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THE FATOU THEOREM FOR NA GROUPS—A NEGATIVE RESULT

BY

JAROSŁAW SOŁOWIEJ (WROCŁAW)

In 1969 E. M. Stein and N. J. Weiss showed that in the case of symmetric spaces the unrestricted almost everywhere convergence of a bounded harmonic function to the boundary fails. The main point of their argument is that due to the action of the maximal compact group K, the unrestricted convergence allows for so many rotations of rectangles that they cannot form a differentiation basis. Somewhat similar phenomena appear in the situation where there is no group of rotations. Rectangles with incomparable sides length parallel to many directions occur while studying approaches to a boundary $X = NA/N_0A$ of the solvable part of the Iwasawa decomposition. These boundaries are not boundaries for the whole semisimple group $G = NAK$ but only for the solvable part NA. They have been intensively studied by E. Damek and A. Hulanicki [DH1] and [DH2]. The aim of this paper is to show why a very natural generalization of "admissible convergence" to the boundary of bounded harmonic functions investigated in [DH2] fails in the case of general boundaries for NA groups.

In the second part of the paper we show sharp pointwise estimates for the Poisson kernel on a boundary as above for harmonic functions with respect to a very regular subelliptic operator. This is a very particular result but it might serve as a general hint what type of estimates should be expected in the general case.

Both parts of the paper originate in questions asked by E. Damek and A. Hulanicki. The author is grateful to both of them for their help. He is also greatly indebted to Jarosław Wróblewski for his generous help concerning the construction of the counterexample.

1. Counterexample

1.1. *Preliminaries.* Let G be a Lie group. We say that a locally compact space X is a *G-space* if there is a continuous map

$$G \times X \ni (s, x) \mapsto sx \in X$$

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such that $(ss')x = s(s'x)$. For bounded measures μ and ν on G and X respectively, the convolution $\mu * \nu$ is a measure on X defined by

$$\int f(x)\, d\mu * \nu(x) = \int f(sx)\, d\mu(s)\, d\nu(x), \qquad f \in C_0(X).$$

Let L be a left-invariant second order, degenerate elliptic differential operator without constant term on G,

$$L = X_1^2 + X_2^2 + \ldots + X_k^2 + X_0.$$

The operator L is the infinitesimal generator of a convolution semigroup of probability measures $\{\mu_t\}_{t>0}$.

Definition 1.1.1. We say that X is an *L-boundary* of G if X is a G-space and there is a probability measure ν on X such that

$$\check{\mu}_t * \nu = \nu \quad \text{for each } t > 0 \tag{1}$$

and $\delta_{S_t} * \nu$ tends weak* to a point mass on X as $t \to \infty$, for almost all trajectories S_t of the diffusion process on G generated by L.

We consider a group S whose Lie algebra is of the form

$$\mathbf{s} = \mathbf{n} \oplus \mathbf{a},$$

where $\mathbf{n}$ is a nilpotent ideal in $\mathbf{s}$ and $\mathbf{a}$ is abelian. We assume that there exists a basis $X_1, \ldots, X_n$ of $\mathbf{n}$ such that for every $H \in \mathbf{a}$,

$$\mathrm{ad}_H(X_j) = \lambda_j(H) X_j,$$

where $\lambda_j \in \mathbf{a}^*$, $j = 1, \ldots, n$. Let $\Delta = \{\lambda_1, \ldots, \lambda_n\}$ and

$$\mathbf{n}^\lambda = \{X \in \mathbf{n} : \mathrm{ad}_H(X) = \lambda(H) X \text{ for all } H \in \mathbf{a}\}.$$

Definition 1.1.2. A subspace $\mathbf{n}'$ of $\mathbf{n}$ is called *homogeneous* if

$$\mathrm{ad}_H(\mathbf{n}') \subset \mathbf{n}' \quad \text{for all } H \in \mathbf{a}.$$

Let $S = \exp \mathbf{s}$, $N = \exp \mathbf{n}$ and $A = \exp \mathbf{a}$. We assume that the smallest Lie subalgebra which contains $X_0, X_1, \ldots, X_n$ is equal to $\mathbf{s}$, which by the Hörmander theorem implies that L is subelliptic.

