C O L L O Q U I U M   M A T H E M A T I C U M

VOL. LXVII   1994   FASC. 1

ON CONVOLUTION OPERATORS WITH SMALL SUPPORT WHICH ARE FAR FROM BEING CONVOLUTION BY A BOUNDED MEASURE

BY

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TO MY UNCLE RUDOLF DONNERMANN,
A RIGHTEOUS GENTILE, WHO MADE THE DIFFERENCE

Let $CV_p(F)$ be the left convolution operators on $L^p(G)$ with support included in F and $M_p(F)$ denote those which are norm limits of convolution by bounded measures in $M(F)$. Conditions on F are given which insure that $CV_p(F)$, $CV_p(F)/M_p(F)$ and $CV_p(F)/W$ are as big as they can be, namely have ℓ^∞ as a quotient, where the ergodic space W contains, and at times is very big relative to $M_p(F)$. Other subspaces of $CV_p(F)$ are considered. These improve results of Cowling and Fournier, Price and Edwards, Lust-Piquard, and others.

Introduction. Let G be a locally compact group with unit e, $F \subset G$ closed and $CV_p(F)$ be the space of left convolution operators Φ on $L^p(G)$, $1 < p < \infty$, with $\operatorname{supp}\Phi \subset F$, equipped with operator norm (see sequel). Let $M(F)$ denote the complex bounded Borel measures on F. If $\mu \in M(G)$ let $\lambda_p\mu \in CV_p(G) = CV_p$ be given by $(\lambda_p\mu)(f) = \mu * f$ for all $f \in L^p(G)$. Define $M_p(F) = \operatorname{norm\,cl} \lambda_p(M(F))$ (where cl denotes closure). Let $PM_p(G)$ be the ultraweak $(u.w)$ closure in $CV_p(G)$ of $\lambda_p(M(G))$ (where $u.w$ is the topology on $CV_p(G)$ generated by the seminorms $\Phi \to |\sum_{n=1}^\infty (\Phi f_n, g_n)|$ with $f_n \in L^p(G)$, $g_n \in L^{p'}(G)$ with $\sum_{n=1}^\infty \|f_n\|_p \|g_n\|_{p'} < \infty$, $1/p + 1/p' = 1$). As is well known (see Herz [Hz1, 2]), if G is amenable or $p = 2$, and in many other cases, $PM_p(G) = CV_p(G)$. $PM_p(G)$ is the dual of the Banach algebra $A_p(G)$ and $(PM_p(G), w^*) = (PM_p(G), u.w)$ (see [Hz1]). If G is abelian and $p = 2$ then $A_2(G) = \mathcal{F}(L^1(\widehat{G})) = A(G)$ where $\mathcal{F}$ denotes Fourier transform.

1991 *Mathematics Subject Classification*: Primary 43A22, 42B15, 22D15; Secondary 43A30, 42A45, 43A07, 44A35, 22D25.

This paper contains some results included in a preprint of ours entitled "On convolution operators which are far from being convolution by a bounded measure", a summary of which appeared in [Gr4].

Throughout this paper we sometimes omit G and instead of $CV_p(G)$, $PM_p(G)$, $A_p(G)$, etc. write CV_p, PM_p, A_p, etc.

Define $PM_p(F) = PM_p(G) \cap CV_p(F)$ and if $P \subset CV_p$ and $x \in G$ let $P_c = \text{norm cl}\{\Phi \in P : \operatorname{supp} \Phi \text{ is compact}\}$, $\sigma(P) = \{x \in G : \lambda_p \delta_x \in P\}$, $E_P(x) = \text{norm cl}\{\Phi \in P : x \notin \operatorname{supp} \Phi\}$, and $W_P(x) = \mathbb{C}\lambda_p\delta_x + E_P(x)$.

It so happens that $M_p(F) \subset W_{P_c}(x)$ for any $x \in \sigma(P)$ if $P \subset CV_p$ is any $u.w$-closed A_p-submodule with $\sigma(P) = F$ ($P = CV_p(F)$ is such).

There are many results in the literature which express the fact that for some closed subset $F \subset G$, $CV_p(F) \sim \lambda_p(M(F)) \neq \emptyset$ ($\sim$ denotes set-theoretical difference), i.e. that there are convolution operators with support included in F which are not expressible as convolution by a bounded measure.

An old result of M. Riesz expresses the fact that if $\mathbb{T}$ is the torus and $\Phi_0 \in PM_p(\mathbb{T}) = P$ is that element for which $\mathcal{F}^*\Phi_0 = 1_{\mathbb{Z}^+}$, then $\Phi_0 \notin \lambda_p(M(\mathbb{T}))$. It can in fact be shown that if $p = 2$, then $\Phi_0 \notin W_P(1)$ (a fortiori $\Phi_0 \notin M_2(\mathbb{T})$), hence $P/W_P(1) \neq \{0\}$.

Most of the results of this paper are concerned with such elements and in fact with the question of when $P/W_P(x)$ is big, for example for $P = CV_p(F)$, and in fact as big as it can be. Since for second countable G, $CV_p(G)$ is a subspace of ℓ^∞, a way to express the above is: For which closed F and for which $x \in F$, if $P = CV_p(F)$, does $P/W_P(x)$ have ℓ^∞ as a continuous linear image (i.e. have ℓ^∞ as a quotient).

If F is too thin this cannot happen. In fact, if G is abelian and F is compact and scattered (i.e. every closed subset has an isolated point) then Loomis's lemma [Lo] insures that $P = CV_2(F) = W_P(x) = M_2(F)$ for all $x \in F$.

In improving theorems of B. Brainerd and R. E. Edwards [BE], Figà-Talamanca and Gaudry [FG], J. F. Price [Pr], M. Cowling and J. Fournier [CF] prove that for any infinite locally compact group G, if $|1/q - 1/2| < |1/p - 1/2|$, there is some $\Phi \in CV_q(G)$ such that $\Phi \notin CV_p(G)$. Any such Φ clearly cannot be in $\lambda_q(M(G))$.

If $p = 2$ then Ching Chou has proved for $P = CV_2(G)$ using C^*-algebraic methods (which are not available if $p \neq 2$) that $P/W_P(e)$ has ℓ^∞ as a quotient, and $P_c/W_{P_c}(e)$ is not norm separable (see [Ch2], Thm. 3.3, Cor. 3.6 and also [Ch1]), if G is nondiscrete and second countable.

A particular case of Theorem 6 of this paper implies that if G is second countable nondiscrete, $P = PM_p(G)$ and $1 < p < \infty$, then for all $x \in G$, $P_c/W_{P_c}(x)$ and $P/W_P(x)$ (a fortiori $P_c/M_p(G)$) have ℓ^∞ as a quotient, improving results of [Gr2], p. 173.

The main contribution of this paper is, however, in controlling supports. They seem to yield results which are new even for the torus $\mathbb{T}$ and the real line $\mathbb{R}$.

Our main results improve substantially, in a sense, results of Edwards and Price [EP] (for connections with existing results see Section II). They imply, for second countable G, that if F is closed such that for some $a, b \in G$, and nondiscrete closed subgroup $H \subset G$, $\mathrm{int}_{aHb}(F) \neq \emptyset$ and $P = CV_p(F)$ then $P_c/W_{P_c}(x)$ (a fortiori $P_c/M_p(F)$) has ℓ^∞ as a quotient for all $x \in \mathrm{int}_{aHb}(F)$ ($\mathrm{int}_H(F)$ is the interior of F in H). Furthermore, the same is the case if G contains the real line $\mathbb{R}$ (or $\mathbb{T}$) and $S \subset \mathbb{R}$ is an ultrathin symmetric set and $xS \subset F$, provided $p = 2$. Moreover, our main results completely avoid considering whether F is a set of synthesis.

A combination of Theorems 6 and 12 yields

THEOREM. *Let G be second countable. Let P and Q be A_p-submodules of $PM_p(G)$ such that P is w^*-closed, Q is normed closed and $P_c \subset Q \subset P$ and $\sigma(P) = F$ ($P = PM_p(F)$ is such).*

(a) *If $H \subset G$ is a nondiscrete closed subgroup, $a, b \in G$ and $x \in \mathrm{int}_{aHb}(F)$, then $Q/W_Q(x)$ (a fortiori $Q_c/M_p(F)$ and $CV_p(F)/M_p(F)$ if such a, b, x, H exist) has ℓ^∞ as a quotient and $TIM_Q(x)$ contains the big set $\boldsymbol{\mathcal{F}}$.*

(b) *If $p = 2$ and G contains $\mathbb{R}$ (or $\mathbb{T}$) as a closed subgroup, $S \subset \mathbb{R}$ is an ultrathin symmetric set and $xS \subset F$ then $Q/W_Q(x)$ (a fortiori $Q/M_2(F)$) has ℓ^∞ as a quotient and $TIM_Q(x)$ contains the big set $\boldsymbol{\mathcal{F}}$.*

Here $TIM_Q(x) = \{\psi \in Q^* : \psi(\lambda_p\delta_x) = 1 = \|\psi\|,\ \psi(E_Q(x)) = 0\}$ (from topological invariant mean on Q at x, this being justified by Prop. 1 and Section 0). $\boldsymbol{\mathcal{F}} \subset \ell^{\infty *}$ is the big set given by $\boldsymbol{\mathcal{F}} = \{\eta \in \ell^{\infty *} : 1 = \|\eta\| = \eta(1),\ \eta(c_0) = 0\}$ where $c_0 = \{x = (x_n) \in \ell^\infty : \lim_{n\to\infty} x_n = 0\}$ and $1 \in \ell^\infty$ is the constant 1 sequence. We note that, as is well known, $\boldsymbol{\mathcal{F}}$ is a w^*-compact perfect subset of $\ell^{\infty *}$ which is as big as it can be, i.e. $\mathrm{card}\,\boldsymbol{\mathcal{F}} = \mathrm{card}\,\ell^{\infty *} = 2^c$, where c is the cardinality of the continuum.

Remark. The onto operator $t : Q/W_Q(x) \to \ell^\infty$ constructed in this theorem is such that the into w^*-w^* and norm isomorphism t^* satisfies $t^*(\boldsymbol{\mathcal{F}}) \subset TIM_Q(x)$.

We note that the Cantor middle third set F is an ultrathin symmetric set in $\mathbb{R} = G$, thus for $P = PM_2(F)$, $P/W_P(0)$, and a fortiori $P/M_2(F)$, has ℓ^∞ as a quotient.

Yet, there exist perfect Helson S-sets $F \subset \mathbb{R}^n$ (or $\mathbb{T}^n$) for $n \geq 1$, which are continuous curves if $n \geq 2$ (or even smooth curves if $n \geq 3$) by results of J.-P. Kahane [Ka2]. And for such, if $P = PM_2(F)$ then $P = W_P(x) = M_2(F) = \lambda_2(M(F))$ for all $x \in F$.

Could it be that, if F is a perfect Helson S-set, $P = PM_2(F)$ is not big enough, hence $P/M_2(F)$ cannot, by default, be big? Our Theorem I.1

insures that this is not the case and this, for all G amenable as discrete, and all $1 < p < \infty$. We have

THEOREM I.1. *Let G be amenable as a discrete group and P a w^*-closed A_p-submodule of $PM_p(G)$. If $\sigma(P) = F$ contains some compact perfect metrizable set then P and P_c have ℓ^∞ as a quotient and P does not have the WRNP.*

That $PM_2(F)$ does not have the RNP if F contains a compact perfect set and G is arbitrary abelian is a known result (see F. Lust-Piquard [P2] for much more). The fact that $PM_p(F)$ does not have even the WRNP is a new result, even for G abelian second countable and $p = 2$. Moreover, Theorem 1 cannot be much improved since $P = PM_2(T)$ is isometric to ℓ^∞. If $F \subset \mathbb{T}$ is a perfect Helson S-set then $PM_2(F) = M(F)$ does not contain ℓ^∞ (but only has ℓ^∞ as a quotient).

If $F \subset G$ is closed let $PM_{p*}(F) = w^*\text{-cl}\,\text{lin}\{\lambda_p \delta_x : x \in F\}$, $C_p(F) = C_p(G) \cap PM_p(F)$, $C_{p*}(F) = C_p(G) \cap PM_{p*}(F)$ where, following Delaporte [De2], let $C_p(G) = \{\Phi \in CV_p : \Phi\Phi' \in PF_p \text{ for } \Phi' \in PF_p\} \subset PM_p$ where $PF_p = \text{norm}\,\text{cl}\,\lambda_p(L^1(G))$. Furthermore, let $PM_{pc}(F) = PM_p(F) \cap (PM_p(G))_c$ and $PM_{p*c}(G) = PM_{p*}(F) \cap (PM_p(G))_c$.

COROLLARY A. *Let G be second countable, $F \subset G$ closed and Q be any of the eight spaces $(PM_{p*}(F))_c \subset PM_{p*c}(F) \subset C_{p*}(F) \subset PM_{p*}(F)$ or $(PM_p(F))_c \subset PM_{pc}(F) \subset C_p(F) \subset PM_p(F)$. If either* (a) *or* (b) *of the main theorem hold for x and F then $Q/W_Q(x)$ has ℓ^∞ as a quotient and $TIM_Q(x)$ contains $\boldsymbol{\mathcal{F}}$.*

We next define, following Delaporte [De1] the β (strict) topology on $CV_p(G)$ by $\Phi_\alpha \to \Phi$ iff $\|(\Phi_\alpha - \Phi)\Phi'\| \to 0$ for all $\Phi' \in PF_p$ and get

COROLLARY B. *Let G be second countable, $Q \subset C_p(G)$ a β-closed A_p-submodule of PM_p ($Q_\beta(\Phi) = \beta\text{-cl}(A_p \cdot \Phi)$ for $\Phi \in C_p(G)$ is such) and $F = \sigma(Q)$. If* (a) *or* (b) *of the main theorem hold then $Q/W_Q(x)$ and $Q_c/W_{Q_c}(x)$ (a fortiori $Q_c/M_p(F)$) has ℓ^∞ as a quotient and $TIM_Q(x)$ contains $\boldsymbol{\mathcal{F}}$.*

We further improve, in a sense, a result of R. E. Edwards and J. F. Price about elements which belong to $\bigcap_q \{PM_q(F) \sim \lambda_q(M(F))\}$.

In the end we show an easy method to construct sets $F \subset G$, for abelian G, such that if $P = PM_2(F)$ then $P/M_2(F)$ has ℓ^∞ as a quotient yet $P = W_P(x)$ for many x. We further note that Theorems 6 or 12 imply that the function algebra $A'_p(F) = A_p/P_0$ is not Arens regular for certain sets F.

