COLLOQUIUM MATHEMATICUM

VOL. LXVII 1994 FASC. 2

SOME REMARKS ON HOLOMORPHIC EXTENSION IN INFINITE DIMENSIONS

BY

PHAM KHAC BAN (HANOI)

In finite-dimensional complex analysis, the extension of holomorphic maps has been investigated by many authors. In recent years some authors have considered this problem in the infinite-dimensional case. The aim of the present note is to study the extension of holomorphic maps with values in some Banach complex manifolds.

Let X be a Banach complex manifold. We say that X has the *holomorphic extension property* (briefly HEP) if every holomorphic map from a Riemann domain Ω over a Banach space having a Schauder basis to X can be extended holomorphically to $\widehat{\Omega}$, the envelope of holomorphy of Ω. Now as in [10], X is said to satisfy the *weak disc condition* if every sequence $\{f_n\}$ of holomorphic maps from the unit open disc D into X, convergent in $H(D^*, X)$, where $D^* = \{\lambda \in D : \lambda \neq 0\}$, is convergent in $H(D, X)$, the space of holomorphic maps from D into X equipped with the compact-open topology.

In this note we shall prove the following two theorems.

THEOREM A. *Let X be a Banach manifold satisfying the weak disc condition. Then X has the HEP.*

THEOREM B. *Let X be a pseudoconvex Banach manifold having C^1-partitions of unity such that every holomorphic map from D^* into X can be extended holomorphically to D. Then X has the HEP.*

Here X is called *pseudoconvex* if

$$\widehat{K}_P := \{z \in X : \varphi(z) \le \sup_K \varphi \text{ for all plurisubharmonic functions } \varphi \text{ on } X\}$$

is compact for each compact subset K of X.

Cover X by a locally finite system of coordinates $\{(U_i, \varphi_i)\}$. Let $\{V_i\}$ be another open cover of X such that $V_i \subset U_i$, $\operatorname{dist}(V_i, \partial U_i) > 0$ and $\varphi_i(V_i)$ is isomorphic to a ball in a Banach space for every i.

1991 *Mathematics Subject Classification*: Primary 58B12.

By hypothesis there exists a C^1-partition of unity $\{h_i\}$ such that $h_i = 1$ on V_i and $\operatorname{supp} h_i \subset U_i$ for every i. Let $p : TX \to X$ be the tangent bundle of X. For each $u \in TX$, put

$$\|u\| = \sum h_i(pu)\|D\varphi_i(pu)(u)\|.$$

Denote by ϱ_X the integral distance on X associated with $\|\cdot\|$. Then ϱ_X defines the topology of X (cf. [1]).

Proof of Theorem A. Let $f : \Omega \to X$ be a holomorphic map, where Ω is a Riemann domain over a Banach space B with a Schauder basis. By S_X we denote the sheaf of germs of holomorphic maps on $\widehat{\Omega}$ with values in X and by S_X^f the domain of existence of f. Then S_X^f is the component of S_X containing the set $\{(x, f_x) : x \in \Omega\}$, where f_x denotes the germ of f at x. We have the commutative diagram

$$\begin{array}{ccc} \Omega & \xrightarrow{f} & X \\ {\scriptstyle e}\big\downarrow & \overset{\alpha}{\searrow} & \big\uparrow{\scriptstyle \widetilde{f}} \\ \widehat{\Omega} & \underset{\beta}{\longleftarrow} & S_X^f \end{array}$$

in which e, α, β are canonical maps and $\widetilde{f}$ is the canonical extension of f.

(i) First we shall prove that S_X^f satisfies the weak disc condition. Given a sequence $\{u_n\} \subset H(D, S_X^f)$ convergent to u in $H(D^*, S_X^f)$. By hypothesis $\{\widetilde{f}u_n\}$ converges to h in $H(D, X)$. Take a neighbourhood U of $h(0)$ in X which is isomorphic to an open subset of a Banach space E. Then we can assume that $\widetilde{f}u_n(\delta D) \subset U$ for $n \geq 1$.

Put $K = \operatorname{cl}\operatorname{conv}(\bigcup_{n\geq 1} \widetilde{f}u_n((\delta/2)D))$. Let F be the canonical Banach space spanned by K. Then

$$\widetilde{f}u_n : \partial((\delta/2)D) \to F$$

is continuous for $n \geq 1$. Hence by the Cauchy formula

$$\widetilde{f}u_n(z) = \frac{1}{2\pi i} \int\limits_{|\lambda|=\delta/2} \widetilde{f}u_n(\lambda)(\lambda - z)^{-1}\, d\lambda$$

it follows that $\widetilde{f}u_n : (\delta/2)D \to F$ is holomorphic for all $n \geq 1$.