Definition 1.1.3. A Borel function F is *L-harmonic* if

$$\langle F, L^+ \phi \rangle = 0 \quad \text{for } \phi \in C_c^\infty(S),$$

where $L^+ = X_1^2 + \ldots + X_k^2 - X_0$.

We have the following simple characterization of L-harmonic functions.

Theorem 1.1.1. *A bounded Borel function F is L-harmonic if and only if $F * \check{\mu}_t = F$ for every $t > 0$.*

For the proof see e.g. [D].

Let $X_0 = Y_0 + Z_0$, where $Y_0 \in \mathbf{n}$ and $Z_0 \in \mathbf{a}$. We define

$$\varDelta_0 = \{\lambda \in \varDelta : \langle \lambda, Z_0 \rangle \geq 0\}, \qquad \varDelta_1 = \varDelta \setminus \varDelta_0,$$

and

$$\mathbf{n}_0(L) = \bigoplus_{\lambda \in \varDelta_0} \mathbf{n}^\lambda.$$

In [DH3] all the boundaries for the pair (S, L) have been described. They are S-spaces (with the natural action of S) $X = S/N_0 A$, where $N_0 = \exp \mathbf{n}_0$ and $\mathbf{n}_0$ is a homogeneous subalgebra of $\mathbf{n}$ containing $\mathbf{n}_0(L)$. Moreover, the Poisson kernel ν has smooth bounded density

$$d\nu(x) = P^X(x)dx.$$

Therefore by (1) and Theorem 1.1.1 for $f \in L^p(X), 1 \leq p \leq \infty$, the function

$$F(s) = \int_X f(sx)\, d\nu(x) \tag{2}$$

is L-harmonic on S. We call it the *Poisson integral* of f. Now we define convergence to the boundary X, which is modeled on the definition of admissible convergence in the case of symmetric spaces, but somewhat more general, as is natural in the context of NA groups. Let K be a compact subset of S and $yAK = \{yaz : \ a \in A, z \in K\}$ for a fixed $y \in N$.

DEFINITION 1.1.4. We say that s_n *tends admissibly to the boundary* X if $s_n \in yAK$ and

$$\lim_{n\to\infty} \langle \lambda, \log a(s_n) \rangle = -\infty \qquad \text{for every } \lambda \in \varDelta_1,$$

where $a(s) = a$ for $s = xa$ with $x \in N$ and $a \in A$.

Let $\mathbf{p}$ be the natural projection from S onto X,

$$\mathbf{p} : S \ni s \to se \in S/N_0 A = X,$$

where $e = N_0 A$. There is a natural question connected with the above definition. Does the Poisson integral $F(s_n)$ tend to its boundary value $f(\mathbf{p}(y))$ when s_n stays in yAK? The answer is affirmative and easy if f is e.g. a continuous function with compact support on X. Also the following theorem holds:

THEOREM 1.1.2 ([DH3]). *Let $f \in L^p(X)$ for some $p > 1$. For every y_0 in N_0 there is a subset X_{y_0} in X such that the Lebesgue measure of $X \setminus X_{y_0}$ is* 0, *and if $y = y_1 y_0$ and $\mathbf{p}(y) \in X_{y_0}$ then*

$$\lim_{\langle \lambda, \log a\rangle \to -\infty, \lambda \in \varDelta_1} \int_X f(y_1 y_0 azx) P^X(x)\, dx = f(\mathbf{p}(y))$$

uniformly in $z \in K$, for every compact subset K of S.

Of course one would like to eliminate the dependence of the exceptional set X_{y_0} on y_0. It would be sufficient to have the same X_{y_0} for y_0 in every compact set $K \subset N_0$. This in turn would follow if we could enlarge our approach region to

$$\bigcup_{\mathbf{p}(y)=x, y\in K_0} yAK = \Gamma_x$$

for two fixed compact sets $K \subset S$ and $K_0 \subset N_0$. This type of convergence with $s \in \Gamma_x$ and $\lim\langle\lambda, \log a(s)\rangle = -\infty$ is called strong admissible in [SW]. Though strong admissible convergence of the Poisson integral of a function f in $C_{\mathrm{c}}(X)$ to f does hold, it can fail on a set of positive measure for $f \in L^\infty(X)$. We show this in the next section when N is a Heisenberg group.

