems 1–3 for sectors. With obvious modifications, our main results hold for functions harmonic or superharmonic on sectors. An easy way to see this is to use extension and approximation theorems. Theorems 2 and 3 can be generalized to sectors by observing that if u is superharmonic on a sector $S_0 = \{re^{i\theta} : r > 0, |\theta| < \theta_0\}$, where $0 < \theta_0 \le \pi$, and if $0 < \theta_1 < \theta_0$, then there exists a superharmonic function $\widetilde{u}$ on $\mathbb{C}$ such that $\widetilde{u} = u$ on the set $S_1 = \{re^{i\theta} : r > 1,\ |\theta| < \theta_1\}$ (see, for example, [2, Theorem 2]). Theorem 1 can be similarly generalized by using the fact that if h is harmonic on S_0, then there exists a harmonic function $\widetilde{h}$ on $\mathbb{C}$ such that $|\widetilde{h} - h| < 1$ on S_1 (see Gauthier *et al.* [7, Theorem 4]). A generalization of this harmonic approximation result to higher dimensions (see Gauthier *et al.* [8, Theorem 1]) allows a corresponding generalization of Theorem 1′ for harmonic functions on cones.

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