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THE SUPPORT OF A FUNCTION WITH THIN SPECTRUM

BY

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We prove that if $E \subseteq \widehat{G}$ does not contain parallelepipeds of arbitrarily large dimension then for any open, non-empty $S \subseteq G$ there exists a constant $c > 0$ such that $\|f1_S\|_2 \geq c\|f\|_2$ for all $f \in L^2(G)$ whose Fourier transform is supported on E. In particular, such functions cannot vanish on any open, non-empty subset of G. Examples of sets which do not contain parallelepipeds of arbitrarily large dimension include all $\Lambda(p)$ sets.

0. Introduction. Many authors have studied the behaviour of functions whose Fourier coefficients vanish on long intervals. For example, if $f = \sum a_k e^{in_k t} \in L^2[0, 2\pi]$ where $n_{k+1} - n_k \geq q > 0$ and $f \not\equiv 0$, then f cannot vanish on any interval of length $2\pi/q$ [14, p. 222]. Similarly, if the Fourier transform of an integrable function f is extended to $\mathbb{R}$ by defining $\widehat{f}(y) \equiv \int_0^{2\pi} e^{-iyx} f(x)\, dx$, and the upper density of $\{y : \widehat{f}(y) = 0\}$ is d, then f cannot be supported on any interval of length less than d [8, p. 13].

These results resemble the uncertainty principle in that the smaller $\{\widehat{f} \neq 0\}$ is, the more $\{f \neq 0\}$ must "spread out". It is natural to ask how little one can assume about $\{\widehat{f} \neq 0\}$ and still obtain interesting conclusions on $\{f \neq 0\}$. In particular, it seems natural to ask: If E has arbitrarily large gaps and $f = \sum a_k e^{in_k t} \in L^2$, $f \not\equiv 0$, then can f vanish on any set with non-empty interior? Of course, the answer is no if E is a finite set or $E = \mathbb{Z}^+$. In this note we will obtain a stronger conclusion than just that f cannot vanish, but under a stronger assumption.

Definition. A subset P of $\mathbb{Z}$ is called a *parallelepiped of dimension* N if P is the sum of N two-element sets and P has 2^N elements.

Arithmetic progressions of length 2^N are examples of parallelepipeds of dimension N. Lacunary sets in $\mathbb{Z}$ and many other "thin" sets, such as $\Lambda(p)$ sets (see Section 2), are examples of sets which do not contain parallelepipeds of arbitrarily large dimension.

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It is a consequence of results of [10] and [12] that if E does not contain parallelepipeds of arbitrarily large dimension then no non-trivial L^2 function whose Fourier transform is supported on E can vanish on any non-empty open set. With the same hypothesis we will prove that for every non-empty open set S there is a constant $c > 0$ such that $\|f1_S\|_2 \ge c\|f\|_2$ for all $f \in L^2$ with $\operatorname{supp} \widehat{f} \subseteq E$. This builds on the work of [11] and [3]. Of course, this stronger conclusion cannot hold if we only assume that E has arbitrarily large gaps since such sets can contain arbitrarily long blocks of consecutive integers.

An interesting consequence of our theorem is that the support of the Fourier transform of a non-trivial, properly supported measure must contain arbitrarily large parallelepipeds.

1. Main result. Our results will actually be proved in the general framework of a compact abelian group G with dual group Γ. The notion of parallelepiped generalizes to subsets of Γ in the obvious way.

DEFINITIONS. A subset E of Γ is said to be *strictly-2-associated* with a subset S of G if there is a constant $c > 0$ (called a *constant of strict-2-associatedness* for E and S) such that $\|f1_S\|_2 \ge c\|f\|_2$ for all $f \in L^2_E = \{f \in L^2(G) : \operatorname{supp} \widehat{f} \subseteq E\}$.

Let X and E be subsets of Γ. We say that E is X-*subtransversal* if whenever $\chi, \psi \in E$, $\chi \ne \psi$, then $\chi\psi^{-1} \notin X$.