1.2. *A boundary of the Heisenberg group.* Let $N = \mathbb{R}^3$ be the Lie group with multiplication given by

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + xy').$$

Then $\mathbf{n} = \mathbb{R}^3$ with the bracket

$$[(x, y, z), (x', y', z')] = (0, 0, xy' - x'y)$$

is the Lie algebra of N. Let $S = NA$ be a semidirect product of N and A, where A acts on N by the automorphisms

$$\delta_a((x, y, z)) = (ax, ay, a^2z), \qquad a > 0.$$

The function

$$|(x, y, z)| = |x| + |y| + |z|^{1/2}$$

is a homogeneous norm on N with the property

$$|\delta_a(x, y, z)| = a|(x, y, z)|.$$

We denote by $\mathbf{s}$ the Lie algebra of S. Then $\mathbf{s}$ is a semidirect product of $\mathbf{n}$ and the Lie algebra $\mathbf{a}$ of A with the bracket

$$[(x, y, z, a), (x', y', z', a')] = (ax' - a'x, ay' - a'y, 2az' - 2a'z + xy' - x'y, 0).$$

Let $E_1, E_2, E_3 \in \mathbf{s}$ be the standard basis of the Heisenberg Lie algebra, i.e. $[E_1, E_2] = E_3$ and $E_0 = -\partial_a$, where ∂_a is differentiation on $\mathbb{R}$, and let

$$L = E_1^2 + E_2^2 + E_3^2 + E_0.$$

Then $N_0(L) = \{0\}$. Now $\mathbf{n}_0 = \{(x, 0, 0) : x \in \mathbb{R}\}$ is a homogeneous subalgebra of $\mathbf{n}$, and let

$$N_0 = \exp \mathbf{n}_0.$$

Let $X = S/N_0A = N/N_0$. In our case we may write $X = \{(x_2, x_3) : x_2, x_3 \in \mathbb{R}\}$. If $f \in L^\infty(X)$ then by (2) the function

$$F(s) = \int_X f(sx)P^X(x)\, dx \tag{3}$$

is L-harmonic on S. From the Harnack inequality it is easy to see that $P^X(x) \neq 0$ for $x \in X$, thus there exist constants $C > 0$ and $r > 0$ such that

$$P^X(x) \geq C\chi_{B(r)}(x) \tag{4}$$

for all $x \in X$, where $B(r) = \{(x_2, x_3) \in X : |x_2| + |x_3|^{1/2} < r\}$. Without loss of generality we may assume that $r = 1$. From (4) and (3) we have for $f \in L^\infty(\mathbb{R}^2)$, $f \geq 0$,

$$F(ya) = \int\limits_{\mathbb{R}^2} f(yax)P^X(x)\,dx \geq C \int\limits_{\mathbb{R}^2} f(yax)\chi_{B(1)}(x)\,dx.$$

Changing variables, we obtain

$$F(ya) \geq C\frac{1}{|B(a)|} \int\limits_{\mathbb{R}^2} f(yx)\chi_{B(a)}(x)\,dx,$$

where

$$yx = (x_2 + y_2, x_3 + y_3 + y_1x_2)$$

is the action of an element $y = (y_1, y_2, y_3) \in N$ on $x = (x_2, x_3) \in X$.

Let $R(a/2)$ be a rectangle with sides of length $a/2$ and $a^2/4$ centered at 0, such that $R(a/2) \subset B(a)$. Then $|B(a)| = (32/3)|R(a/2)|$. Thus we may replace $B(a)$ by $R(a/2)$, and take $f(x) = \chi_T(x)$ for a measurable set T. Then

$$\begin{aligned} F(ya) &\\ &\geq C\frac{1}{|R(a/2)|} \int\limits_{\mathbb{R}^2} \chi_T(x_2 + y_2, x_3 + y_3 + y_1x_2)\chi_{R(a/2)}(x_2, x_3)\,dx_2\,dx_3. \end{aligned} \tag{5}$$