Now for each $n \geq 1$, consider the map

$$\widetilde{u}_n : (\delta/2)D \to \liminf_{0\in W} H^\infty(W, F)$$

where $H^\infty(W, F)$ is the Banach space of bounded holomorphic functions on W with values in F, defined by

$$\widetilde{u}_n(\lambda) = [u_n(\lambda) \circ \theta_{\gamma u_n(\lambda)}]_0$$

where $\gamma = \widehat{p}\beta$, $\widehat{p} : \widehat{\Omega} \to B$ defining Ω as a Riemann domain over B and $\theta_v(r) = v + r$.

Then $\{\widetilde{u}_n\}$ converges to $\widetilde{u}$ in $H((\delta/2)D^*, \liminf_{0\in W} H^\infty(W,F))$, and hence in $H((\delta/2)D, \operatorname{lim\,ind}_{0\in W} H^\infty(W,F))$. Since the above inductive limit is regular there exists a neighbourhood W of zero in B such that $\{\widetilde{u}_n\}$ is contained and bounded in $H^\infty(W,F)$. Extend u holomorphically to $0 \in D$ by setting $u(0) = \widetilde{u}(0) \circ \theta^{-1}_{\gamma u(0)}$. It remains to check that $\{u_n\}$ converges to u in $H(D, S^f_X)$.

For this consider the neighbourhood $\widetilde{W}$ of $u(\partial((\delta/2)D))$ given by

$$\widetilde{W} = \bigcup\{(\gamma u(\lambda) + x, [u(\lambda)]_{\gamma u(\lambda)+x}) : |\lambda| = \delta/2,\ x \in W\}.$$

Take n_0 such that $u_n(\partial((\delta/2)D)) \subset \widetilde{W}$ for all $n > n_0$. Now for each $|\lambda| = \delta/2$ and $n > n_0$ there exists a neighbourhood $W(n,\lambda)$ of $\gamma u(\lambda)$ in $\gamma u(\lambda)+W$ such that $u_n(\lambda)x = u(\lambda)x$ for all $x \in W(n,\lambda)$. Hence $u_n(\lambda)x = u(\lambda)x$ for all $x \in \gamma u(\lambda) + W$ and $|\lambda| = \delta/2$. This shows that the above equality holds for all $|\lambda| \le \delta/2$. Hence $u_n \to u$ in $H(D, S^f_X)$.

(ii) Now by (i), S^f_X is pseudoconvex.

(iii) Since B has a Schauder basis, S^f_X is the domain of existence of a holomorphic function [5]. This implies that the canonical map $\alpha : \Omega \to S^f_X$ can be extended holomorphically to $\widehat{\Omega}$. Hence $\beta : S^f_X \to \widehat{\Omega}$ is a biholomorphism and $\widetilde{f}\beta^{-1}$ is a holomorphic extension of f to $\widehat{\Omega}$. The theorem is proved.

Remark. In the finite-dimensional case Theorem A was proved by Shiffman in [10]. Our proof is different.

Proof of Theorem B. By Theorem A it suffices to show that X satisfies the weak disc condition. Consider a sequence $\{f_n\} \subset H(D,X)$ converging to f in $H(D^*,X)$. By hypothesis $f \in H(D,X)$. Put

$$K = \bigcup_{n\ge1} f_n(\partial((1/2)D)) \quad\text{and}\quad Z = (\overline{K})^\wedge_P.$$

Since $\overline{K}$ is compact and X is pseudoconvex, Z is also compact. Moreover, by the maximum principle for plurisubharmonic functions we have

$$\Big(\bigcup_{n\ge1} f_n((1/2)\partial D)\Big)^\wedge_P \supseteq \bigcup_{n\ge1} f_n((\delta/2)\overline{D}).$$

Hence

$$Z \supseteq \bigcup_{n\ge1} f_n((1/2)\overline{D}).$$

To complete the proof of Theorem B it remains to show that Z has a hyperbolic neighbourhood W. Indeed, let d_W denote the hyperbolic distance of W. Then

$$\begin{aligned} d_W(f_n(0), f(0)) &\le d_W(f_n(0), f_n(z)) + d_W(f_n(z), f(z)) + d_W(f(z), f(0)) \\ &\le 2d_W(0, z) + d_W(f_n(z), f(z)) \end{aligned}$$

and these inequalities imply that $f_n \to f$ in $H(D, X)$.