The reader who will go through the proof of our Theorem 1.1 of [Gr2], which is used in Theorem 6, and that of Theorem 12 will note our indebtedness to H. Rosenthal's fundamental ℓ^1 theorem (see [Ro]).

We have the following open questions:

(a) Characterize closed sets in $\mathbb{R}^n$ (or $\mathbb{T}^n$) for which $PM_{p*}(F)/M_p(F)$ or $PM_p(F)/M_p(F)$ have ℓ^∞ as a quotient.

(b) A brilliant result of T. Körner [Ko] as improved by Saeki [S] shows that every nondiscrete abelian G contains a compact Helson set F which disobeys synthesis. If $P = PM_2(F)$, does $P/W_P(x)$ have ℓ^∞ as a quotient for some $x \in F$?

(c) Do there exist perfect subsets F of $\mathbb{R}$ or $\mathbb{T}$ such that if $P = PM_p(F)$ then $P/M_p(F)$ (or $P/W_P(x)$ for some $x \in F$) is an infinite-dimensional norm separable Banach space?

Many thanks are due F. Lust-Piquard for her kind advice on the editing of this paper.

0. Definitions and notations

(a) *Notations and remarks on locally compact groups.* Throughout let G be a locally compact group with identity e and fixed left Haar measure $\lambda = dx$ and $L^p(G)$, $1 \le p \le \infty$, the usual complex-valued function spaces (see Hewitt–Ross [HR], Vol. I) with norm $\|f\|_p = (\int |f|^p\,dx)^{1/p}$ if $p < \infty$ and $\|f\|_\infty = \operatorname{ess\,sup}|f(x)|$. If $F \subset G$ is closed let $C_{\mathrm{c}}(F)$, $C_0(F)$, $UC(F)$, $C(F)$ denote the spaces of complex continuous functions on F: with compact support, which tend to 0 at infinity, are bounded two-sided uniformly continuous, are bounded, respectively (all equipped with $\|\ \|_\infty$ norm). $WAP(G) \subset C(G)$ denotes the weakly almost periodic functions on G. If $f \in C(G)$ then $\operatorname{supp} f = \operatorname{cl}\{x : f(x) \neq 0\}$ where cl denotes closure (in G in this case). $\mathbb{C}$ denotes the field of complex numbers.

If F is locally compact, $M(F)$ denotes the space of complex bounded measures on F with variation norm [HR]. Thus $M(F) = C_0(F)^*$, where X^* always denotes the dual of the normed space X. If $F = G$ we sometimes suppress G and write C_{c}, C_0, C, L^p, etc. instead of $C_{\mathrm{c}}(G)$, $C_0(G)$, $C(G)$, $L^p(G)$, etc.

If f, g are complex functions on G and $\mu \in M(G)$ let: $f^\vee(x) = f(x^{-1})$, $f^\sim(x) = \overline{f(x^{-1})}$, $(f*g)(x) = \int g(y^{-1}x)f(y)\,dy$, $(\mu*f)(x) = \int f(y^{-1}x)\,d\mu(y)$ whenever these make sense as in [HR]. If $U \subset G$, $1_U(x) = 1$ or 0 according as $x \in U$ or $x \notin U$, and $\lambda(U)$ is the Haar measure of U. Further, let $(l_x f)(y) = f(xy)$ and $(r_x f)(y) = f(yx)$ for $x, y \in G$.

If $F, H \subset G$ then $\mathrm{int}_H(F)$ denotes the interior of F in H. Thus $x \in \mathrm{int}_H(F)$ iff $x \in U \cap H \subset F$ for some open set U in G. Set $G \sim U = \{x \in G : x \notin U\}$.

We follow all other notations on groups and convolutions from Hewitt–Ross [HR].

(b) *Notations and remarks on A_p, PM_p, CV_p.* We generally follow Herz [Hz1, 2] for notations on A_p, PM_p, CV_p except when otherwise stated. For the reader's benefit and in the interest of clarity we state below some definitions and results, some of which are stated in [Hz1, 2] in slightly different form than needed here.

$A_p(G)$: For $1 < p < \infty$, let $A_p = A_p(G)$ be, as in [Hz1], the Banach algebra of functions f on G which can be represented as $f = \sum_{n=1}^{\infty} u_n * v_n$ where $u_n \in L^{p'}(G)$ and $v_n \in L^p(G)$, with $\sum_{n=1}^{\infty} \|u_n\|_{p'} \|v_n\|_p < \infty$, $1/p + 1/p' = 1$, with norm $\|f\|_{A_p}$ being the infimum of the last sum over all representations of f.

$S^p_A(x)$: If $x \in G$ define $S^p_A(x) = \{v \in A_p : v(x) = \|v\| = 1\}$ and $S^p_A(e) = S^p_A$ (the set of "states" of A_p) where $\|v\|$ stands for $\|v\|_{A_p}$ or other norms, obvious from the context.

If $E \subset G$ is closed set $I_E = \{v \in A_p : v = 0 \text{ on } E\}$ and $J_E = \{v \in A_p \cap C_c(G) : E \cap \operatorname{supp} v = \emptyset\}$. If $J \subset A_p$ is any closed ideal whose zero set is $Z(J) = \{x \in G : u(x) = 0 \text{ for all } u \in J\} = E$ then $J_E \subset J \subset I_E$ (see [Hz1]).

$CV_p(G)$ (denoted by $CONV_p(G)$ in [Hz1]) is the algebra of bounded convolution operators Φ on $L^p(G)$ with operator norm. Thus if $\Phi \in CV_p$ then $\Phi(f * v) = (\Phi f) * v$ for all $f \in L^p$ and $v \in C_c(G)$. If $PM_p(G) = u.w$-cl $\lambda_p(M(G))$ then $A_p(G)^* = PM_p(G)$ and $(PM_p, u.w) = (PM_p, w^*)$ (i.e. the $u.w$-topology restricted to PM_p coincides with the w^*-topology, see [Hz1], p. 116, Pier [Pi], p. 94, Prop. 10.3). Furthermore, both PM_p and CV_p are A_p-modules (see [Hz1] and Derighetti [Der], pp. 8–9) and if $\Phi \in PM_p$ and $u, v \in A_p$ then $(u \cdot \Phi, v) = (\Phi, uv)$. If G is amenable or $p = 2$, and in many other cases, $PM_p(G) = CV_p(G)$.

If $\mu \in M(G)$ then $\lambda_p \mu \in PM_p$ is given by $(\lambda_p \mu)(f) = \mu * f$ for $f \in L^p$. We will omit p at times, and write $\lambda(\mu)$. If $x \in G$, then $\delta_x \in M(G)$ is the point mass at x and we write δ_x instead of $\lambda_p \delta_x$ at times.

If $\Phi \in CV_p(G)$, define the support of Φ ([Hz1], p. 116), $\operatorname{supp} \Phi$, by: If $u \in L^p$ then $x \notin \operatorname{supp} u$ iff there is a neighborhood V of x such that $\int uv\, dx = 0$ for all $v \in C_c$ with $\operatorname{supp} v \subset V$. Define $\operatorname{supp} \Phi$ by: $x \notin \operatorname{supp} \Phi$ iff there is a neighborhood U of e such that $x \notin \operatorname{supp} \Phi(u)$ for all $u \in C_c$ with $\operatorname{supp} u \subset U$.

It is shown in [Hz1], p. 120, that for $\mu \in M(G)$, $\operatorname{supp} \lambda_p \mu = \operatorname{supp} \mu$ (as a measure) and $(\lambda_p \mu, v) = \int v\, d\mu$ for $v \in A_p$. Moreover, if $v \in A_p$ and $\Phi \in CV_p$ then $\operatorname{supp} u \cdot \Phi \subset \operatorname{supp} u \cap \operatorname{supp} \Phi$ ([Hz1], p. 118).

Let $P \subset CV_p(G)$. Define

$$\sigma(P) = \{x : \lambda_p \delta_x \in P\}, \qquad P_c = \operatorname{ncl}\{\Phi \in P : \operatorname{supp} \Phi \text{ is compact}\}$$

and $PM_{pc} = PM_{pc}(G) = (PM_p(G))_c$, where ncl denotes norm closure. Clearly $\sigma(P) = \sigma(P \cap PM_p)$. Furthermore, let

$$E_P(x) = \operatorname{ncl}\{\Phi \in P : x \notin \operatorname{supp} \Phi\}, \qquad W_P(x) = \mathbb{C}\lambda_p \delta_x + E_P(x).$$

($E_P(x)$ is sometimes called the *null-ergodic space* of P at x.) Note that $P_{\rm c} \subset PM_p(G)$ always holds. Indeed, if $\Phi \in CV_p$ has compact support then by [Hz1], p. 117, Φ is the ultrastrong (hence $u.w$) closure of $\{\lambda_p(w) : w \in C_{\rm c}(G),\ \|\lambda_p(w)\| \le \|\Phi\|\}$ and PM_p is $u.w$-closed in CV_p.

For closed $F \subset G$, define

$$\begin{aligned} CV_p(F) &= \{\Phi \in CV_p : \operatorname{supp}\Phi \subset F\},\\ PM_p(F) &= \{\Phi \in PM_p : \operatorname{supp}\Phi \subset F\},\\ PM_{p*}(F) &= w^*\text{-cl}\,\text{lin}\{\lambda_p\delta_x : x \in F\},\\ M_p(F) &= \text{ncl}\{\lambda_p\mu : \mu \in M(F)\}. \end{aligned}$$

If $P \subset CV_p(G)$ is an A_p-submodule and $x \in \sigma(P)$ let

$$\begin{aligned} TIM_P(x) = \{\psi \in P^* : \psi(\lambda_p\delta_x) = 1 = \|\psi\|,&\\ \psi(u\cdot\Phi) = \psi(\Phi) \text{ for all } \Phi \in P,\ u \in S_A^p(x)\}& \end{aligned}$$

and $TIM_P(e) = TIM_P$. (TIM from topologically invariant mean.) If $p = 2$ and G is abelian and $P = PM_2(G)$ with $\mathcal{F}^*P = L^\infty(\widehat{G})$, where $\mathcal{F}$ denotes Fourier transform, then

$$\begin{aligned} TIM_P(a) = \mathcal{F}^{**}\Big\{\psi \in L^\infty(\widehat{G})^* : \psi(h) = \psi((\overline{a}f) * h),&\\ \text{for all } 0 \le f \in L^1(\widehat{G}) \text{ with } \int f\,dx = 1, \text{ and } h \in L^\infty(\widehat{G})\Big\}.& \end{aligned}$$

We stress that we usually omit G and write PM_p, PM_{pc}, CV_p, etc. instead of $PM_p(G)$, $PM_{pc}(G)$, $CV_p(G)$, etc.

(c) *Some remarks on Banach spaces*. Let ℓ^∞ be the space of complex bounded sequences $x = (x_n)$ with $\|x\| = \sup|x_n|$. Define by c (resp. c_0): $\{x = (x_n) \in \ell^\infty : \lim_n x_n \text{ exists (resp. } \lim_n x_n = 0)\} \subset \ell^\infty$.

Let $\boldsymbol{\mathcal{F}} = \{F \in \ell^{\infty*} : F(1) = 1 = \|F\|,\ F = 0 \text{ on } c_0\}$.

Note that $\boldsymbol{\mathcal{F}}$ is as "big" as it can be, since $\beta\mathbb{N} \sim \mathbb{N} \subset \boldsymbol{\mathcal{F}}$ and $\operatorname{card}\boldsymbol{\mathcal{F}} = 2^{\operatorname{card}\mathbb{R}} = \operatorname{card}\ell^{\infty*}$, where $\mathbb{R}$ denotes the real line. $\boldsymbol{\mathcal{F}}$ is a convex w^*-compact perfect subset of $(\ell^{\infty*}, w^*)$.

Note that if G is second countable then $CV_p(G)$ is isometric to a subspace of ℓ^∞ (if $\{f_n\}$ and $\{g_n\}$ are norm dense sequences in the unit ball of $L^p(G)$ and $L^{p'}(G)$ respectively, then, for $\Phi \in CV_p$, let $(t\Phi)(n,m) = (\Phi f_n, g_m) \in \ell^\infty(\mathbb{N}\times\mathbb{N}) \subset \ell^\infty$).

Hence the assertion "$P/W_P(x)$ and $P/M_p(F)$ have ℓ^∞ as a quotient" means, since $P \subset \ell^\infty$, that these spaces are as big (and as complex) as they can be. If Y is any norm separable Banach space, or any dual of such, then since $Y \subset \ell^\infty$, there is some subspace $X \subset P/W_P(x)$ which has Y as a quotient.

We follow Rudin [Ru2] in notations for normed spaces. If X is a Banach space and $K \subset X^*$ we say the K *contains* $\boldsymbol{\mathcal{F}}$ if there is an *onto* linear bounded map $t : X \to \ell^\infty$ such that $t^* : \ell^{\infty *} \to X^*$, which is easily seen to be a w^*-w^*-continuous norm isomorphism into (see sequel), satisfies in addition $t^*(\mathcal{F}) \subset K$.

Let X, Y be Banach spaces, $t : X \to Y$ a bounded linear map (an operator for short). t is an isomorphism (into or onto) if for some $a, b > 0$, $a\|x\| \le \|tx\| \le b\|x\|$ for all $x \in X$. If $t : X \to Y$ is an onto operator then $X/t^{-1}(0) \approx Y$ (are isomorphic Banach spaces). Furthermore, if $Y_0 \subset Y$ is a closed subspace then $X/t^{-1}(Y_0) \approx Y/Y_0$, since if $s : Y \to Y/Y_0$ is the canonical map then $(st)^{-1}(0) = t^{-1}(Y_0)$.

Hence if $t : X \to \ell^\infty$ is an onto operator then X has the quotient $X/t^{-1}(0)$ isomorphic to ℓ^∞, and conversely. In this case for any closed subspace $W \subset t^{-1}(0)$, X/W has $X/t^{-1}(0) \approx \ell^\infty$ as a quotient (X/W is always equipped with the quotient norm).

If $X \subset Y$ are Banach spaces and X has ℓ^∞ as a quotient so does Y, since any operator $t : X \to \ell^\infty$ admits an extension operator $t_1 : Y \to \ell^\infty$ by the injectivity of ℓ^∞ (see [LT]).

If X is a Banach space and $K \subset X^*$ then $w^*\text{-seq}\,\text{cl}\,K = \{y^* \in X^* : y^* = w^*\text{-}\lim x_n^* \text{ for some } \textit{sequence } \{x_n^*\} \subset K\}$. ($y^* = w^*\text{-}\lim_n x_n^*$ iff $y^*(x) = \lim_n x_n^*(x)$ for all $x \in X$.) This is the w^*-*sequential closure* of K in X^*.