If there are distinct characters χ and ψ in E such that $\chi\psi^{-1}$ belongs to a finite subgroup X, then the polynomial $\chi - \psi$ vanishes on the annihilator of X, an open subgroup of G, and hence E is not strictly-2-associated with all non-empty open subsets of G. If G is connected then the only finite subgroup of Γ is the trivial subgroup, so all subsets of Γ are X-subtransversal for all finite subgroups X. Thus when G is a connected group the assumption of X-subtransversality in our theorems is vacuous.

THEOREM 1. *Suppose $E \subseteq \Gamma$ is X-subtransversal for all finite subgroups X of Γ. If E does not contain parallelepipeds of arbitrarily large dimension then E is strictly-2-associated with all non-empty open subsets of G.*

Remarks. In [3] we obtained a stronger result, but under stronger assumptions (see the remarks after Corollary 2.2). Some of the same ideas are used in this proof.

Several preliminary lemmas are needed.

LEMMA 1.1. *Suppose $E \subseteq \Gamma$ and U is an open, non-empty subset of G such that E and U are strictly-2-associated. Let S be any open subset of G containing $\overline{U}$. There is a finite subgroup X of Γ, depending on S and E, so that whenever the set $E \cup \{\chi\}$, $\{\chi\} \in \Gamma$, is X-subtransversal, then*

$E \cup \{\chi\}$ is strictly-2-associated with S, with constant of strict-2-associatedness independent of χ.

Proof. Choose a neighbourhood V of the identity so that $UV \subseteq S$. Given $v \in V$ and $f \in L^2_{E\cup\{\chi\}}(G)$, let $f_v(x) = f(xv) - \chi(v)f(x)$. Observe that

$$\|1_S f\|_2^2 = \frac{1}{m(V)} \int\limits_V \int\limits_S |f(x)|^2 \, dm(x)\, dm(v) \geq \frac{1}{4m(V)} \int\limits_V \int\limits_U |f_v(x)|^2 \, dm(x)\, dm(v).$$

Assume that whenever $h \in \mathrm{Trig}_E(G)$ then $\|1_U h\|_2^2 \geq c_1 \|h\|_2^2$. Since $f_v \in \mathrm{Trig}_E(G)$ for all $v \in V$, we have

$$\|1_S f\|_2^2 \geq \frac{c_1}{4m(V)} \int\limits_V \|f_v\|_2^2 \, dm(v).$$

The proof of the lemma is completed in the same manner as were the proofs of Lemmas 2.2 and 3.4 of [3]. ∎

COROLLARY 1.2. *Suppose $E \subseteq \Gamma$ is X-subtransversal for all finite subgroups X of Γ. If there is a finite set F and an open, non-empty set U so that $E\backslash F$ and U are strictly-2-associated, then E is strictly-2-associated with any open set S containing $\overline{U}$.*

Proof. Assume $F = \{\chi_1, \ldots, \chi_N\}$. Being a compact Hausdorff space, G is normal. Thus it is possible to choose open sets $S_1, \ldots, S_{N-1}$ satisfying

$$\overline{U} \subset S_1 \subset \overline{S}_1 \subset S_2 \subset \ldots \subset \overline{S}_{N-1} \subset S.$$

By the previous lemma $(E\backslash F) \cup \{\chi_1\}$ is strictly-2-associated with S_1, hence $(E\backslash F) \cup \{\chi_1, \chi_2\}$ is strictly-2-associated with S_2, and so by induction E is strictly-2-associated with S. ∎

LEMMA 1.3. *Let S and S_1 be open, non-empty subsets of G with $\overline{S}_1 \subseteq S$. Given $c > 0$ there is a finite symmetric set $F = F(c, S_1, S) \subseteq \Gamma$, containing the identity, so that if the sets $\{E_i\}_{i\in I} \subseteq \Gamma$ are strictly-2-associated with S_1, with constant of strict-2-associatedness c, and $E_i E_j^{-1} \cap F = \emptyset$ for all $i, j \in I$, $i \neq j$, then $\bigcup_{i\in I} E_i$ is strictly-2-associated with S.*

Proof. Since G is normal there is a continuous function $g : G \to [0, 1]$ with $g(S_1) = 1$ and $g(S^{\mathrm{c}}) = 0$. Let P be a polynomial with

$$\|P - g\|_\infty^2 < \varepsilon \equiv c^2/12.$$

Set $F = (\operatorname{supp} \widehat{P})(\operatorname{supp} \widehat{P})^{-1}$. If $f \in \mathrm{Trig}_{\cup E_i}(G)$, $f = \sum_{i\in I} f_i$ with