We see that $(x_2+y_2, x_3+y_3+y_1x_2)$ in (5) means that we translate T by the vector $[-y_2, -y_3]$ and next *rotate*. From now on we assume that we *rotate* only in the range from $\pi/4$ to $\pi + \pi/4$, which means $|y_1| \leq 1$. By these operations we get $T^{y_1}_{(y_2,y_3)}$ and thus

$$F(ya) \geq C\frac{|T^{y_1}_{(y_2,y_3)} \cap R(a/2)|}{|R(a/2)|}.$$

Now instead of translating and *rotating* T we fix T and translate and rotate $R(a/2)$. This does not change anything. Hence, to prove that the Fatou theorem is not true in our case it is enough to take for T the set constructed in the proof of Theorem 1.3.2.

1.3. *Differentiation basis*

1.3.1. *Introduction*

DEFINITION 1.3.1. We say that $\mathcal{B} = \bigcup_{x\in R^2} \mathcal{B}(x)$ is a *differentiation basis* if for each $x \in \mathbb{R}^2$, $\mathcal{B}(x)$ is a collection of bounded measurable sets with positive measure containing x and such that there is at least one sequence $\{R_k\} \subset \mathcal{B}(x)$ with $\delta(R_k) \to 0$, where $\delta(\cdot)$ is diameter.

Let $\mathcal{B} = \bigcup_{x\in\mathbb{R}^2} \mathcal{B}(x)$ be a differentiation basis.

DEFINITION 1.3.2. We say that $\mathcal{B}$ is a *density basis* if for each measurable set A and for almost every $x \in \mathbb{R}^2$ we have

$$\lim_{k\to\infty} |A \cap R_k|/|R_k| = \chi_A(x), \tag{6}$$

where $\{R_k\}$ is any sequence of $\mathcal{B}(x)$'s contracting to x.

Now we define a differentiation basis in the following manner. Let

$$\mathcal{B} = \bigcup_{x\in\mathbb{R}^2} \mathcal{B}(x), \tag{7}$$

where $\mathcal{B}(x) = \{$all open rectangles containing x with sides of length a and a^2, where $0 < a < 1$ and the angle between the side of length a and the x-axis ranges from $\pi/4$ to $\pi/4+\pi\}$.

We will prove that $\mathcal{B}$ is not a density basis. It is enough to find measurable sets T and $M \subset T^{c}$, $|M| > 0$, such that for each $x \in M$ there exists a sequence $\{R_k\} \subset \mathcal{B}(x)$ contracting to x such that

$$|T \cap R_k|/|R_k| > C > 0,$$

where C is a constant independent of k. To do this we use the construction of a Perron tree based on I. J. Schoenberg's paper [Sch], to which we refer for more details.

1.3.2. *Construction of a Perron tree.* We define by induction a family of sets $\{T_n\}_{n\geq 2}$, each of which will be called a Perron tree.

If we have a triangle S of height 1 having its base on a line $y = k$ for some k, then we let S sprout by the following procedure: Extend the lateral sides of S upward beyond its vertex until they reach the line $y = k + 2$, next join the points from the line $y = k+2$ to the vertices of S lying on the line $y = k$.

Let T_2 be an isosceles triangle ABC of height 2 having its base AB on the line $y = 0$ and right-angled at C, and let $S_2^1 = T_2 \cap \{y \geq 1\}$. We let S_2^1 sprout and obtain a closed polygon which is called T_3. If we have T_n then the intersection $T_n \cap \{y \geq n-1\}$ is composed of 2^{n-2} triangles, denoted by $S_n^1, S_n^2, \ldots, S_n^{2^{n-2}}$ from right to left. If we let sprout $S_n^i, i = 1, \ldots, 2^{n-2}$, we obtain T_{n+1}.

1.3.3. *Some properties of the Perron tree.* Let T_n^i denote the triangle similar to S_n^i, obtained from S_n^i by extending its lateral sides down to the line $y = 0$, and let $\widehat{T_n^i}$ denote the triangle similar to T_n^i, obtained from T_n^i by extending its lateral sides down two times.

For our construction we need the following theorem and lemma.

