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UNIQUENESS FOR A CLASS OF COOPERATIVE SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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Let

$$\dot{x} = f(t, x) \tag{1}$$

be a system of ordinary differential equations, with $f = (f_1, \ldots, f_n) : I \times U \to \mathbb{R}^n$, where I is an open interval in $\mathbb{R}$ and $U \subset \mathbb{R}^n = \{(x^1, \ldots, x^n)\}$ is an open subset. Recall that the function f satisfies the Carathéodory conditions if the following hold:

(C1) $f(t, \cdot)$ is continuous for each fixed $t \in I$.

(C2) $f(\cdot, x)$ is measurable for each fixed $x \in U$.

(C3) There exists an integrable function $m : I \to [0, \infty)$ such that $|f(t, x)| \le m(t)$ for each $(t, x) \in I \times U$.

It is well known (see e.g. Thm. 1.1 on p. 43 in [1]) that if f satisfies the Carathéodory conditions then for each $(t_0, x_0) \in I \times U$ there exists at least one solution to the initial value problem

$$\begin{aligned} \dot{x} &= f(t, x), \\ x(t_0) &= x_0 \end{aligned} \tag{2}$$

defined on an open interval $J \subset I$ containing t_0. (Of course, here by *solution* we understand an absolutely continuous function satisfying (1) a.e.)

The Carathéodory conditions alone do not imply uniqueness of solutions to (2).

System (1) is called *cooperative* (or *quasimonotone*) if the following is satisfied:

(A1) For all $i \ne j$ the function f_i is nondecreasing with respect to x^j.

Our next assumption will be:

(A2) $\sum_{i=1}^{n} f_i(t, x) = 0$ for all $t \in I$, $x \in U$.

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THEOREM. *Let f satisfy* (C1) *through* (C3) *along with* (A1) *and* (A2). *Then for any* $(t_0, x_0) \in I \times U$ *there exists precisely one solution to the initial value problem* (2).

Before proceeding with the proof of the Theorem let us introduce the following notation: for $x, y \in U$ we write $x \le y$ if $x^i \le y^i$ for each i, and $x < y$ if $x \le y$ and $x \ne y$.

Let (t_0, x_0) be fixed. First of all, notice that the result is a local one, so we may assume U to be an open parallelepiped in $\mathbb{R}^n$. This enables us to avoid pathologies described in [12] and in Chapter II of [10]. By an argument similar to that used in the proof of Thm. 1.2 on pp. 45–47 in [1] (compare also Thm. 16.2 in [10]) there exists a *maximum solution* $x_{\max}(\cdot)$ of (2) such that for any solution $x(\cdot)$ of (2) the inequality $x(t) \le x_{\max}(t)$ holds for all t in the common interval of existence. Analogously, there exists a *minimum solution* $x_{\min}(\cdot)$ of (2).

Notice that by (A2) we have

$$\sum_{i=1}^{n} \frac{d}{dt}(x_{\max})^i(t) = \sum_{i=1}^{n} \frac{d}{dt}(x_{\min})^i(t) = 0$$

for a.e. t in their respective intervals of existence, so the absolutely continuous real functions $\Sigma_{\max} := \sum_{i=1}^{n}(x_{\max})^i$ and $\Sigma_{\min} := \sum_{i=1}^{n}(x_{\min})^i$ have their derivatives equal to 0 a.e. This implies $\Sigma_{\min}(t) = \Sigma_{\max}(t)$ as long as both are defined. Suppose to the contrary that for some τ one has $x_{\min}(\tau) < x_{\max}(\tau)$. From this it follows that $\Sigma_{\min}(\tau) < \Sigma_{\max}(\tau)$. The contradiction obtained proves the Theorem.

Concluding remarks. 1. Condition (A2) can be generalized to:

(A2′) There exists a C^1 first integral $H : U \to \mathbb{R}$ for (1) such that $(\partial H/\partial x^i)(x) > 0$ for each $x \in U$ (see [6], [7], or [5]).

The proof remains much the same.

2. In many papers dealing with cooperative systems satisfying (A2) or (A2′) some additional conditions have been assumed guaranteeing the uniqueness of solutions (see [8], [9], [11]). In the light of our Theorem those hypotheses are redundant.

3. We have the continuous dependence of the unique solution on the initial value (compare pp. 58–60 in [1]). Therefore, whenever U is convex and system (1) is autonomous, it generates a continuous local flow that is *monotone* (for the definition and properties of monotone flows the reader is referred to [4], see also [2], [3]).

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