$f_i \in \mathrm{Trig}_{E_i}(G)$, then
$$\|Pf\|_2^2 = \sum_{i\in I} \|Pf_i\|_2^2.$$
Thus
$$\|1_S f\|_2^2 \ge \|gf\|_2^2 \ge \tfrac{1}{2}\|Pf\|_2^2 - \|(P-g)f\|_2^2 \ge \tfrac{1}{2}\sum_{i\in I}\|Pf_i\|_2^2 - \varepsilon\|f\|_2^2.$$
But also
$$\|Pf_i\|_2^2 \ge \tfrac{1}{2}\|gf_i\|_2^2 - \varepsilon\|f_i\|_2^2 \ge \tfrac{1}{2}\|1_{S_1} f_i\|_2^2 - \varepsilon\|f_i\|_2^2 \ge \tfrac{1}{2}c^2\|f_i\|_2^2 - \varepsilon\|f_i\|_2^2$$
since the sets $\{E_i\}_{i\in I}$ are strictly-2-associated with S_1 with constant of strict-2-associatedness c.

Hence
$$\begin{aligned}\|1_S f\|_2^2 &\ge \sum_{i\in I}(\tfrac{1}{4}c^2\|f_i\|_2^2 - \tfrac{1}{2}\varepsilon\|f_i\|_2^2) - \varepsilon\|f\|_2^2 \\ &\ge \tfrac{1}{4}c^2\|f\|_2^2 - \tfrac{3}{2}\varepsilon\|f\|_2^2 \ge \tfrac{1}{8}c^2\|f\|_2^2. \ \blacksquare\end{aligned}$$

DEFINITIONS. Let F be a finite subset of Γ. For $\chi, \psi \in \Gamma$ we say that χ is F-*equivalent* to ψ if for some positive integer m there is a sequence $\chi = \chi_1, \chi_2, \ldots, \chi_m = \psi$ with $\chi_{i+1}\chi_i^{-1} \in F$ for $i = 1, \ldots, m-1$. If $\chi_i \in E$ for $i = 1, \ldots, m$ we say χ is (E, F)-*equivalent* to ψ.

When F is a symmetric subset of Γ containing the identity this is an equivalence relation.

We will say E has the *uniformly large gap property* provided for each finite symmetric subset F of Γ, containing the identity, there is an integer $s > 0$ such that if $\chi\psi^{-1} \notin F^s$ then χ and ψ are not (E, F)-equivalent.

LEMMA 1.4 (Theorem 3.1 of [3]). *Suppose $E \subset \Gamma$ does not contain arbitrarily large parallelepipeds. Then E has the uniformly large gap property.*

LEMMA 1.5. *Suppose E does not contain arbitrarily large parallelepipeds and is X-subtransversal for all finite subgroups X of Γ. Let S be an open subset of G and assume $E' \subset E$ is strictly-2-associated with the open set S_1 whose closure is contained in S. Then there is a finite set $F_1 = F_1(E, E', S, S_1)$ so that whenever $E'' = \{\chi_i\}_{i\in I} \subset E$ satisfies $\chi_i\chi_j^{-1} \notin F_1$ if $i \ne j$, then $E' \cup E''$ is strictly-2-associated with S.*

Proof. Choose S_2 open with $\overline{S}_1 \subset S_2 \subset \overline{S}_2 \subset S$. By Lemma 1.1 there is a constant $c > 0$ which is a constant of strict-2-associatedness for S_2 and each of the sets $E' \cup \{\chi\}$, $\chi \in \Gamma$. By Lemma 1.3 we obtain the finite set $F = F(c, S_2, S)$ with the property that if $\{E_i\}_{i\in I}$ are strictly-2-associated with S_2 with constant of strict-2-associatedness c, and $E_iE_j^{-1} \cap F = \emptyset$, then $\bigcup_{i\in I} E_i$ is strictly-2-associated with S.