THEOREM 1.3.1 ([Sch]). *Let PQV be a triangle similar to ABC, of the same height n as T_n, and having its base PQ on the line $y=0$. If we divide its base PQ into 2^{n-2} equal parts and dissect PQV accordingly into the triangles $\tau_1^n, \tau_2^n, \ldots, \tau_{2^{n-2}}^n$, then an appropriate horizontal translation will make T_i^n coincide with τ_i^n, $i=1,\ldots,2^{n-2}$. Moreover, at no stage of the construction of T_n is there an overlap between the 2^{n-2} triangles of new growth of T_n, so that T_n is bounded by a simple closed polygon, and*

$$|T_n| = 2n.$$

The proof of this theorem is in [Sch]. The following lemma and the idea of its proof come from [G].

LEMMA 1.3.1. *Let $P_n = \bigcup_{i=1}^{2^{n-2}} (\widehat{T_n^i} - T_n^i)$. Then*

$$|P_n| \geq n^2.$$

P r o o f. Extend the lateral sides of the triangle ABC (see construction of the Perron tree) down until they reach the line $y=-n$; the two points R and S so obtained together with A and B compose a trapezium $ABRS$. Since $|ABRS| = n^2+2n-4 \geq n^2$ for $n \geq 2$, it is enough to show that $ABRS \subset P_n$.

Let $x \in ABRS$, and for $i=1,\ldots,2^{n-2}$ let s_i be the vertex of S_n^i which lies on the line $y=n$, and β_i be the angle between the line passing through the points s_i, x and the x-axis. Then

$$\pi/4 \leq \beta_{2^{n-2}} < \beta_{2^{n-2}-1} < \ldots < \beta_2 < \beta_1 \leq \pi + \pi/4. \tag{8}$$

Let α_i, α_i' be the angles between the lateral sides of S_n^i and the x-axis. Then using Theorem 1.3.1 we have

$$\begin{aligned} \pi/4 = \alpha_1 < \alpha_1' = \alpha_2 < \alpha_2' = \alpha_3 < \ldots \\ < \alpha'_{2^{n-2}-1} = \alpha_{2^{n-2}} < \alpha'_{2^{n-2}} = \pi + \pi/4. \end{aligned} \tag{9}$$

Now it is very easy to see from (8) and (9) that there exists $1 \leq j \leq 2^{n-2}$ such that $\alpha_j \leq \beta_j \leq \alpha_j'$, but this exactly means that $x \in \widehat{T_n^j} - T_n^j$. ∎

We will use the following notation: T_n^* means that we shrink the Perron tree T_n in the ratio n to 1, thus T_n^* has height 1 and width $2(n-2)/n < 2$, T_n^{**} means that we shrink T_n^* in the ratio 2^n to 1. We do the same with P_n. By Theorem 1.3.1, $|T_n^*| = 2/n$; moreover, T_n^{i**} has height $1/2^n$ and base of length $8/2^{2n}$.

THEOREM 1.3.2. *The differentiation basis $\mathcal{B}$ defined by* (7) *is not a density basis.*

P r o o f. Let $A = [-1,1] \times [-1,1]$ and let $\{\varepsilon_n\}_{n\geq 1}$ be a decreasing sequence of positive numbers such that $\sum_{i=1}^{\infty} \varepsilon_i \leq 1/2$. Choose n_1 such that

$|T_{n_1}^*| \le \varepsilon_1$. Next divide A into $2^{n_1} \times 2^{n_1}$ equal squares. Into each such small square we put a copy of $T_{n_1}^{**} \cup P_{n_1}^{**}$. Let $\mathbf{T}_1$ be union of all the $T_{n_1}^{**}$'s in A and $\mathbf{P}_1$ the union of all the $P_{n_1}^{**}$'s in A. We do the same with $\varepsilon_2, \varepsilon_3, \ldots$ and obtain sequences $\{\mathbf{T}_k\}_{k\ge1}$ and $\{\mathbf{P}_k\}_{k\ge1}$ such that

$$|\mathbf{T}_k| \le \varepsilon_k \tag{10}$$

and

$$|\mathbf{P}_k| \ge 1 \tag{11}$$

for $k \ge 1$. Let

$$T = \bigcup_{k=1}^{\infty} \mathbf{T}_k, \qquad P = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \mathbf{P}_k.$$