If $B \subset X$ then $\text{ncl}\,B$ is the norm closure of B in X; if $B \subset X^*$ then $w^*\text{-cl}\,B$ is the w^*-closure of B in X^*. $\text{lin}\,B$ is the linear span of B.

I. When P has ℓ^∞ as a quotient. Let $P \subset PM_p(G)$ be a w^*-closed A_p-module with $\sigma(P) = F$. If F contains some compact perfect set then P cannot be norm separable since $\|\lambda_p\delta_x - \lambda_p\delta_y\| \ge 1$ if $x, y \in F$ and $x \ne y$. If G is second countable then P is the dual of the (norm) separable space A_p/J (where $J = (P)_0$). By Stegall's theorem ([DU], p. 195), P does not have the RNP.

The dual X^* of a Banach space X has the weak RNP (WRNP) iff X does not contain an isomorph of ℓ^1. R. C. James has constructed separable Banach spaces X which do not contain ℓ^1 and such that X^* is not norm separable, i.e. X^* has the WRNP but not the RNP ([DU], p. 214).

We will prove that if G_{d} (i.e. G with discrete topology) is amenable and F contains a compact perfect metrizable set then P cannot be a James space, in fact P cannot have the WRNP, thus has ℓ^∞ as a quotient. This result, which is new even if $p = 2$ and G is abelian, cannot be much improved since $PM_2(\mathbb{T})$ is isometric to ℓ^∞.

We use in the proof a beautiful result of E. Saab [Sa] which states that the dual space X^* has the WRNP iff for every w^*-compact set $M \in X^*$, the restriction of any $x^{**} \in X^{**}$ to (M, w^*) has a point of continuity.

THEOREM 1. *Let G be amenable as a discrete*, *locally compact group. Let P be a w^*-closed A_p-submodule of $PM_p(G)$. If $\sigma(P)$ contains some compact* *perfect metrizable set then P does not have the WRNP, and both P_{c} and P have ℓ^∞ as a quotient.*

Proof. Let $K \subset \sigma(P)$ be perfect, metrizable and compact and let the countable set $S_1 \subset K$ be dense in K. Let H be the group generated (algebraically) by S_1. Then 1_H is a positive definite function on G_{d} (which is amenable). Hence the linear functional F defined on the dense subspace $\{\sum_{i=1}^n \alpha_i \lambda\delta_{x_i} : \alpha_i \in \mathbb{C},\ x_i \in G,\ n \geq 1\}$ of $PF_2(G_{\mathrm{d}})$ (with $PM_2(G_{\mathrm{d}})$ norm) by $(F, \sum_i \alpha_i\lambda\delta_{x_i}) = \sum_i \alpha_i 1_H(x_i)$ is continuous and in fact $\|F\| = \|1_H\|_{B_\lambda(G_{\mathrm{d}})} = 1$. Thus

$$\Big|\Big(F, \sum_{i=1}^n \alpha_i\lambda\delta_{x_i}\Big)\Big| \leq \Big\|\sum_{i=1}^n \alpha_i\lambda\delta_{x_i}\Big\|_{PF_2(G_{\mathrm{d}})}$$

for all $\alpha_i \in \mathbb{C}$, $x_i \in G$, $n \geq 1$. It is, however, well known (see for example [DR2], pp. 437–438) that

$$\Big\|\sum_{i=1}^n \alpha_i\lambda\delta_{x_i}\Big\|_{PF_2(G_{\mathrm{d}})} \leq \Big\|\sum_{i=1}^n \alpha_i\lambda\delta_{x_i}\Big\|_{PF_2(G)}.$$

But Herz's main theorem [Hz3] shows that the embedding $PM_p \subset PM_2$ is a contraction if G is amenable. It follows that

$$\Big|\Big(F, \sum_{i=1}^n \alpha_i\lambda\delta_{x_i}\Big)\Big| = \Big|\sum_{i=1}^n \alpha_i 1_H(x_i)\Big| \leq \Big\|\sum_{i=1}^n \alpha_i\lambda\delta_{x_i}\Big\|_{PM_p(G)}.$$

Thus F can be extended as a continuous linear functional F_0 on P. Consider now F_0 restricted to the w^*-compact set $K_P = \{\lambda\delta_x : x \in K\}$, a subset of the unit ball of P. (Recall that $x \to \lambda_p\delta_x$ from G to (PM_p, w^*) is bicontinuous and one-to-one.) Then, if $S = H \cap K$, for each $x \in K$ we have $(F_0, \lambda\delta_x) = (F, \lambda\delta_x) = 1_S(x)$. The set S is a countable dense subset of K, since $S_1 \subset S$. And by the known remark that follows this proof, $K \sim S$ is also dense in K. Hence 1_S, as a *function on* K, has *no point of continuity*. Consequently, the functional F_0 restricted to the w^*-compact set (K_P, w^*) has no point of w^*-continuity. By Theorem 1 of E. Saab [Sa] the Banach space P does *not* have the WRNP. (Since P is w^*-closed, it is the dual of the Banach space $A_p(G)/J$ where $J = \{v \in A_p : (\Phi, v) = 0 \text{ for all } \Phi \in P\}$.) Hence by a well known theorem of H. Rosenthal, A_p/J contains a subspace L isomorphic to ℓ^1 ([Ro], p. 808). Thus L^* is isomorphic to ℓ^∞. Let now $t : P \to L^*$ be the restriction map $(t\Phi, x) = (\Phi, x)$ for all $x \in L$. Then t is an *onto* continuous linear map. Clearly $PM_{p*}(K) \subset P_{\mathrm{c}}$, and $PM_{p*}(K)$ (hence by injectivity P_{c}) has ℓ^∞ as a quotient. ∎

Remarks. 1. If K is locally compact with no isolated points and $S \subset K$ is countable then $K \sim S = S_1$ is dense in K. If not, $K \sim \mathrm{cl}\, S_1 \neq \emptyset$ is open and $K \sim \mathrm{cl}\, S_1 \subset K \sim S_1 = S$. Thus $K \sim \mathrm{cl}\, S_1$ is countable and locally compact. By Baire's theorem there is some $x_0 \in K \sim \operatorname{cl} S_1$ which is open in $K \sim \operatorname{cl} S_1$, hence in K. But K has no isolated points.

2. It can be shown that if G is not metrizable and $\sigma(P)$ contains any nonvoid Baire set in G (as in [HR], (11.1)) and G_{d} is amenable then P does not have the WRNP.

3. (a) If $F \subset G$ is a perfect Helson S-set then $PM_2(F)$ does not contain an isomorph of ℓ^∞, since $M(F)$ and all its closed subspaces are (while ℓ^∞ is not) weakly sequentially complete. Yet $PM_2(F)$ has ℓ^∞ as a quotient. $PM_2(F)$ is not, though, a quotient of ℓ^∞ since by H. Rosenthal's theorem ([DU], p. 156), $PM_2(F)$ would be reflexive, hence by a theorem of Glicksberg F would be finite.

(b) If $F \subset \mathbb{R}$ is an ultrathin symetric set (see Thm. 12) then $c_0 \subset PM_2(F)$ by Y. Meyer's theorem (see [P3], p. 201) hence $\ell^\infty \subset PM_2(F)$ (see [DU], p. 23).

II. When $P/W_P(x)$ has ℓ^∞ as a quotient

(a) *The case that* $1 < p < \infty$. The main result of this section is Theorem 6. We need in its proof some properties of norm closed A_p-submodules P of $CV_p = CV_p(G)$ (or of $PM_p = PM_p(G)$) and of the null-ergodic subspaces of P at x, $E_P(x)$.

PROPOSITION 1. *Let P be a norm closed A_p-submodule of $CV_p(G)$ and $a \in G$. Let $S \subset S_A^p(a)$ have the property that for any neighborhood V of a there is some $v \in S$ such that* $\operatorname{supp} v \subset V$. *Then* $E_P(a) = \operatorname{ncl}\{\Phi - u \cdot \Phi : u \in S_A^p(a),\ \Phi \in P\} = \operatorname{ncl}\{\Phi - u \cdot \Phi : u \in S,\ \Phi \in P\}$. *Consequently*, $TIM_P(a) = \{\psi \in P^* : \psi(\delta_a) = 1 = \|\psi\|,\ \psi(E_P(a)) = 0\}$.

Remark. If G is amenable then $S = S_A^2(a) \subset S_A^p(a)$ satisfies the above condition.

Proof of Proposition 1. If $\Phi \in P$ and $a \notin \operatorname{supp} \Phi$ let $v \in S$ be such that $\operatorname{supp} \Phi \cap \operatorname{supp} v = \emptyset$. Then by [Hz1], Prop. 10, p. 118, $\operatorname{supp} v \cdot \Phi = \emptyset$. But $v \cdot \Phi \in PM_p$. Thus $v \cdot \Phi = 0$ by [Hz1], p. 101, and $\Phi = \Phi - v \cdot \Phi$. Thus $E_P(a) \subset \operatorname{ncl}\{\Phi - u \cdot \Phi : u \in S,\ \Phi \in P\}$.

Let now $\Phi \in P$ and $u \in S_A^p(a)$. Let $v \in A_p \cap C_{\mathrm{c}}$ be such that $v = 1$ on a neighborhood V of a. Then for any $w \in A_p$ with $\operatorname{supp} w \subset V$ one has

$$((\Phi - u \cdot \Phi) - v \cdot (\Phi - u \cdot \Phi), w) = (\Phi - u \cdot \Phi, w - vw) = 0$$

since $v = 1$ on supp w. It follows by [Hz1], p. 101, that $a \notin \operatorname{supp}[\Phi - u \cdot \Phi - v \cdot (\Phi - u \cdot \Phi)]$. Thus if we show that $(v - uv) \cdot \Phi \in E_P(a)$ it will follow that $\Phi - u \cdot \Phi \in E_P(a)$. Now $(v - uv)(a) = 0$ and $v - uv \in A_p$. Since points are sets of synthesis ([Hz1], pp. 91–92), there is a sequence $v_n \in A_p \cap C_{\mathrm{c}}$ such that $v_n = 0$ on a neighborhood V_n of a and such that $\|v_n - (v - uv)\| \to 0$. But then $a \notin v_n \cdot \Phi$ ([Hz1], p. 118), $v_n \cdot \Phi \in E_P(a)$, and $\|v_n \cdot \Phi - (v - uv)\cdot \Phi\| \to 0$, since CV_p is an A_p-module. Thus $(v - uv) \cdot \Phi \in E_P(a)$. ∎

PROPOSITION 2. *Let P be a norm closed A_p-submodule of CV_p. Then* $E_{P_c}(a) = \mathrm{ncl}\{\Phi \in P : a \notin \operatorname{supp}\Phi,\ \operatorname{supp}\Phi$ *is compact*$\}$.

Proof. By definition $E_{P_c}(a) = \mathrm{ncl}\{\Phi \in P_c : a \notin \operatorname{supp}\Phi\}$. It is thus enough to show that any $\Phi_0 \in P_c$ such that $a \notin \operatorname{supp}\Phi_0$ can be approximated in norm by elements $\Phi \in P$ with compact support such that $a \notin \operatorname{supp}\Phi$.

Since $\Phi_0 \in P_c$, there are $\Phi_n \in P_c$ with compact support such that $\|\Phi_n - \Phi\| \to 0$. Let $v_0 \in A_p \cap C_c$ satisfy $v_0 = 1$ on a neighborhood of a and $\operatorname{supp} v_0 \cap \operatorname{supp}\Phi_0 = 0$. Thus $\operatorname{supp} v_0 \cdot \Phi_0 = \emptyset$ ([Hz1], p. 118) and $v_0 \cdot \Phi_0 \in PM_p$. Thus $v_0 \cdot \Phi_0 = 0$.

Let now $v_n \in A_p \cap C_c$ be such that $v_n = 1$ on a neighborhood of $\operatorname{supp}\Phi_n \cup \operatorname{supp} v_0$. Then $v_n \cdot \Phi_n = \Phi_n$ and $(v_n - v_0)\cdot\Phi_n = \Phi_n - v_0\cdot\Phi_n \to \Phi_0 - v_0\cdot\Phi_0 = \Phi_0$, in norm. But $a \notin \operatorname{supp}(v_n - v_0)$, hence $a \notin \operatorname{supp}(v_n - v_0)\cdot\Phi_n$. ∎

PROPOSITION 3. *Let P, Q be A_p-submodules of $PM_p(G)$ such that P is w^*-closed, Q is norm closed and $P_c \subset Q \subset P$. Let $F = \sigma(P)$. Then for any $a \in F$, $M_p(F) \subset \mathbb{C}\lambda\delta_a \oplus E_Q(a)$, and the sum is direct.*

Proof. It is enough to prove that for any probability measure $\mu \in M(F)$, $\lambda\nu = \lambda(\mu - \mu\{a\}\delta_a)$ belongs to $E_{P_c}(a) \subset E_Q(a)$ (we write λ instead of λ_p). Let $\varepsilon > 0$. There is by regularity a compact $K \subset F \sim \{a\}$ such that $\|\nu - \nu_K\|_{M(F)} < \varepsilon$ where $\nu_K(B) = \nu(K \cap B)$ for all Borel subsets $B \subset G$. But then $\|\lambda(\nu - \nu_K)\|_{PM_p} \le \|\nu - \nu_K\|_{M(F)} < \varepsilon$. Since supp $\lambda\nu_K = \operatorname{supp}\nu_K$ (as a measure), $a \notin \operatorname{supp}\lambda\nu_K$. It is hence enough to show that $\lambda\nu_K \in P$. It will then follow, since supp $\lambda\nu_K$ is compact, that $\lambda\nu_K \in P_c$, hence $\lambda\nu_K \in E_{P_c}(a)$.

Let $\nu_0 = \nu(K)^{-1}\nu_K$. There is a net ν_α of convex combinations of $\{\delta_x : x \in K\} \subset P$ such that $\int v\,d\nu_\alpha \to \int v\,d\nu_0$ for all $v \in C_0(G)$, a fortiori for $v \in A_p$. Thus $\lambda\nu_\alpha \to \lambda\nu_0$ in (PM_p, w^*). Thus $\lambda\nu_0 \in P$ since P is w^*-closed and $\lambda\nu_K \in P$.