Set $F_1 = F^s$ where s is chosen as in the previous lemma. Let E_i denote the elements of $E' \cup E''$ which belong to the (E,F)-equivalence class which contains χ_i, and let $E_0 = E' \cup E'' \setminus \bigcup_{i\in I} E_i$. If $i \in I$ then $E_i \subseteq E' \cup \{\chi_i\}$, and $E_0 \subset E'$. Thus each set E_i with $i \in I \cup \{0\}$ is strictly-2-associated with S with constant of strict-2-associatedness c. By definition of the equivalence relation, $E_i E_j^{-1} \cap F = \emptyset$ if $i \neq j$ and so by Lemma 1.3, $\bigcup_{i\in I\cup\{0\}} E_i = E' \cup E''$ is strictly-2-associated with S. ∎

Before completing the proof of the theorem we need some additional terminology.

Definitions. A subset E of Γ is said to *tend to infinity* if for every finite set $\Delta \subseteq \Gamma$ there is a finite set $F \subseteq E$ such that if χ, ψ are distinct elements of $E \setminus F$ then $\chi\psi^{-1} \notin \Delta$.

For subsets of $\mathbb{Z}$ this means that for each positive integer N only finitely many points of E differ in absolute value by at most N.

As in [11] we define sets of class M_n inductively as follows:

M_0 = class of subsets of Γ which tend to infinity;

M_n = class of those subsets of Γ which for each finite set Δ are the union of two sets, one a finite union of sets in class M_{n-1} and the other a set of the form $\{\chi_i\}$ where $\chi_i \chi_j^{-1} \notin \Delta$ if $i \neq j$.

In [11] it is shown that any subset of $\mathbb{Z}$ which does not contain parallelepipeds of dimension n belongs to the class M_{n-2}. Essentially the same proof works for subsets of Γ.

Proof of Theorem 1. We will prove the following statement by induction on k: Let $E'' \subset E$ belong to M_k. For all non-empty open sets S_1 and S, with $\overline{S}_1 \subset S$, and for all subsets E' of E which are strictly-2-associated with S_1, it is the case that $E' \cup E''$ is strictly-2-associated with S.

As E belongs to class M_n for some n, this clearly suffices to prove the theorem.

So first assume E'' is in class M_0, and let S, S_1 and E' be as above. Choose S_2 open with $\overline{S}_1 \subset S_2 \subset \overline{S}_2 \subset S$. Choose the finite set $F_1 = F_1(E, E', S_2, S_1)$ as in Lemma 1.5. Since E'' tends to infinity there is a finite set F so that if $\chi, \psi \in E'' \setminus F$, $\chi \neq \psi$, then $\chi\psi^{-1} \notin F_1$. By Lemma 1.5, $E' \cup (E'' \setminus F)$ is strictly-2-associated with S_2. By Corollary 1.2, $E' \cup E''$ is strictly-2-associated with S, which completes the first step in the induction argument.

Now assume the induction statement is true for $k = n$ and suppose E'' is in class M_{n+1}. Let S_1, S be non-empty, open sets with $\overline{S}_1 \subset S$, and suppose $E' \subset E$ is strictly-2-associated with S_1. Choose S_2 open with $\overline{S}_1 \subset S_2 \subset \overline{S}_2 \subset S$, and choose the finite set $F_1 = F_1(E, E', S_2, S_1)$ as in Lemma 1.5. Since E'' belongs to M_{n+1}, $E'' = E^* \cup \{\chi_i\}$ where E^* is a finite union of sets in class M_n and if $i \neq j$, then $\chi_i\chi_j^{-1} \notin F_1$. By Lemma 1.5, $E' \cup \{\chi_i\}$ is strictly-2-associated with S_2. Now assume $E^* = \bigcup_{i=1}^m E_i$ where each set E_i is in class M_n. Suppose S_3 is an open set satisfying $\overline{S}_2 \subset S_3 \subset \overline{S}_3 \subset S$. By the induction assumption $E' \cup \{\chi_i\} \cup E_1$ is strictly-2-associated with S_3, and by applying the induction assumption $m-1$ more times it follows that $E' \cup \{\chi_i\} \cup E^* = E' \cup E''$ is strictly-2-associated with S. ■

2. Applications. The reader will have noticed that we have actually proved the following stronger result:

THEOREM 2. *Suppose $E \subseteq \Gamma$ belongs to the class M_n for some n, E has the uniformly large gap property and E is X-subtransversal for all finite subgroups X of Γ. Then E is strictly-2-associated with all open, non-empty subsets of G.*

As a consequence we can prove

COROLLARY 2.1. *In either of the following two cases E is strictly-2-associated with all open, non-empty subsets of G:*

(a) *$E \subseteq \Gamma$ tends to infinity and is X-subtransversal for all finite subgroups X.*

(b) *$E \subseteq \mathbb{Z}$ belongs to the class M_n for some n.*

Proof. (a) Sets which tend to infinity clearly have the uniformly large gap property.