Then in view of (10),

$$|T| = \Big|\bigcup_{k=1}^{\infty} \mathbf{T}_k\Big| \le \sum_{k=1}^{\infty} |\mathbf{T}_k| \le 1/2,$$

and by (11),

$$|P| \ge \liminf_{k\to\infty} |\mathbf{P}_k| \ge 1.$$

Let $M = P \setminus T$. Then

$$|M| = |P \setminus T| \ge 1 - 1/2 = 1/2.$$

Every $x \in M$ lies in infinitely many $\mathbf{P}_k$'s. Let $x \in \mathbf{P}_k$ for some k. Then there exists $1 \le i \le 2^{n_k-2}$ such that $x \in \widehat{T_{n_k}^{i**}} - T_{n_k}^{i**}$. We take a rectangle $B_k \in \mathcal{B}(x)$ with sides of length $4/2^{n_k}$ and $16/2^{2n_k}$ such that $B_k \supset \widehat{T_{n_k}^{i**}}$. Then $T \cap B_k \supset T_{n_k}^{i**}$ and

$$|T \cap B_k| \ge |T_{n_k}^{i**}| = 4/2^{3n_k}, \qquad |B_k| = 64/2^{3n_k}.$$

This implies

$$|T \cap B_k|/|B_k| \ge 1/16. \tag{12}$$

Since x lies in infinitely many $\mathbf{P}_k$'s, there exists a sequence $\{B_k\}_{k\ge1} \subset \mathcal{B}(x)$ such that (12) is valid for all B_k. On the other hand, $\chi_T(x) = 0$. Of course B_k is contracting to x. This means that condition (6) fails on the set M of positive measure and consequently $\mathcal{B}$ is not a density basis. ∎

2. Estimation

2.1. In this part we restrict our attention to the group $S = NA$ with A being one-dimensional. Then the action δ_a of A on $\mathbf{n}$ becomes

$$\delta_a X_j = e^{\lambda_j(\log a)} X_j = a^{d_j} X_j$$

for some numbers $1 \le d_1 \le \ldots \le d_n$. The number $Q = \sum_{j=1}^n d_j$ is called the *homogeneous dimension* of $\mathbf{n}$. The formula

$$\exp \circ \delta_a \circ \exp^{-1}$$

defines dilations on N, i.e. N becomes a homogeneous group [FS]. The dilations on N will also be denoted by δ_a. Denote by $\|\cdot\|$ the Euclidean norm in $\mathbf{n}$, and define the corresponding left-invariant distance

$$\tau(x) = \inf \int_0^1 \|\dot{\gamma}(t)\|\, dt,$$

where the infimum is taken over all C^1 curves γ in N such that $\gamma(0) = e$ and $\gamma(1) = x$. Then $B_r(x) = \{y \in N : \tau(x^{-1}y) < r\}$ is the ball of radius r centered at x. We use $B(r)$ instead of $B_r(e)$. For a function $f \in C_{\mathrm{b}}(N)$ we write $\|f\| = \sup\{|f(x)| : x \in N\}$.

Let $f \in C_{\mathrm{c}}^\infty(B(1))$ be a nonnegative function such that $\int_N f(x)\, dx = 1$, and $\tau * f(x) = \int \tau(xy^{-1}) f(y)\, dy$. Then for any left-invariant vector field X we have (see [H1])

$$|X^2(\tau * f)(x)| \le \int |Xf(y)| \|\mathrm{Ad}_y X\|\, dy \tag{13}$$

for all $x \in N$.

2.2. *A maximum principle.* On the group S we will investigate operators of the form

$$\mathcal{L} = \sum_{j=1}^k a^{2d_j} X_j^2 + a^2 \partial_a^2 - \kappa a \partial_a,$$

where $\kappa > -1$ is a constant, $X_j \in \mathbf{n}$ for $j = 1, \ldots, k$, and ∂_a is differentiation along $\mathbb{R}$.