To show that $\mathbb{C}\lambda\delta_a \oplus E_Q(a)$ is a direct sum let V_α be a base of neighborhoods at a and $v_\alpha \in A_p$ be such that supp $v_\alpha \subset V_\alpha$ and $\|v_\alpha\| = 1 = v_\alpha(a)$. Then $\|v_\alpha \cdot \Phi\| \to 0$ for any $\Phi \in P$ such that $a \notin \operatorname{supp}\Phi$, hence for any $\Phi \in E_P(a)$, and a fortiori for any $\Phi \in E_Q(a)$. Yet $v_\alpha \cdot \lambda\delta_a = \lambda\delta_a$. ∎

Remark. It follows that $\mathbb{C}\lambda\delta_a \oplus E_Q(a)$ is a norm closed subspace of Q.

The crux of the proof of the main Theorem 6 is in fact included in the proof of the next:

THEOREM 4. *Let G be a second countable locally compact group. Let P, Q be A_p-submodules of $PM_p(G)$ such that P is w^*-closed, Q is norm closed, and $P_c \subset Q \subset P$. Let S_P and S_Q be separable linear subspaces of P and Q respectively, and define $W_P = \mathrm{ncl}(E_P + S_P)$, $W_Q = \mathrm{ncl}(E_Q + S_Q)$ and*

$F = \sigma(P)$. If $e \in \mathrm{int}_H(F)$ for some closed nondiscrete subgroup $H \subset G$, then P/W_P and Q/W_Q have ℓ^∞ as a quotient and both TIM_P and TIM_Q contain $\mathcal{F}$.

Remark. The main part of Theorem 4 is the Q/W_Q part. To prove it we need first to prove the P/W_P part and using this, we show that Q/W_Q has ℓ^∞ as a quotient.

Proof of Theorem 4. Let

$$J = (P)_0 = \{v \in A_p : (v, \Phi) = 0 \text{ for all } \Phi \in P\}.$$

Then by duality $P = (A_p/J)^*$ and J is a closed ideal whose zero set is $F = \sigma(P)$. (Thus $J_F \subset J \subset I_F$, see [Hz1], p. 101.)

If $v \in A_p$ set $v' = v + J \in A_p/J$ with quotient norm $\|v'\| = \inf\{\|v - u\| : u \in J\}$. If $\Phi \in P$ then $(\Phi, v') = (\Phi, v)$ is well defined and $|(\Phi, v')| \le \|\Phi\|\|v'\|$. Let $v \in S_A^p = \{v \in A_p : 1 = \|v\| = v(e)\}$. Then for any $u \in J$, since $e \in F$, $1 = v(e) + u(e) \le \|v - u\|$.

Thus $1 - v(e) \le \inf\{\|v - u\| : u \in J\} = \|v'\| \le \|v\| = 1$ and $v'(e) = 1 = \|v'\|$, where we define for $x \in F$, $v'(x) = (\lambda\delta_x, v') = v(x)$. (For $x \in F$, $w' \to w'(x)$ is a multiplicative linear functional on the Banach algebra A_p/J.)

Since S_A^p is a convex set, it follows from the above that $(S_A^p)' = \{v' : v \in S_A^p\}$ is a convex subset of the unit sphere of A_p/J.

Since $P \subset PM_p(G)$ and also $P = (A_p/J)^*$, both algebras A_p and A_p/J (which is a function algebra on F) act on P, namely:

$$(v', \Phi, w') = (\Phi, v'w') = (\Phi, vw) = (v \cdot \Phi, w) \qquad \text{for } v, w \in A_p \text{ and } \Phi \in P,$$

the last two expressions being independent of which representatives v, w of v', w' (resp.) we chose, since $\Phi \in P$. Thus $v' \cdot \Phi = v \cdot \Phi$ as elements of $(A_p/J)^*$, which is identified with $P \subset PM_p$, and $\|v' \cdot \Phi\| \le \|v'\|\|\Phi\|$.

By our Theorem 5 on p. 123 and the remark on p. 122, both of [Gr1], there is some $\psi_0 \in w^*\text{-cl}\, S_A^p \subset PM_p^*$ such that $1 = (\psi_0, \lambda\delta_e) = \|\psi_0\|$ and $(\psi_0, u \cdot \Phi) = (\psi_0, \Phi)$ for all $u \in S_A^p$ and $\Phi \in PM_p$ (we consider $A_p \subset PM_p^*$). The restriction ψ_0^1 of ψ_0 to P will satisfy, since $\lambda\delta_e \in P$ and $\|\psi_0^1\| \le 1$, that $1 = (\psi_0^1, \lambda\delta_e) = \|\psi_0^1\|$ and $(\psi_0^1, u \cdot \Phi) = (\psi_0, u' \cdot \Phi) = (\psi_0^1, \Phi)$ for all $u \in S_A^p$ (i.e. $u' \in (S_A^p)'$) and $\Phi \in P$. Now, considering $(S_A^p)' \subset P^*$, we have $\psi_0^1 \in w^*\text{-cl}(S_A^p)' \subset P^*$. In fact, if $v_\alpha \in S_A^p$ are such that $(\Phi, v_\alpha) \to (\psi_0, \Phi)$ for all $\Phi \in PM_p$, then for $\Phi \in P$ and $u_\alpha \in J$, we have $(\Phi, v'_\alpha) = (\Phi, v_\alpha + u_\alpha) = (\Phi, v_\alpha) \to (\psi_0, \Phi) = (\psi_0^1, \Phi)$.

Considering $(S_A^p)' \subset P^*$ define $\mathbf{A} = \{\psi \in w^*\text{-}\mathrm{cl}(S_A^p)' : u' \cdot \psi = \psi \text{ for all } u' \in (S_A^p)'\} \subset P^*$ where $(u' \cdot \psi, \Phi) = (\psi, u' \cdot \Phi)$ for $\Phi \in P$, $u' \in A_p/J$ and $\psi \in P^*$.

Clearly $\mathbf{A} \ne \emptyset$ since $\psi_0^1 \in \mathbf{A}$. Now G is second countable, hence A_p (a fortiori A_p/J) is norm separable (since $L^p(G)$ is such). Thus S_A^p and a fortiori $(S_A^p)'$ is norm separable. *We stress* that we only need the norm separability of $(S_A^p)'$ in A_p/J, and not the fact that G is second countable, in the proof.

Let hence $a_n \in S_A^p$ be such that $\{a_n'\}$ is dense in $(S_A^p)'$. If $u' \in (S_A^p)'$ and $\|a_{n_k}' - u'\| \to 0$ then $\|(a_{n_k}' - u') \cdot \Phi\| \to 0$ for any $\Phi \in P$. Hence if $\psi \in P^*$ and $a_n' \cdot \psi = \psi$ for all n then $u' \cdot \psi = \psi$ for all $u' \in (S_A^p)'$. This shows that $\mathbf{A} = \{\psi \in w^*\text{-cl}(S_A^p)' : a_n' \cdot \psi = \psi \text{ for all } n\} = \{\psi \in w^*\text{-cl}(S_A^p)' : S_n^{**}(\psi) = 0 \text{ for all } n\}$ where $S_n : A_p/J \to A_p/J$ are the operators defined by $S_n(v') = a_n' v' - v'$.

We will need in the proof the following which we state as "Claim 5" for further use:

CLAIM 5. *Under the above assumptions*, $\mathbf{A} \cap w^*\text{-seq}\,\text{cl}(S_A^p)' = 0$.

Proof. Assume that $\psi_1 \in \mathbf{A}$ is such that for some sequence $v_n \in S_A^p$, we have $(v_n', \Phi) \to (\psi_1, \Phi)$ for all $\Phi \in P$. Let $r : A_p(G) \to A_p(H)$ be the restriction map $(rv)(x) = v(x)$ for all $x \in H$. Then by Herz [Hz1], p. 92, Theorems 1a and 1b, r is onto, $\|r\| \le 1$ and $r^* : A_p(H)^* \to PM_p(H)$ is an *into* isometric inclusion (where $PM_p(H) = \{\Phi \in PM_p(G) : \text{supp}\,\Phi \subset H\}$; see [Hz1], Theorem A, p. 91, and note that H need not be a set of synthesis).

Let V_0, U be open in G such that $\text{cl}\,V_0$ is compact, $H \cap \text{cl}\,V_0 \subset F$, $U = U^{-1}$ and $e \in U \subset U^3 \subset V_0$. Let $v_0 = \lambda(U)^{-1} 1_U * 1_U \in A_p(G)$. Then $v_0(e) = 1 = \|v_0\|$, $v_0 = 0$ off U^2, thus $\text{supp}\,v_0 \subset \text{cl}\,U^2 \subset U^3 \subset V_0$ and $v_0 \in S_A^p$.

We claim that the sequence $r(v_0 v_n)$ is a weak Cauchy sequence in the Banach space $A_p(H)$. Let in fact $T \in A_p(H)^*$ and $w \in J$. Then $(v_0 \cdot r^*T, w) = (T, r(v_0 w)) = (T, 0) = 0$, since $w = 0$ on F and $\text{supp}\,r(v_0 w) = \text{supp}(rv_0)(rw) \subset V_0 \cap H \subset F$. Thus $r(v_0 w)(x) = 0$ for all $x \in H$.

But then $v_0 \cdot r^*T \in J^0 = \{\Phi \in PM_p(G) : (\Phi, v) = 0 \text{ for all } v \in J\}$. However, $J = (P)_0$ and $J^0 = ((P)_0)^0 = P$ since P is w^*-closed. Hence $v_0 \cdot r^*T \in P$ for any $T \in A_p(H)^*$. It follows that for any such T,

$$(T, r(v_0 v_n)) = (v_0 \cdot r^*T, v_n) = (v_0 \cdot r^*T, v_n') \to (\psi_1, v_0 \cdot r^*T).$$

Hence $r(v_0 v_n)$ is a weak Cauchy sequence in the closed subspace of $A_p(H)$ given by $A_K^p(H) = \text{ncl}\{v \in A_p(H) : \text{supp}\,v \subset K\}$ where $K = H \cap \text{cl}\,V_0$ is compact.

But $A_K^p(H)$ is weakly sequentially complete by a joint result of ours and M. Cowling ([Gr1], p. 131, Lemma 18) (proved by F. Lust-Piquard for compact groups H and $K = H$, [P1], p. 265, Theorem 4). It follows that there is some $u_0 \in A_K^p(H) \subset A_p(H)$ such that for all $T \in A_p(H)^*$, $(T, u_0) \leftarrow (T, r(v_0 v_n)) = (v_0 \cdot r^*T, v_n) \to (\psi_1, v_0 \cdot r^*T)$. Thus

$$(*) \qquad (T, u_0) = (\psi_1, v_0 \cdot r^*T) \quad \text{for all } T \in A_p(H)^*.$$

But if *in addition* $r^*T \in P$ then, since $v_0 \in S_A^p$ and $\psi_1 \in \mathbf{A}$, $(\psi_1, v_0 \cdot r^*T) = (\psi_1, v_0' \cdot r^*T) = (\psi_1, r^*T)$.

Let now $a \in F \cap H$ and $T = \delta_a \in A_p(H)^*$ (a slight abuse of notation). If $v \in A_p(G)$ then $(r^*\delta_a, v) = (\delta_a, rv) = v(a)$. Thus, $r^*\delta_a = \lambda\delta_a \in P$, since $a \in F = \sigma(P)$. Hence by $(*)$,

$$(**) \qquad (\psi_1, r^*\delta_a) = (\psi_1, v_0 \cdot r^*\delta_a) = (\lambda\delta_a, u_0) = u_0(a) \quad \text{if } a \in F \cap H.$$

Thus $u_0(e) = (\psi_1, v_0 \cdot r^*\delta_e) = \lim(\delta_e, r(v_0 v_n)) = 1$ since $v_0 v_n \in S_A^p$. But if $a \neq e$ and $a \in F \cap H$, there is some $w \in S_A^p$ such that $a \notin \operatorname{supp} w$. Thus $w \cdot r^*\delta_a = 0$, by pairing with any $v \in A_p(G)$. Hence by $(**)$, $0 = (\psi_1, w \cdot r^*\delta_a) = (\psi_1, r^*\delta_a) = u_0(a)$. It follows that u_0 is not a continuous function, since H is *not discrete* by assumption. Yet $u_0 \in A_p(H) \subset C_0(H)$, which is a contradiction. This proves Claim 5.

Continuing the proof of Theorem 4 we first show that P/W_P has ℓ^∞ as a quotient. Let $\{\Phi_n\}$ be a dense sequence in S_P. Clearly $\psi_0^1 \in \mathbf{A} = \{\psi \in w^*\text{-cl}\,(S_A^p)' : S_n^{**}\psi = 0 \text{ for all } n\}$ where $S_n(v') = a_n' v' - v'$ if $v' \in A_p/J$. Let $\beta_n = (\psi_0^1, \Phi_n)$ and $\mathbf{A}_0 = \{\psi \in \mathbf{A} : (\psi, \Phi_n) = \beta_n \text{ for all } n\}$. Then $\psi_0^1 \in \mathbf{A}_0$. Thus $\mathbf{A}_0 \neq \emptyset$ and $\mathbf{A}_0 \cap w^*\text{-seq}\,\text{cl}(S_A^p)' = \emptyset$, since $\mathbf{A}_0 \subset \mathbf{A}$. Clearly $\mathbf{A}_0 \subset w^*\text{-cl}(S_A^p)'$.

We apply now our Theorem 1.4 of [Gr2], p. 158 (see introduction), to conclude that there exists an *onto* operator $t : P \to \ell^\infty$ such that if $S = \text{ncl}\,\text{lin}\{\bigcup_{n=1}^\infty S_n^* P\}$ then $t(S + S_P) \subset c = \{a = (a_n) \in \ell^\infty : \lim a_n \text{ exists}\}$ and $t^* : \ell^{\infty *} \to P^*$ is a w^*-w^*-continuous norm isomorphism into such that $t^*(\boldsymbol{\mathcal{F}}) \subset \mathbf{A}_0$. The operator $t : P \to \ell^\infty$ is given by $(t\Phi)(n) = (\Phi, u_n)$ where $\{u_n'\} \subset (S_A^p)'$ is a particular sequence, which is isomorphic to a canonical ℓ^1 basis. If $X_0 = \text{ncl}\,\text{lin}\{u_n'\}$ in A_p/J then X_0 is isomorphic to ℓ^1 and $t = i^*$, where $i : X_0 \to A_p/J$ is the embedding map.

Note that $S_n^*\Phi = a_n' \cdot \Phi - \Phi$ for $\Phi \in P$ and if $u \in S_A^p$ and $\|a_{n_i}' - u'\| \to 0$ then $\|(u' \cdot \Phi - \Phi) - (a_{n_i}' \cdot \Phi - \Phi)\| \to 0$. Hence $S = \text{ncl}\,\text{lin}\{u' \cdot \Phi - \Phi : u \in S_A^p, \Phi \in P\}$. But as was seen above, $u \cdot \Phi = u' \cdot \Phi$ if $\Phi \in P = (A_p/J)^*$ and $u \in S_A^p$. Applying now Proposition 1 we see that $S = E_P$, hence $t(E_P + S_P) \subset c$, i.e. $E_P + S_P \subset t^{-1}(c)$.