By [1] it suffices to show that there exists a neighbourhood W of Z in X for which $\sup\{\|\sigma'(0)\| : \sigma \in H(D, W)\} < \infty$. Otherwise we can find a decreasing neighbourhood basis $\{W_n\}$ of Z such that for each $n \ge 1$ there exists a holomorphic map σ_n from D to W_n such that $\|\sigma'(0)\| \ge n$. As in [3] there exists a sequence $\{\beta_n\}$ of holomorphic maps from nD to X such that $\beta_n(nD) \subseteq \sigma_n(D)$, $\|\beta'_n(0)\| = 1$ for $n \ge 1$ and $\{\beta_n\}$ is equicontinuous on every compact set in $\mathbb{C}$. Since $Z = \bigcap_{n\ge 1} W_n$, by the compactness of Z it follows that $\{\beta_n\}$ contains a subsequence $\{\alpha_n\}$ converging to a holomorphic map $\alpha : \mathbb{C} \to Z$. Obviously $\alpha \ne \text{const}$ because $\|\alpha'(0)\| = 1$.

By hypothesis α can be extended to a holomorphic map $\widehat{\alpha}$ from $\mathbb{C}P^1$ into X. Now take a holomorphic function σ on D^* which cannot be extended to a holomorphic map from D into $\mathbb{C}P^1$. By hypothesis $\gamma = \widehat{\alpha}\sigma$ extends to a holomorphic map $\widehat{\gamma} : D \to X$. Since $\widehat{\alpha}$ is nonconstant, it follows that $\widehat{\alpha}$ is a finite map. Hence we can find a neighbourhood U of $\widehat{\gamma}(0)$ such that $\widehat{\gamma}^{-1}(U)$ is a bounded domain in $\mathbb{C}$. Thus σ extends holomorphically to $0 \in D$. This is impossible so the theorem is proved.

Remark. In the case when X is a holomorphically convex space with $\dim X < \infty$ such that every holomorphic map from D^* into X extends holomorphically to D, Theorem B was proved by D. D. Thai in [12].

Acknowledgements. The author would like to thank Professor N. V. Khue and Dr. B. D. Tac for suggestions that led to an improvement of the presentation of this note.

REFERENCES

[1] P. K. Ban, *Banach hyperbolicity and the extension of holomorphic maps*, Acta Math. Vietnam. 16 (1991), 187–200.

[2] T. J. Barth, *Convex domains and Kobayashi hyperbolicity*, Proc. Amer. Math. Soc. 79 (1980), 556–558.

[3] R. Brody, *Compact manifolds and hyperbolicity*, Trans. Amer. Math. Soc. 235 (1978), 213–219.

[4] F. Docquier und H. Grauert, *Levisches Problem und Rungescher Satz für Teilgebiete Steinscher Mannigfaltigkeiten*, Math. Ann. 140 (1960), 94–123.

[5] L. Gruman et C. O. Kiselman, *Le problème de Levi dans les espaces de Banach à base*, C. R. Acad. Sci. Paris Sér. A 274 (1972), 1296–1298.

[6] A. Hirschowitz, *Prolongement analytique en dimension infinie*, Ann. Inst. Fourier (Grenoble) 22 (2) (1972), 255–292.
[7] S. M. Ivashkovich, *Hartogs' phenomenon for holomorphically convex Kähler manifolds*, Izv. Akad. Nauk SSSR Ser. Mat. 50 (1986), 866–873 (in Russian).
[8] S. Kobayashi, *Hyperbolic Manifolds and Holomorphic Maps*, Dekker, New York, 1970.
[9] P. Noverraz, *Pseudo-convexité, Convexité Polynomiale et Domaines d'Holomorphie en Dimension Infinie*, North-Holland Math. Stud. 3, North-Holland, Amsterdam, 1973.
[10] B. Shiffman, *Extension of holomorphic maps into Hermitian manifolds*, Math. Ann. 194 (1971), 249–258.
[11] B. D. Tac, *Extending holomorphic maps in infinite dimensions*, Ann. Polon. Math. 54 (1991), 241–253.
[12] D. D. Thai, *On the D^*-extension and the Hartogs extension*, Ann. Scuola Norm. Sup. Pisa 18 (1991), 13–38.

DEPARTMENT OF MATHEMATICS
PEDAGOGICAL INSTITUTE HANOI I
HANOI, VIETNAM

Reçu par la Rédaction le 5.8.1992