(b) Subsets of $\mathbb{Z}$ which belong to the class M_n have zero uniform density [11] and such sets are easily seen to have the uniformly large gap property. As T is connected all subsets are X-subtransversal for all finite subgroups of $\mathbb{Z}$. ■

DEFINITIONS. Let $0 < p < \infty$. A subset E of Γ is called a *$\Lambda(p)$ set* if there is some $q < p$ and constant $c(q)$ such that $\|f\|_p \leq c(q)\|f\|_q$ whenever $\operatorname{supp} \widehat{f}$ is a finite subset of E.

Let $1 \leq p < 2$. We say that E is a *p-Sidon set* if there is a constant c so that $\|\widehat{f}\|_p \leq c\|f\|_\infty$ whenever $\operatorname{supp} \widehat{f}$ is a finite subset of E. A 1-Sidon set is usually called a *Sidon set*.

Lacunary sets in $\mathbb{Z}$ are examples of sets which are $\Lambda(p)$ for all $p > 0$ and are p-Sidon for all $1 \leq p < 2$. For other examples the reader is referred to [6], [9] or [13]. Subsets of Γ which are $\Lambda(p)$ for some $p > 0$ or p-Sidon for some $1 \leq p < 2$ cannot contain parallelepipeds of arbitrarily large dimension ([4], [6]) and thus we have

COROLLARY 2.2. *If $E \subseteq \Gamma$ is a $\Lambda(p)$ set for some $p > 0$ which is X-subtransversal for all finite subgroups X of Γ, then E is strictly-2-associated with all open, non-empty subsets of G.*

Previously ([3]) we extended work of [1], [2] and [11] by showing that any $\Lambda(p)$ set for some $p > 2$, which was X-subtransversal for all finite subgroups, was strictly-2-associated with all sets of positive measure. As there are no known examples of $\Lambda(p)$ sets which are not $\Lambda(2+\varepsilon)$ sets for some $\varepsilon > 0$, Corollary 2.2 is only of formal interest.

No arithmetic characterization of $\Lambda(p)$ sets or p-Sidon sets is known. One reason for the interest in studying properties of sets which do not contain large parallelepipeds is because it is unknown if the absence of parallelepipeds of arbitrarily large dimension characterizes $\Lambda(2)$ sets.

We will give one other application of our theorem. Klemes [7] used other methods to prove a weaker version of this result.

COROLLARY 2.3. *Let G be a connected group. Suppose $\mu \in M(G)$, $\mu \neq 0$ and μ is supported on a proper compact set K. Then* $\operatorname{supp}\widehat{\mu}$ *contains parallelepipeds of arbitrarily large dimension.*

Remark. Host and Parreau [5] proved a theorem resembling this, but with stronger assumptions and conclusions.

Proof of Corollary 2.3. The regularity of Haar measure m ensures that there is an open set O containing K with $m(O) < 1$. Let V be a neighbourhood of the identity chosen so that $K\overline{V} \subseteq O$, and let S be the complement of $K\overline{V}$. Then obviously S is an open set of positive measure. Choose $F \in L^\infty(G)$ supported on V, with $F * \mu \not\equiv 0$.

If $\operatorname{supp}\widehat{\mu}$ does not contain parallelepipeds of arbitrarily large dimension then $\operatorname{supp}\widehat{\mu}$ is strictly-2-associated with S, and as $\operatorname{supp}\widehat{F*\mu} \subseteq \operatorname{supp}\widehat{\mu}$, it follows that $F * \mu$ cannot vanish identically on S. But $F * \mu$ is supported on $K\overline{V} = S^{\mathrm{c}}$, which is a contradiction. ∎

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