Thus $W_P = \text{ncl}(E_P + S_P)$ has the property that P/W_P has $P/t^{-1}(c) \approx \ell^\infty/c$ (norm isomorphism) as a quotient (see Section 0). But as is well known, ℓ^∞/c contains an isometric copy Y of ℓ^∞ (by folklore, or see [Gr2], p. 161). The identity map $i_0 : Y \to Y$ has (since ℓ^∞ is injective, [DU], p. 155) a linear bounded extension $i_1 : \ell^\infty/c \to Y$. Hence $P/t^{-1}(c)$, and a fortiori P/W_P, has ℓ^∞ as a quotient.

We now prove the Q/W_Q case. Since $S_Q \subset Q \subset P$ we will choose in the above proof $S_Q = S_P$ and let $\{\Phi_n\} \subset S_Q$ be dense in S_Q. Let $t : P \to \ell^\infty$ be the onto operator constructed above such that $t(E_P + S_Q) \subset c$, $t^* : \ell^{\infty *} \to P$ satisfies $t^*(\boldsymbol{\mathcal{F}}) \subset \mathbf{A}_0$ and $P/t^{-1}(c) \approx \ell^\infty/c$ is isomorphic. Let $\Pi : \ell^\infty \to \ell^\infty/c$ be the canonical map. Then for any $u \in S_A^p$ and $\Phi \in P$,

$\Pi t(u\cdot\Phi-\Phi)=0$, thus $\Pi t(u\cdot\Phi)=\Pi t\Phi$. Now $u\cdot\Phi\in P_{\rm c}\subset Q$ since $A_p\cap C_{\rm c}(G)$ is dense in $A_p(G)$. It follows that $\Pi t(Q)=\Pi t(P)=\ell^\infty/c$.

Let now ϱ be the map Πt restricted to Q. Since $E_Q\subset E_P$ we have $\varrho(E_Q)\subset\Pi t(E_P)=\{0\}$. Also, since $S_Q\subset Q$, $\varrho(S_Q)=\{0\}$. Hence $E_Q+S_Q\subset\varrho^{-1}(0)$, and $W_Q\subset\varrho^{-1}(0)$. But $\varrho(Q)=\ell^\infty/c$, hence $Q/\varrho^{-1}(0)\approx\ell^\infty/c$. But, as above, ℓ^∞/c contains a copy of ℓ^∞ and thus has ℓ^∞ as a quotient. It follows that Q/W_Q has $Q/\varrho^{-1}(0)$, and a fortiori ℓ^∞, as a quotient.

Define now $i:Q\to P$ as the inclusion map, $i\Phi=\Phi$. Clearly $\varrho(\Phi)=\Pi ti(\Phi)$ if $\Phi\in Q$. Let $TI_P=\{\psi\in P^*:u\cdot\psi=\psi$ for all $u\in S_A^p\}$ (TI for topologically invariant), and let TI_Q be defined similarly.

We claim that i^* restricted to TI_P is a w^*-w^*-continuous isometry such that $i^*(TI_P)\subset TI_Q$ and $i^*(TIM_P)\subset TIM_Q$. To show this let $\psi\in TI_P$ and $\Phi_0\in P$, $\|\Phi_0\|=1$, such that $(\psi,\Phi_0)\ge\|\psi\|-\varepsilon$. Let $u\in S_A^p\cap C_{\rm c}(G)$. Then $u\cdot\Phi_0\in P_{\rm c}\subset Q$, $\|u\cdot\Phi_0\|\le\|\Phi_0\|=1$ and $(i^*\psi,u\cdot\Phi_0)=(\psi,u\cdot\Phi_0)=(\psi,\Phi_0)$. But $\|i^*\|\le1$. Thus $\|i^*\psi\|=\|\psi\|$ if $\psi\in TI_P$. Let now $\psi\in TI_P$, $u\in S_A^p$ and $\Phi\in Q$. Then $(i^*\psi,u\cdot\Phi)=(\psi,u\cdot\Phi)=(\psi,\Phi)=(\psi,i\Phi)=(i^*\psi,\Phi)$, and $i^*(TI_P)\subset TI_Q$.

If now $\psi\in TIM_P$, then $(\psi,\delta_e)=1=\|\psi\|$. But $\lambda\delta_e\in P_{\rm c}\subset Q$. Hence $(i^*\psi,\lambda\delta_e)=(\psi,\lambda\delta_e)=1=\|\psi\|=\|i^*\psi\|$, which proves our claim.

Recall now that $\{\Phi_n\}$ is dense in $S_Q\subset Q$ and $(\psi_0^1,\Phi_n)=\beta_n$. If $\psi\in TIM_P$ and $\psi(\Phi_n)=\beta_n$ for all n, then $(i^*\psi,\Phi_n)=(\psi,\Phi_n)=\beta_n$. Hence $i^*\{\psi\in TIM_P:(\psi,\Phi_n)=\beta_n$ for all $n\}\subset\{\psi\in TIM_Q:(\psi,\Phi_n)=\beta_n$ for all $n\}$.

But $t^*:\ell^{\infty*}\to P^*$ was a w^*-w^*-continuous norm isomorphism into such that $t^*(\boldsymbol{\mathcal F})\subset\mathbf{A}_0\subset\{\psi\in TIM_P:(\psi,\Phi_n)=\beta_n$ for all $n\}$.

It follows that i^*t^* restricted to the linear span of $\boldsymbol{\mathcal F}$ (which coincides with $c_0^\perp\subset\ell^{\infty*}$ and is hence w^*-closed) is a w^*-w^*-continuous norm isomorphism into TI_Q such that $i^*t^*(\boldsymbol{\mathcal F})\subset\{\psi\in TIM_Q:(\psi,\Phi_n)=\beta_n$ for all $n\}$. ∎

Remark. If we make use of the full force of Theorem 1.6 of [Gr2] we can see that even Q/W_Q' has ℓ^∞ as a quotient where W_Q' is a much larger space than W_Q (where even W_Q'/W_Q has ℓ^∞ as a quotient). In fact, if $AC=\mathbb{C}1\oplus\operatorname{ncl}\operatorname{lin}\{f-f_n:f\in\ell^\infty,\ n\ge1\}\subset\ell^\infty$ where $f_n(k)=f(n+k)$ for $k\ge1$, then $W_Q'=\varrho^{-1}(AC/c)$ and $Q/\varrho^{-1}(AC/c)\approx\ell^\infty/AC$ and this last has ℓ^∞ as a quotient (see details in [Gr2], p. 162). Here $\varrho:Q\to\ell^\infty$ is defined in the above proof.

One of the main results of this section is Theorem 6. The reader will note that the crux of its proof is contained in Theorem 4.

THEOREM 6. *Let G be a second countable locally compact group. Let P and Q be A_p-submodules of $PM_p(G)$ such that P is w^*-closed, Q is norm closed, $P_{\rm c}\subset Q\subset P$ and $\sigma(P)=F$. Let S_P and S_Q be separable linear subspaces of P and Q respectively, and define* $W_P(d)=\operatorname{ncl}(E_P(d)+S_P)$

and $W_Q(d) = \mathrm{ncl}(E_Q(d) + S_Q)$ *for* $d \in F$. *Let* $H \subset G$ *be a nondiscrete closed subgroup, and* $a, b \in G$. *Then for any* $d \in \mathrm{int}_{aHb}(F)$, $Q/W_Q(d)$, $P/W_P(d)$ *and* $CV_p(F)/W_P(d)$ *have* ℓ^∞ *as a quotient, and* $TIM_P(d)$ *and* $TIM_Q(d)$ *contain* $\boldsymbol{\mathcal{F}}$. *Consequently*, $P_c/M_p(F)$, $Q/M_p(F)$, $P/M_p(F)$ *and* $CV_p(F)/M_p(F)$ *have* ℓ^∞ *as a quotient if such* a, b, d, H *exist.*

Remarks to Theorem 6. 1. The onto operator $t : Q/W_Q(d) \to \ell^\infty$ constructed is such that the into w^*-w^* and norm isomorphism t^* satisfies $t^*(\boldsymbol{\mathcal{F}}) \subset TIM_Q(d)$. This also applies to Theorem 12.

2. The fact that $Q/W_Q(d)$ has ℓ^∞ as a quotient is a strictly stronger fact than that $Q/M_p(F)$ has such, since even $W_Q(e)/M_p(G)$ has ℓ^∞ as a quotient if G is abelian, $p = 2$ and $Q = PM_{2c}(G)$ by C. Chou [Ch1]. More such examples are given at the end of this section.

3. The fact that $P_c/W_{P_c}(d)$ and $P/W_P(d)$ have ℓ^∞ as a quotient does not imply, directly from the injectivity of ℓ^∞, that $Q/W_Q(d)$ has ℓ^∞ as a quotient. In fact, if G is abelian and $\mathcal{F} : L^1(\widehat{G}) \to A_2(G)$ is Fourier transform, let $P = PM_2(G)$ and $Q = \mathcal{F}^{*-1}(C(\widehat{G}))$. Then $P_c \subset Q \subset P$ and $P_c \neq Q \neq P$ if $\widehat{G}$ is not discrete or compact, since $UC(\widehat{G}) \neq C(\widehat{G}) \neq L^\infty(\widehat{G})$. Furthermore, $\mathcal{F}^*(E_{P_c}(e)) = \mathrm{ncl}\,\mathrm{lin}\{f - l_x f : x \in \widehat{G},\ f \in UC(\widehat{G})\}$ and $\mathcal{F}^*(E_Q(e)) = \mathrm{ncl}\,\mathrm{lin}\{f - \Phi * f : f \in C(\widehat{G}),\ 0 \le \Phi \in L^1(\widehat{G}),\ \int \Phi\, dx = 1\}$ where ncl is in $L^\infty(\widehat{G})$ norm. Then $\mathcal{F}^*(E_{P_c}(e)) \neq \mathcal{F}^*(E_Q(e))$ (see [LR]). Let $W_Q = \mathbb{C}\lambda\delta_e \oplus E_Q(e)$, $W_{P_c} = \mathbb{C}\lambda\delta_e \oplus E_{P_c}(e)$. Then $W_{P_c} \neq W_Q$ and the fact that P_c/W_{P_c} has ℓ^∞ as a quotient does not imply the same for Q/W_Q, directly from the injectivity of ℓ^∞. This is the reason that we phrased Theorem 6 in terms of the module Q with $P_c \subset Q \subset P$.

4. The space $W_Q(d)$ is not usually w^*-closed. In fact, if $Q = PM_2(G)$ then $W_Q(e) = \mathbb{C}\lambda\delta_e \oplus E_Q(e)$ satisfies $w^*\text{-cl}\, W_Q(e) = Q$ if G is not discrete, as can be easily seen from the fact that $L^\infty(\widehat{G})$ does not admit invariant means which belong to $L^1(\widehat{G})$.

5. It has been proved by H. Rosenthal that if some operator $T : C(K) \to X$ is not weakly compact where K is Stonean compact Hausdorff and X a Banach space then X contains a copy of ℓ^∞ (see [DU], p. 156, Thm. 10, p. 180, for W^*-algebras, and p. 23, Cor. 6). One may hence be tempted to show that the canonical map $q : Q \to Q/W_Q$ is not weakly compact and apply Rosenthal's theorem. Unfortunately, $Q = P_c$ may be very different from an L^∞ space or a W^*-algebra.

Even in the case that G is abelian and $p = 2$ and we take $P = PM_2(G)$ then $P_c \approx UC(\widehat{G})$ (are isometric, via $\mathcal{F}^*$). Thus P_c is not even a dual Banach space if $\widehat{G}$ is not discrete. Furthermore, if $G = \mathbb{R}$ and $F = \{x \in \mathbb{R} : 0 \le x \le 1\}$ then $\mathcal{F}^*(PM_2(F)) \subset UC(\mathbb{R})$ is not even a pointwise subalgebra of $UC(\mathbb{R})$.

But moreover, if $p \neq 2$ and even if G is abelian, then $PM_p(G)$ is strikingly different from $PM_2(G)$. It has been shown by Y. Benyamini and P. K. Lin in [BL] that if G_1, G_2 are compact abelian and $PM_p(G_1)$, $PM_p(G_2)$ are isometric as Banach spaces then G_1, G_2 are isomorphic (while $PM_2(G)$ is isometric to ℓ^∞ for any infinite abelian compact metric group G).

6. All our results express the fact that Q/W_Q and P/W_P have ℓ^∞ as a quotient. In this connection R. Haydon has constructed in [Ha] a Banach space $C(K)$, K compact Hausdorff, which has ℓ^∞ as a quotient, does *not* contain an isomorph of ℓ^∞ and yet has the Grothendieck property (i.e. w^*-convergent sequences in $C(K)^*$ converge weakly). See also Talagrand [T].

Proof of Theorem 6. For $a \in G$, let $r_a, l_a : A_p \to A_p$ be defined by $(r_a v)(x) = v(xa)$ and $(l_a v)(x) = v(ax)$. Then l_a, r_a are isometric algebra isomorphisms of A_p ([Hz1], p. 97). To see that, note that by [HR], Vol. I, p. 292, $l_a(v * \check{u}) = (l_a v) * \check{u}$ and $r_a(v * \check{u}) = v * (r_a \check{u})$, where $\check{u}(x) = u(x^{-1})$. But $(r_a\check{u})^\vee = l_{a^{-1}}u$. Thus $\|l_a v\|_{p'}\|u\|_p = \|v\|_{p'}\|u\|_p = \|v\|_{p'}\|l_{a^{-1}}u\|_p = \|v\|_{p'}\|(r_a\check{u})^\vee\|_p$ for any $u \in L^{p'}$, $v \in L^p$. Hence $\|l_a w\|_{A_p} \le \|w\|_{A_p}$ and $\|r_a w\|_{A_p} \le \|w\|_{A_p}$. Using this for $l_{a^{-1}}$ and $r_{a^{-1}}$ we get equality for all $w \in A_p$. Also clearly, for all $a, b \in G$, $r_b l_a = l_a r_b$, $l_a l_b = l_{ba}$ and $r_a r_b = r_{ab}$.

It is readily checked, and known, that if $J = (P)_0 = \{v \in A_p : (\Phi, v) = 0 \text{ for all } \Phi \in P\}$ then $Z(J) = F$ and $Z(l_x r_y J) = x^{-1}Fy^{-1}$. Furthermore, if for $I \subset A_p$ we define $I^0 = \{\Phi \in PM_p : (\Phi, v) = 0 \text{ for all } v \in I\}$ then $(l_x r_y J)^0 = l^*_{x^{-1}} r^*_{y^{-1}}(P)$. Since $(\Phi, l_x r_y v) = 0$ for all $v \in J$ iff $l^*_x r^*_y \Phi \in J^0 = P$ iff $\Phi \in r^{*-1}_y l^{*-1}_x P = l^*_{x^{-1}} r^*_{y^{-1}} P$. (Clearly the last is w^*-closed since r^*_a, l^*_a are w^*-w^*-continuous isometries onto.) Thus

$$(*) \qquad a^{-1}Fb^{-1} = Z(l_a r_b J) = \sigma((l_a r_b J)^0) = \sigma(l^*_{a^{-1}} r^*_{b^{-1}} P).$$

Clearly $r^*_b(\lambda\delta_x) = \lambda\delta_{xb}$ and $l^*_a(\lambda\delta_x) = \lambda\delta_{ax}$. Also, for any $v \in A_p$ and $\Phi \in PM_p$ one has $l^*_a r^*_b(v \cdot \Phi) = (l_{a^{-1}} r_{b^{-1}} v) \cdot (l^*_a r^*_b \Phi)$, thus $l^*_a r^*_b R$ is an A_p-submodule if R is such. To check this for r_b let $u \in A_p$. Then $(r^*_b(v \cdot \Phi), u) = (\Phi, v(r_b u)) = (\Phi, r_b((r_{b^{-1}} v)u)) = ((r_{b^{-1}} v) \cdot r^*_b \Phi, u)$. This readily implies that

$$(**) \qquad \operatorname{supp}(l^*_a r^*_b \Phi) = a(\operatorname{supp}\Phi)b \quad \text{ for all } \Phi \in PM_p\,.$$

In fact, if $x \in \operatorname{supp}\Phi$ and $v_a \cdot \Phi \to \lambda\delta_x$ in w^*, then $\lambda(\delta_{axb}) \leftarrow l^*_a r^*_b(v_a \cdot \Phi) = (l_{a^{-1}} r_{b^{-1}} v_a) \cdot (l^*_a r^*_b \Phi)$, and by [Hz1], pp. 119–120, $axb \in \operatorname{supp}(l^*_a r^*_b \Phi)$. If $y \in \operatorname{supp}(l^*_a r^*_b \Phi)$ then $x = a^{-1}yb^{-1} \in \operatorname{supp}(l^*_{a^{-1}} r^*_{b^{-1}} l^*_a r^*_b \Phi) = \operatorname{supp}\Phi$ and $y = axb$.

We now claim that $l^*_a r^*_b P_c = (l^*_a r^*_b P)_c \subset l^*_a r^*_b Q \subset l^*_a r^*_b P$ and that

$$(***) \qquad l^*_a r^*_b E_R(x) = E_{l^*_a r^*_b R}(axb) \quad \text{ if } R = P, P_c \text{ or } Q.$$

In fact, if $\Phi \in P$ then $x \notin \operatorname{supp}\Phi$ iff $axb \notin a(\operatorname{supp}\Phi)b = \operatorname{supp}(l^*_a r^*_b \Phi)$ by $(**)$. Clearly $\operatorname{supp}\Phi$ is compact if $\operatorname{supp}(l^*_a r^*_b \Phi) = a(\operatorname{supp}\Phi)b$ is compact. Thus $(***)$ follows.

Assume first that $d=s\in \mathrm{int}_H(F)$ and let $t=s^{-1}$. Then $s\in sV\cap H\subset F$ for some neighborhood V of e. Thus $e\in V\cap H\subset tF=\sigma(l_t^*P)$ by $(**)$.

Apply now Theorem 4 to the w^*-closed A_p-module l_t^*P (since $e\in \mathrm{int}_H(tF)$). We deduce that $R_1=l_t^*P/l_t^*(W_P(d))$ and $R_2=l_t^*Q/l_t^*(W_Q(d))$ have ℓ^∞ as a quotient since by $(***)$, $l_t^*(W_R(d))=\mathrm{ncl}(E_{l_t^*R}(e)+l_t^*S_R)$ for $R=P$ or Q, and $l_t^*S_R$ is separable.

Also, $\mathit{TIM}_{l_t^*R}$ contains $\boldsymbol{\mathcal{F}}$ if $R=P$ or Q.

Now, again since $l_s^*:PM_p\to PM_p$ is a w^*-w^*-continuous isometry onto, we see that R_1 and R_2 are norm isomorphic to $l_s^*l_t^*P/l_s^*l_t^*W_P(d)=P/W_P(d)$ and $l_s^*l_t^*Q/l_s^*l_t^*W_Q(d)$ respectively. Hence $P/W_P(d)$ and $Q/W_Q(d)$ have ℓ^∞ as a quotient.

Now l_y^{**} is also a w^*-w^*-continuous norm isometry of PM_p^* onto PM_p^*. Thus $(l_s^{**})^{-1}(\mathit{TIM}_{l_t^*R})=l_t^{**}(\mathit{TIM}_{l_t^*R})$ contains $\boldsymbol{\mathcal{F}}$. But since $t=s^{-1}$, $l_t^{**}\,\mathit{TIM}_U=\mathit{TIM}_{l_s^*U}(s)=\mathit{TIM}_R(s)$ if $U=l_t^*R$ where $R=P$ or $R=Q$, since if $\psi\in \mathit{TIM}_U$ then $(l_t^{**}\psi,\lambda\delta_s)=(\psi,\lambda\delta_e)=1$, and $l_tS_A^p=S_A^p(s)=\{v\in A_p: v(s)=1=\|v\|\}$. Hence $\mathit{TIM}_P(s)$ and $\mathit{TIM}_Q(s)$ contain $\boldsymbol{\mathcal{F}}$.

This proves the theorem if $d\in \mathrm{int}_H(F)$.

Assume now that $d\in \mathrm{int}_{aHb}(F)$. Let V be a neighborhood of e such that $d\in dV\cap aHb\subset F$. Then $a^{-1}db^{-1}\in a^{-1}dVb^{-1}\cap H\subset a^{-1}Fb^{-1}$. Hence $a^{-1}db^{-1}\in \mathrm{int}_H(a^{-1}Fb^{-1})$. But by the first part with $s=a^{-1}db^{-1}$ and since by $(*)$, $a^{-1}Fb^{-1}=\sigma(l_{a^{-1}}^*r_{b^{-1}}^*P)$, we see that $U=l_{a^{-1}}^*r_{b^{-1}}^*R/W_R$ has ℓ^∞ as a quotient for $R=P$ or Q, where we apply $(***)$ with $x=a^{-1}$, $y=b^{-1}$ and $W_R=\mathrm{ncl}(E_{l_x^*r_y^*R}(xdy)+l_x^*r_y^*S_R)=\mathrm{ncl}\,l_{a^{-1}}^*r_{b^{-1}}^*(E_R(d)+S_R)$. Also $\mathit{TIM}_V(xdy)$ contains $\boldsymbol{\mathcal{F}}$, where $V=l_x^*r_y^*R$ with $R=P$ or Q. Now since l_x^*, r_x^* are w^*-w^*-continuous isometries onto, by taking images by $l_a^*r_b^*$ we get $R/l_a^*r_b^*W_R\approx U$, hence $R/l_a^*r_b^*W_R$ has ℓ^∞ as a quotient with $R=P$ or Q, since $l_a^*r_b^*W_R=\mathrm{ncl}(E_R(d)+S_R)=W_R(d)$, as follows readily from $(***)$. Thus $P/W_P(d)$ and Q/W_Q have ℓ^∞ as a quotient.

If now L is any normed space such that $R/W_R(d)\subset L$ (with $R=P$ or Q) then by the injectivity of ℓ^∞, L has ℓ^∞ as a quotient. Since $Q\subset P\subset CV_p(F)$ we see that $CV_p(F)/W_R(d)$ has ℓ^∞ as a quotient with $R=P$ or Q.

If we apply the theorem with $S_P=S_Q=\mathbb{C}\lambda\delta_d$ then by Proposition 3, $M_p(F)\subset\mathbb{C}\lambda\delta_d\oplus E_Q(d)=W_Q(d)$. Since $Q/W_Q(d)$ has ℓ^∞ as a quotient so does $Q/M_p(F)$, hence so do $P/M_p(F)$, $CV_p(F)/M_p(F)$ and (since $Q=P_{\mathrm{c}}$ is allowed) $P_{\mathrm{c}}/M_p(F)$.

Now the argument above for $\mathit{TIM}_P(s)$ and $\mathit{TIM}_Q(s)$ carries over by taking images by $l_{a^{-1}}^*r_{b^{-1}}^*$. Hence we conclude that $\mathit{TIM}_P(s)$ and $\mathit{TIM}_Q(s)$ contain $\boldsymbol{\mathcal{F}}$. ■

R e m a r k. We have in fact also proved that if $\{\Phi_n\}\subset S_Q$ is dense in S_Q and $\psi_0\in \mathit{TIM}_Q(d)$ and $(\psi_0,\Phi_n)=\beta_n$ then even the set $\{\psi\in \mathit{TIM}_Q(d):(\psi,\Phi_n)=\beta_n \text{ for all } n\}$ contains $\boldsymbol{\mathcal{F}}$ (see the proof of Theorem 4). The reader should also note the remarks after the statement of Theorem 4.

The norm closed A_p-submodules of PM_p in the next corollary are defined in the introduction.

For the rest of the paper denote for simplicity $W_P(x) = \mathbb{C}\lambda_p\delta_x + E_P(x)$.

COROLLARY 7. *Under the assumptions on* G, H, a, b, *as in Theorem* 6, *let* $F \subset G$ *be closed and let* Q *be any of the eight spaces* $(PM_{p*}(F))_c \subset PM_{p*c}(F) \subset C_{p*}(F) \subset PM_{p*}(F)$, $(PM_p(F))_c \subset PM_{pc}(F) \subset C_p(F) \subset PM_p(F)$. *Then for any* $d \in \mathrm{int}_{aHb}(F)$, $Q/W_Q(d)$ (*a fortiori* $Q/M_p(F)$ *and* $CV_p(F)/M_p(F)$ *if such* a, b, d, H *exist*) *has* ℓ^∞ *as a quotient and* $TIM_Q(d)$ *contains* $\boldsymbol{\mathcal{F}}$.

Remark. If G is amenable then $PM_{pc}(F) = (PM_p(F))_c$ and $PM_{p*c}(F) = (PM_{p*}(F))_c$.

Our next corollary deals with the case that $P \subset CV_p(G)$ (which may properly include $PM_p(G)$).

COROLLARY 8. *Under the assumptions as in Theorem* 6 *except that* $P \subset CV_p(G)$ *is an ultraweakly closed* A_p*-submodule* ([Hz1], p. 116, [Der], pp. 9–10), *the conclusions of Theorem* 6 *hold for* P_c.

Proof. Clearly $P_c \subset PM_p$. Let $Q = P \cap PM_p$. Since PM_p is ultraweakly closed ([Hz1], p. 91) so is Q. And since $(PM_p, w^*) = (PM_p, u.w)$, Q is a w^*-closed A_p-submodule of PM_p ([Pi], p. 95). Clearly $P_c \subset Q \subset P$ and $Q_c = P_c$. Now $F = \sigma(P) = \sigma(Q)$. We apply our Theorem 6 to Q. ∎

We apply next our results to β (strictly) closed A_p-submodules of PM_p (with a view to further applications).

DEFINITION. Following the beautiful thesis of Delaporte [De1] we define the β (or *strict*) *topology* on $CV_p(G)$ by: $\Phi_\alpha \to \Phi$ in β iff $\|(\Phi_\alpha - \Phi)\Phi'\| \to 0$ for all $\Phi' \in PF_p(G)$, where $PF_p(G)$ and $C_p(G)$ are defined in the introduction.

It is shown in [De1], Thm. 2.1, that $C_p(G)$ is a norm closed subalgebra and an A_p-submodule of $CV_p(G)$ and $PM_{pc}(G) \subset C_p(G) \subset PM_p(G)$. Furthermore, $(CONV_p(G), \beta)$ and $(C_p(G), \beta)$ are complete and if $W_p(G)$ is the dual of ($PF_p(G)$, norm) then $W_p(G)$ can be identified with the dual of $(C_p(G), \beta)$ in analogy with the well known theorems of Buck for the abelian case. (To be consistent with [De1] we note that right and left can be interchanged by [De1], p. 8.)

LEMMA 9. *Let* $Q \subset C_p(G)$ *be a* β*-closed* A_p*-module* (*if* $\Phi \in C_p(G)$ *then* $Q_\beta(\Phi) = \beta\text{-cl}(A_p \cdot \Phi)$ *is such*). *If* $P = w^*\text{-cl}\, Q$ *then* $P_c \subset Q \subset P$, *hence* $\sigma(Q) = \sigma(P)$.

Proof. Let $\Phi \in P_c \subset C_p(G)$ have compact support. Let $u \in A_p \cap C_c(G)$ be such that $u \cdot \Phi = \Phi$. Let $\Phi_\alpha \in Q$, $\Phi_\alpha \to \Phi$ in w^* and let $w \in W_p(G) =$

$(C_p(G),\beta)'$ (the dual of $(C_p(G),\beta)$ by [De1], Thm. 2.8). Then $\langle w, u\cdot(\Phi_\alpha - \Phi)\rangle \to 0$. Hence $\Phi = \sigma(C_p, W_p)$-$\lim u\cdot\Phi_\alpha$. Now $\Phi \in P_{\rm c} \subset C_p(G)$. Hence $\Phi \in \beta$-cl$(A_p\cdot Q) \subset Q$ (by duality). Hence $P_{\rm c}\subset Q$ (since norm $\ge \beta$, Q is also norm closed).

As for $Q_\beta(\Phi)$ let β-$\lim u_\alpha\cdot\Phi = \Psi$, with $u_\alpha \in A_p$. Let $w\in W_p(G) = (C_p(G),\beta)'$ and $u\in A_p(G)$. Then $\langle w, u(u_\alpha\Phi - \Psi)\rangle = \langle wu, u_\alpha\Phi-\Psi\rangle \to 0$. Since $u\cdot\Psi\in C_p(G)$ we get $u\cdot\Psi \in \sigma(C_p,W_p)$-cl$(A_p\cdot\Phi) = \beta$-cl$(A_p\cdot\Phi)$ (since $C_p(G)$ is β-closed in PM_p, β-cl$(A_p\cdot\Phi)\subset C_p(G)$, and by duality). ∎

Remark. If G is in addition amenable then Delaporte has shown in [De1], Cor. 4.4, that $Q = P\cap C_p(G)$.

COROLLARY 10. *Let G be second countable, $H\subset G$ a closed nondiscrete subgroup, and $a,b\in G$. Let $Q\subset C_p(G)$ be a β-closed A_p-submodule and $\sigma(Q) = F$. Then for any $d\in \mathrm{int}_{aHb}(F)$, $Q/W_Q(d)$ and $Q_{\rm c}/W_{Q_{\rm c}}(d)$ (a fortiori $Q_{\rm c}/M_p(F)$ if such a,b,d,H exist) have ℓ^∞ as a quotient and both $TIM_Q(d)$ and $TIM_{Q_{\rm c}}(d)$ contain $\mathcal{F}$.*

Proof. Let $P = w^*$-cl Q. Then $P_{\rm c}\subset Q\subset P$ and $P_{\rm c} = Q_{\rm c}$, by Lemma 9. The rest is just Theorem 6. ∎

Are there any (many) elements which belong simultaneously, for *all* $1 < p < \infty$, to $PM_p(F)\sim\lambda_p(M(F))$? We improve, in a sense, Theorem 5.7 of Edwards and Price [EP] (see sequel) in the next corollary.

If G is amenable then Herz's Theorem C [Hz3] allows one to conclude that $PM_p(G)\subset PM_q(G)$ if $1<p\le q\le 2$ or $2\le q\le p<\infty$, with contraction of norms. Furthermore, if $\Phi\in PM_p$ then by the definition of support, supp Φ in $PM_p(G)$ is the same as supp Φ in $PM_q(G)$. Thus $PM_p(F)\subset PM_q(F)$ if p,q are as above. Consider in the sequel $PM_p(G)$ for all $1<p<\infty$ to be a subset of $PM_2(G)$. The above observations will now yield

COROLLARY 11. *Let G be amenable and second countable, $a,b\in G$, and $H\subset G$ a closed nondiscrete subgroup. If F is closed and $\mathrm{int}_{aHb}(F)\ne\emptyset$ then, for $1<p\le 2$,*

$$(PM_p(F))_{\rm c}\sim\lambda_p(M(F)) = \bigcap_{p\le q\le 2}\{(PM_q(F))_{\rm c}\sim\lambda_q(M(F))\},$$

has cardinality c, and the same conclusion holds for $2\le p<\infty$ with the intersection taken over $2\le q\le p$.

Proof. Our main theorem shows that card$(PM_p(F))_{\rm c}\sim M_p(F) = c$ since $(PM_p(F))_{\rm c}/M_p(F)$ has ℓ^∞ as a quotient and card $PM_p(G) = c$. Since $\lambda_p(M(F)) = \lambda_q(M(F))$ for all p,q, the rest follows. ∎

Remark. Gaudry and Inglis have proved in [GI] that $PM_p(\mathbb{T})$ is not norm dense in $PM_q(\mathbb{T})$ if $1<p<q\le 2$.

Connections with existing results. In improving theorems of B. Brainerd and R. E. Edwards [BE], Figà-Talamanca and Gaudry [FG] and J. F. Price [Pr] (see also [DR]) have proved that if G is noncompact abelian or G is any compact group, then there exists an operator Φ in $PM_p(G)$ *for all* $1 < p < \infty$ which is not convolution by a bounded measure. Furthermore, Cowling and Fournier prove in [CF], p. 65, that for *any* locally compact G, if $|q^{-1} - 2^{-1}| < |p^{-1} - 2^{-1}|$ there is some $\Phi \in CV_q(G)$ such that $\Phi \notin CV_p(G)$. Any such Φ clearly cannot be convolution by a bounded measure. Cowling and Fournier are even able to control, in a sense, the support of Φ (see further remarks).

Furthermore, it has been proved by J. Dieudonné ([Pi], p. 85, Thm. 9.6) that if G is a nonamenable group then for every $1 < p < \infty$ there exists a positive unbounded Radon measure μ on G such that the convolution operator $(\lambda\mu)(f) = \mu * f$ for $f \in L^p$ is in $CV_p(G)$. Clearly $\lambda\mu \notin \lambda(M(G))$.

All the above constructed operators belong to $PM_{\mathrm{pc}}(G)$ (some even to $M_p(G)$). If we apply our Theorem 6 with $P = PM_p(G)$ we get a much more powerful result, in a sense, namely that if G is second countable nondiscrete then $PM_{\mathrm{pc}}(G)/\mathbb{C}\lambda_p\delta_x \oplus E_{P_{\mathrm{c}}}(x)$ for all $x \in G$ (a fortiori $PM_{\mathrm{pc}}/M_p(G)$) has even (the nonseparable) ℓ^∞ as a quotient. Hence $\{\Phi \in PM_{\mathrm{pc}}(G) : \Phi \notin M_p(G)\}$ is big. This also improves Ching Chou's result in [Ch2] who shows that if G is second countable and $P = CV_2(G)$ then $P/\mathbb{C}\lambda_2\delta_e + E_P(e)$ has ℓ^∞ as a quotient and $P_{\mathrm{c}}/\mathbb{C}\lambda_2\delta_e + E_{P_{\mathrm{c}}}(e)$ is not norm separable (see [Ch2], Thm. 3.3 and Cor. 3.6). Chou uses C^*-algebra methods which are not available if $p \neq 2$.

As for controlling the supports of elements $\Phi \in CV_p(G)$ such that $\Phi \notin \lambda(M(G))$, the following should be noted:

If G is any locally compact group and $U \subset G$ is any open set with $F = \mathrm{cl}\, U$ compact, R. E. Edwards and J. F. Price construct in [EP], p. 269, *norm separably* many elements $\Phi \in PM_{\mathrm{pc}}(G)$ such that $\operatorname{supp}\Phi \subset F$ and $\Phi \notin \lambda(M(G))$, *provided* U admits a Rudin–Schapiro (URS) sequence ([EP], p. 267). In fact, moreover, part of their Theorem 5.7 shows that there are in this case $k_n \in C_{\mathrm{c}}(G)$ with $\operatorname{supp} k_n \subset U$ and there is a continuum of sequences $\omega = (\omega_n) \in \ell^1_+$ such that the series $\sum_{n=1}^\infty \omega_n \lambda(k_n) = \Phi_\omega \in PM_p(F)$ (convergence in PM_p norm for every $1 < p < \infty$), yet $\Phi_\omega \notin \lambda(M(G))$. If G is *abelian*, URS sequences exist for all open sets U. But if G is arbitrary, necessary and sufficient conditions for the existence of URS sequences for a given open set U seem to be unknown (see [EP], p. 267 and pp. 288–293).

Furthermore, Cowling and Fournier are even able to control the support of the elements $\Phi \in CV_q$, $\Phi \notin CV_p$ mentioned above by showing that $\operatorname{supp}\Phi$ can be chosen in any symmetric Cantor set ([Ka3], p. 35) if $G = \mathbb{T}$ (the torus), or if G is not unimodular $\operatorname{supp}\Phi$ can be chosen in the kernel of the modular function.

If we restrict ourselves to fixed p, and G second countable, then our Theorem 6 yields a much more powerful result, in a sense. We only assume that F is closed and $\operatorname{int}_{aHb} F \neq \emptyset$ for some nondiscrete closed subgroup H, and some $a, b \in G$ (which holds e.g. if $\operatorname{int} F \neq \emptyset$; note that we need no Rudin–Schapiro sequences at all), and we get the existence of plenty ($PM_{\mathrm{pc}}(F)/M_p(F)$ is not even separable) of elements in $PM_{\mathrm{pc}}(F)$ which do not belong to $\lambda(M(F))$. Our results, however, do not cover the case that G is discrete or that $F \subset \mathbb{T}$ is a Cantor symmetric set unless $p = 2$.

In regard to Theorem 1, it has been shown by Lust-Piquard [P2] that if G is abelian and $F \subset G$ contains a perfect set then $CV_p(F)$ does not have the RNP.

(b) *The case that* $p = 2$. If $p = 2$ and F contains an "ultrathin symmetric" (see sequel) subset of $\mathbb{R}$ or $\mathbb{T}$ then the result of Theorem 6 remains true. Thus F can be much thinner in this case. We use in the proof a powerful result of Y. Meyer ([Me], p. 246), namely:

THEOREM (Y. Meyer). *Let $G = \mathbb{R}$, and $S \subset \mathbb{R}$ an ultrathin symmetric compact set. If $f_k \in A_2(S)$ is such that $\|f_k\|_{A_2(S)} = 1$ for all $k \geq 1$ and $\|f_k\|_{A_2(K)} \to 0$ for all compact $K \subset S$ not containing 0, then $\{f_k\}$ contains a subsequence equivalent to a canonical ℓ^1 basis.*

Here $A_2(F) = A_2(G)/I_F$ is the usual quotient algebra, for any closed $F \subset G$.

We do not know if Y. Meyer's theorem is true if $p \neq 2$. The proof of Theorem 12 works for all $1 < p < \infty$ for which Y. Meyer's theorem holds.

We thank F. Lust-Piquard for pointing out to us Y. Meyer's result (also used in [P3], p. 200).

DEFINITION. Let $t_j > 0$, for $j \geq 1$, be such that $t_n > \sum_{j=n+1}^{\infty} t_j$ for all $n \geq 1$. Let S be the compact set of all reals $t = \sum_{j=1}^{\infty} \varepsilon_j t_j$ where $\varepsilon_j = 0$ or 1. S is then called a *symmetric set*. If in addition $\sum_{k=1}^{\infty} (t_{k+1}/t_k)^2 < \infty$ then S is called an *ultrathin symmetric set* ([Me] or [GMc], p. 333). Clearly any symmetric set contains an ultrathin symmetric subset.

THEOREM 12. *Let G be an arbitrary locally compact group and let $P, Q \subset PM_2(G)$ be $A_2(G)$-modules such that P is w^*-closed, Q is norm closed, $P_{\mathrm{c}} \subset Q \subset P$ and $\sigma(P) = F$ is metrizable. Assume that $\mathbb{R}$, the real line, is a closed subgroup of G, $S \subset \mathbb{R}$ is an ultrathin symmetric set and $a \in G$. If $aS \subset F$ or $Sa \subset F$, then $Q/W_Q(a)$ (a fortiori $Q/M_2(F)$) has ℓ^∞ as a quotient and $TIM_Q(a)$ contains $\boldsymbol{\mathcal{F}}$.*

P r o o f. Clearly $S \subset \mathbb{R} \cap a^{-1}F$ and if $b = a^{-1}$ then $\sigma(l_b^* P) = bF$ and $l_b^* W_P(a) = W_{l_b^* P}(ba) = W_{l_b^* P}(e)$ by the proof of Theorem 6. Hence by the same proof we can assume that $S \subset \mathbb{R} \cap F$, $\sigma(P) = F$ and $a = e$.

Let V_n be a sequence of neighborhoods of e such that $V_{n+1}^2 \subset V_n$, $\mathrm{cl}\, V_n$ is compact for all n, and $V_n \cap F$ is a neighborhood base in F of e. Let $v_n \in A(G)$ (we omit 2 and write $A(G) = A_2(G)$ and $PM_2(G) = PM(G)$) such that $\operatorname{supp} v_n \subset V_n$ and $v_n(e) = 1 = \|v_n\|$.

Let $J = \{v \in A(G) : (\Phi, v) = 0 \text{ for all } \Phi \in P\}$; thus $P = (A(G)/J)^*$. Let $A'(F) = A(G)/J$. Then $A'(F)$ with quotient norm is a function algebra on F which acts on P. In fact (as in the proof of Theorem 4), if $v \in A(G)$ and $v' = v + J$ with $\|v'\| = \inf\{\|v + u\| : u \in J\}$ then for $\Phi \in P$, $v \cdot \Phi = v' \cdot \Phi$ and $\|v' \cdot \Phi\| \le \|v'\|\|\Phi\|$. Also $v_n'(e) = 1 = \|v_n'\|$.

If $\Phi \in P$ and $e \notin \operatorname{supp} \Phi$ then $v_n' \cdot \Phi = 0$ if n is such that $V_n \cap \operatorname{supp} \Phi = \emptyset$ (see [Hz1], p. 118). Thus $\|v_n' \cdot \Phi\| \to 0$ for all $\Phi \in E_P(e)$.

We first show, using Y. Meyer's theorem, that $\{v_n'\}$ contains a subsequence equivalent to a canonical ℓ^1 basis. Let $r : A(G) \to A(\mathbb{R})$ be the restriction map, i.e. $(rv)(x) = v(x)$ for $x \in \mathbb{R}$. Then r is onto and $\|r\| \le 1$ by [Hz1], Thm. 1 (which holds for all $1 < p < \infty$). Thus $\|rv_n\| = 1 = (rv_n)(e)$ and $\operatorname{supp} rv_n \subset V_n \cap \mathbb{R}$.

For closed $L \subset G$, let $I_L = \{v \in A(G) : v = 0 \text{ on } L\}$; similarly, for $L \subset \mathbb{R}$, $I_L^{\mathbb{R}} = \{v \in A(\mathbb{R}) : v = 0 \text{ on } L\}$. These are the biggest ideals in $A(G)$ and $A(\mathbb{R})$ respectively whose zero set is L. Let $A(L) = A(G)/I_L$ and $A^{\mathbb{R}}(L) = A(\mathbb{R})/I_L^{\mathbb{R}}$.

Let $q : A(\mathbb{R}) \to A^{\mathbb{R}}(S)$ be the canonical map. Then $qrv_n(e) = 1 = \|qrv_n\|$ (see Thm. 4). If $K \subset S$ is compact and $e \notin K$ then $V_n \cap K = \emptyset$ for $n \ge n_0$, thus $\|qrv_n\|_{A^{\mathbb{R}}(K)} = 0$. Hence $\|qrv_n\|_{A^{\mathbb{R}}(K)} \to 0$ for all compact $K \subset S$ such that $e \notin K$. It now follows from Y. Meyer's result that there is a subsequence n_j and some $c > 0$ such that

$$\Big\| \sum_{j=1}^{k} \alpha_j (qrv_{n_j}) \Big\|_{A^{\mathbb{R}}(S)} \ge c \sum_{j=1}^{k} |\alpha_j|,$$

for all k and complex $\alpha_1, \ldots, \alpha_k$.

Now if $v \in A(G)$ then $\|v'\|_{A'(F)} = \inf\{\|v + u\| : u \in J\} \ge \inf\{\|v + u\| : u \in I_S\}$ (since $J \subset I_F \subset I_S$) $\ge \inf\{\|rv + ru\| : u \in I_S\} \ge \inf\{\|rv + w\| : w \in I_S^{\mathbb{R}}\}$ (since $rI_S \subset I_S^{\mathbb{R}}$) $= \|qrv\|_{A^{\mathbb{R}}(S)}$.

It now follows that for all $k \ge 1$ and $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$ we have

$$\Big\| \sum_{j=1}^{k} \alpha_j v_{n_j}' \Big\|_{A'(F)} \ge c \sum_{j=1}^{k} |\alpha_j|.$$

We now construct an onto operator $t : P \to \ell^\infty$ as follows:

If $\Phi \in P$ let $(t\Phi)(j) = (\Phi, v_{n_j}')$. Since $\|v_{n_j}'\| = 1 = v_{n_j}'(e)$ for all j, $t(P) \subset \ell^\infty$, $\|t\| \le 1$ and $t(\lambda_2 \delta_e) = 1$ (the constant one sequence in ℓ^∞, by abuse of notation). But t is onto since if $b = (b_n) \in \ell^\infty$ with norm $\|b\|$, define the linear functional on $\mathrm{lin}\{v_{n_j}' : j \ge 1\}$ by $F_0(\sum_{j=1}^k \alpha_j v_{n_j}') = \sum_{j=1}^k b_j \alpha_j$.

Then

$$\Big|F_0\Big(\sum_{j=1}^{k}\alpha_j v'_{n_j}\Big)\Big| \le \|b\|\sum_{j=1}^{k}|\alpha_j| \le \|b\|(1/c)\Big\|\sum_{j=1}^{k}\alpha_j v'_{n_j}\Big\|.$$

By the Hahn–Banach theorem there is an extension $\Phi_0 \in (A(G)/J)^* = P$ of F_0. Thus $(t\Phi_0)(j) = (\Phi_0, v'_{n_j}) = b_j$, i.e. $t\Phi_0 = b$ and t is onto.

We now claim that $t(E_P(e)) \subset c_0 \subset \ell^\infty$, thus $t(W_P(e)) \subset c \subset \ell^\infty$. Let $v_0 \in A \cap C_c(G)$ satisfy $v_0 = 1$ on V_1 be fixed. If $\Phi \in E_P(e)$ then $(t\Phi)(j) = (\Phi, v'_{n_j}) = (\Phi, v_{n_j}) = (\Phi, v_{n_j}v_0) = (v'_{n_j}\cdot\Phi, v'_0) \to 0$ since $\|v'_{n_j}\cdot\Phi\| \to 0$ by the above. Since $t(\lambda_2\delta_e) = 1$ we have $t(W_P(e)) \subset c$.

We now show that if $v \in A(G)$, $v(e) = 1$, and $\Phi \in P$ then $t(v\cdot\Phi) - t(\Phi) \in c_0$. We first prove that for all $v \in A(G)$, $\|v'v'_{n_j} - v(e)v'_{n_j}\| \to 0$. Indeed, if $w = vv_0 - v(e)v_0$ then $w(e) = 0$ and $\|vv_{n_j} - v(e)v_{n_j}\| = \|v_{n_j}w\|$. If $\varepsilon > 0$ let $w_0 \in A \cap C_c(G)$ be such that $w_0 = 0$ on a neighborhood of e and $\|w_0 - w\| < \varepsilon$ (points are sets of synthesis by [Hz1]). But since $\operatorname{supp} v_{n_j} \subset V_{n_j}$ there is some i such that if $j \ge i$ then $v_{n_j}w_0 = 0$ on F. Hence $\|v'_{n_j}w'\| \to 0$. If now $\Phi \in P$ and $v \in A(G)$ with $v(e) = 1$ then

$$t(v\cdot\Phi - \Phi)(j) = |(\Phi, v'v'_{n_j} - v'_{n_j})| \le \|\Phi\|\|v'v'_{n_j} - v'_{n_j}\| \to 0,$$

thus $t(v\cdot\Phi) - t(\Phi) \in c_0$.

We now show that $t^*(\boldsymbol{\mathcal{F}}) \subset TIM_P(e)$. In fact, if $\psi \in \ell^{\infty *}$ satisfies $\psi = 0$ on $c_0 \subset \ell^\infty$ and $v \in A(G)$, $v(e) = 1$, $\Phi \in P$ one has $(t^*\psi, v\cdot\Phi) = (\psi, t(v\cdot\Phi)) = (\psi, t\Phi)$ (by the above claim) $= (t^*\psi, \Phi)$. By our Proposition 1, $t^*\psi(E_P(e)) = 0$. If in addition $(\psi, 1) = 1 = \|\psi\|$ then $\|t^*\psi\| \le 1$ since $\|t\| \le 1$ and $(t^*\psi, \lambda_2\delta_e) = (\psi, t(\lambda_2\delta_e)) = (\psi, 1) = 1$ by the above. Thus $t^*\boldsymbol{\mathcal{F}} \subset TIM_P(e)$. Clearly $t^* : \ell^{\infty *} \to P^*$ is a w^*-w^*-continuous norm isomorphism into (directly or by [Ru2], (4.14)). The rest of the proof is the same as the end of the proof of Theorem 4 with $S_Q = S_P = \mathbb{C}\lambda_2\delta_e$ (the line containing $\lambda_2\delta_e$). ∎

Remarks. 1. By using Theorem 2.7.6 in Rudin [Ru1] one can see that Y. Meyer's powerful theorem used above holds true if $\mathbb{R}$ is replaced by the torus (see [P3], p. 200), hence $\mathbb{R}$ can be replaced by $\mathbb{T}$ in Theorem 12.

2. Corollaries 7, 8, 10 and 11 to Theorem 6 hold true for $p = 2$ with the condition on F being replaced by that of Theorem 12.

3. It has been proved by J.-P. Kahane in [Ka2] that if $n \ge 2$ [resp. $n \ge 3$] there exists a *continuous* [resp. smooth] curve $F \subset \mathbb{T}^n$ which is a Helson S-set, i.e. $PM_2(F) = \lambda_2(M(F))$. A fortiori, if $Q = PM_2(F)$ then for all $x \in F$, $Q = W_Q(x) = M_2(F) = \lambda_2(M(F))$ and $TIM_Q(x)$ contains a unique element. The explicit construction of such curves has been done by O. C. McGehee in [Mc]. If $n > 2k$ there exist Helson sets which are k-dimensional manifolds ([Mc], p. 236). Any infinite compact abelian G contains a perfect Helson S-set ([Ka1], [Ru1]).

F. Lust-Piquard mentioned to us that if G is abelian and $F \subset G$ is a Helson S-set for $p = 2$ it is such for all $1 < p < \infty$ since $PM_2(F) = \lambda(M(F)) \subset PM_p(F) \subset PM_2(F)$, thus $\lambda(M(F)) = PM_p(F)$. Any infinite compact abelian G contains a perfect Helson S-set F. For such F, $PM_p(F) = M(F)$ does not have the WRNP.

4. Let G be abelian second countable and let $\mathcal{F} : L^1(\widehat{G}) \to A(G)$ and $\mathcal{F}_{\mathrm{S}} : M(\widehat{G}) \to B(G)$ be Fourier and Fourier–Stieltjes transforms respectively. If $\mu \in M(G)$ then $\mathcal{F}^*(\lambda_2\mu) = \mathcal{F}_{\mathrm{S}}(\mu)$. If $P = PM_2(F)$ let $\mathcal{F}^*(P_{\mathrm{c}}) = UC(\widehat{G}, F)$, $\mathcal{F}^*(M_2(F)) = \boldsymbol{\mathcal{B}}_2(\widehat{G}, F) = \operatorname{ncl} \mathcal{F}_{\mathrm{S}}(M(F))$, with ncl in $L^\infty(\widehat{G})$. Omit F if $F = G$ (thus for example $\mathcal{F}^* M_2(G) = \boldsymbol{\mathcal{B}}_2(\widehat{G})$).

Assume that $e \in F$, thus $1 \in UC(\widehat{G}, F)$, and let $A_F = \{f \in UC(\widehat{G}, F) : \psi_1(f) = \psi_2(f)$ for any $\psi_1, \psi_2 \in IM_F\}$ where IM_F is the set of invariant means on $UC(\widehat{G}, F)$ omit F if $F = G$; see [Pa]. Let $A = A_G$.

It can be seen that if $P = PM_2(F)$ and $W_{P_{\mathrm{c}}}(e) = \mathbb{C}\lambda_2\delta_e + E_{P_{\mathrm{c}}}(e)$, then $\mathcal{F}^*(W_{P_{\mathrm{c}}}(e)) = A_F$. Thus $\boldsymbol{\mathcal{B}}_2(\widehat{G}, F) \subset A_F$.

As is well known, $UC(\widehat{G})/A$ (a fortiori $UC(\widehat{G})/\boldsymbol{\mathcal{B}}_2(\widehat{G})$, see [Ch1]) has ℓ^∞ as a quotient and IM contains $\boldsymbol{\mathcal{F}}$ if G is infinite. Our result is much stronger in that it allows one to localize to the set F. Thus $UC(\widehat{G}, F)/A_F$ (a fortiori $UC(\widehat{G}, F)/\boldsymbol{\mathcal{B}}_2(\widehat{G}, F)$) has ℓ^∞ as a quotient if $e \in \operatorname{int}_{aH}(F)$ for some nondiscrete closed subgroup H and $a \in G$, and IM_F contains the big set $\boldsymbol{\mathcal{F}}$.

If in addition G contains $\mathbb{R}$ or $\mathbb{T}$ and F only contains an ultrathin symmetric set S, then the same remains true.

These results are new even if $G = \mathbb{R}$ or $\mathbb{T}$ and improve substantially [P3], p. 201, Ex. 3. Moreover, the result on the largeness of IM_F cannot be proved by the conventional methods which use properties of finite intersections of translates of subsets of G (see [Pa]) since $\mathcal{F}^*(PM_2(F))$ need not be a pointwise subalgebra of $L^\infty(\widehat{G})$ (take $G = \mathbb{R}$, $F = [0, 1]$).

5. A closed set $F \subset G$ is *p-ergodic* if, for $Q = PM_p(F)$, $PM_p(F) = \mathbb{C}\lambda\delta_x \oplus E_Q(x)$ for all $x \in F$. 2-ergodic sets for abelian G have been studied by Woodward in [Wo1, 2, 3]. It is clear from [Wo2] that if F contains an open set $U \subset G$ then for any $x \in U$, $PM_2(F) \neq \mathbb{C}\lambda\delta_x \oplus E_Q(x)$. It does not follow from [Wo2] that $PM_p(F)/\mathbb{C}\lambda\delta_x \oplus E_Q(x)$ has ℓ^∞ as a quotient even in this case ($p = 2$).

Important examples of perfect 2-ergodic sets $F \subset \mathbb{R}$ whose every closed subset obeys synthesis and yet every "portion" of which is not a Helson set are the "Sigtuna" sets of I. Katznelson (see [GMc], p. 394). (The closed set E is a "portion" of F if for some *open* interval I of $\mathbb{R}$, $E = I \cap F$.) Could, for such F, $(PM_2(F))_{\mathrm{c}}/M_2(F)$ have ℓ^∞ as a quotient?

6. We now give examples of w^*-closed A_2-submodules $P \subset PM_2(G)$, G abelian, for which $P/M_2(F)$ *has* ℓ^∞ as quotient where $F = \sigma(P)$, yet $P/\mathbb{C}\lambda_2\delta_x \oplus E_P(x) = \{0\}$ for many $x \in F$.

Let G be abelian, $p = 2$, $F = F_1 \cup F_2 \subset G$, F_1, F_2 compact disjoint, where F_1 is a perfect Helson synthesis set, and F is also of synthesis. Clearly $P = PM_2(F) = PM_2(F_1) \oplus PM_2(F_2)$ since if $v_1, v_2 \in A_2(G)$ are such that $v_i = 1$ on a neighborhood O_i of F_i such that $\overline{O}_1 \cap \overline{O}_2 = \emptyset$ and $v_1(O_2) = v_2(O_1) = 0$ then for any $\Phi \in PM_2(F)$, $\Phi = v_1 \cdot \Phi + v_2 \cdot \Phi$ and $v_i \cdot \Phi \in PM_2(F_i)$. The sum is direct since any $\Phi \in PM_2(F_1) \cap PM_2(F_2)$ has void support, hence $\Phi = 0$. Since F_1 is a Helson S-set, $PM_2(F_1) = \lambda_2(M(F_1)) = \mathbb{C}\lambda_2\delta_x \oplus E_{P_1}(x)$ for all $x \in F_1$ where $P_i = PM_2(F_i)$. Since any $\Phi \in P_2$ satisfies $x \notin \operatorname{supp} \Phi$ if $x \in F_1$ it follows that $E_P(x) \supset E_{P_1}(x) + P_2$. Hence for $x \in F_1$, $P = P_1 \oplus P_2 \subset \mathbb{C}\lambda_2\delta_x \oplus E_{P_1}(x) \oplus P_2 \subset \mathbb{C}\lambda_2\delta_x \oplus E_P(x) \subset P$. Thus $P = \mathbb{C}\lambda_2\delta_x \oplus E_P(x)$ for all $x \in F_1$.

If now $\operatorname{int}_{aH}(F_2) \neq \emptyset$ for some nondiscrete closed subgroup $H \subset G$ and $a \in G$ then by our main theorem, $P/M_2(F)$ has ℓ^∞ as a quotient. In fact, moreover, even $P/\mathbb{C}\lambda_2\delta_x \oplus E_P(x)$ has ℓ^∞ as a quotient for all $x \in \operatorname{int}_{aH}(F_2) \subset \operatorname{int}_{aH}(F)$ (similarly if F_2 contains xS where S is an ultrathin symmetric subset of $\mathbb{R} \subset G$).

7. If $P/W_P(d)$ has ℓ^∞ as a quotient where $d \in \sigma(P)$ and P is as in Theorem 6 or 12 then the function algebra $A'_p(F) = A_p/J$ where $J = \{u \in A_p : (\Phi, u) = 0 \text{ for all } \Phi \in P\}$ is not Arens regular. Indeed, if $WAP_P = \{\Phi \in P : \{u \cdot \Phi : \|u\|_{A_p} \leq 1\}$ is weakly relatively compact$\}$ then $\Phi \in WAP_P$ iff $\{u \cdot \Phi : u \in A'_p(F),\ \|u\|_{A'_p(F)} \leq 1\}$ is weakly relatively compact. But $WAP_P \subset W_P(d)$ as in [Gr1], p. 125. Hence $P/W_P(d) \neq \{0\}$ by Theorems 6 or 12, a fortiori $(A'_p(F))^* = P \neq WAP_P$, which implies the result.

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Reçu par la Rédaction le 17.5.1993;
en version modifiée le 25.8.1993