We will need the following maximum principle for functions defined on $S_a = \{xb : x \in N,\ b > a\}$. Let $D(a,b,r) = \{xc : x \in B_r(e),\ a < c < b\}$ and let $C = \sum_{j=1}^k \|X_j^2(\tau * f)\|$. Then by (13), $C < \infty$.

Theorem 2.2.1 ([DH2]). *Let $0 < \varepsilon < 1$, $0 < a_0 < a_1$ and let σ, R be constants satisfying*

$$0 < \sigma \le 2d_1, \qquad \kappa + 1 - \sigma > 0,$$
$$R > \max\left\{ \frac{C}{\sigma(\kappa + 1 - \sigma)} \max(a_1^{2d_1}, a_1^{2d_k}), 2 \right\}.$$

Suppose that F is a twice continuously differentiable function on $D = D(a_0, a_1(2/\varepsilon)^{1/\sigma}, R\varepsilon^{-2d_k/\sigma})$, $\mathcal{L}F \ge 0$ in D, F is continuous in $\overline{D}$ and $|F| < 1$. If $F(xa_0) \le 0$ for $x \in B(R\varepsilon^{-2d_k/\sigma})$ then $F(a_1) \le \varepsilon$.

COROLLARY 2.2.1. *For every $\varepsilon > 0$ and $b > 0$ there exists $R = R(\varepsilon, b) > 0$ such that for all $a < b$,*

$$F(b) \le \max_{x \in B(R)} F(xa) + \varepsilon$$

for all $F \in C^2(S_a) \cap C(\overline{S}_a)$ with $\mathcal{L}F \ge 0$ and $|F| \le 1$.

COROLLARY 2.2.2 (Maximum principle). *If $F \in C^2(S_a) \cap C(\overline{S}_a)$, $\mathcal{L}F \ge 0$ and F is bounded in S_a, then for every $b > a$ and $y \in N$,*

$$F(yb) \le \sup_{x \in N} F(xa).$$

2.3. *Properties of harmonic measures.* The Dirichlet problem in S_a has a solution, i.e. the following theorem is true.

THEOREM 2.3.1 ([DH2]). *For every bounded continuous function f on N there exists a bounded harmonic function F on S_a which is continuous on $\overline{S}_a$ and such that $F(xa) = f(x)$ for $x \in N$.*

By the above theorem and the maximum principle we have

Remark 2.3.1. There exists exactly one function F which satisfies the conditions of Theorem 2.3.1.

For each $s \in S_a$ and $f \in C_{\mathrm{b}}(N)$ let

$$m_s(f) = F(s),$$

where F is the function of Theorem 2.3.1. Them m_s is a well defined linear functional with norm 1. Moreover, if $f \ge 0$ then $m_s(f) \ge 0$, i.e. m_s is a positive functional. If we restrict the domain of m_s to $C_{\mathrm{c}}(N)$ then $\|m_s\|_{C_c(N)} = 1$. Indeed, in view of Corollary 2.2.1 for every $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that

$$F(s) \le \max_{x \in B(R_\varepsilon)} f(x) + \varepsilon.$$

Take $f_\varepsilon \in C_{\mathrm{c}}(N)$ such that $f_\varepsilon(x) = -1$ for $x \in B(R_\varepsilon)$ and $f_\varepsilon(x) = 0$ for $x \in B^{\mathrm{c}}(R_\varepsilon + 1)$. Then $|F(s)| \ge 1 - \varepsilon$.

By the Riesz theorem there exists a probability measure μ_a^s on N such that

$$m_s(f) = \int_N f(y)\, \mu_a^s(dy).$$

Thus

$$F(xb) = \int_N f(y)\, \mu_a^{x,b}(dy) \quad \text{for } x \in N,\ b > a.$$

Since $\mathcal{L}$ is left-invariant,

$$F(xb) = \int_N f(xy)\, \mu_a^{e,b}(dy). \tag{14}$$

We write μ_a^b instead of $\mu_a^{e,b}$, and let $\mathcal{H}_a$ be the set all functions harmonic in S_a and continuous in $\overline{S}_a$.

For every $0 < a < b < c$,

$$\check{\mu}_a^c = \check{\mu}_a^b * \check{\mu}_b^c;$$

this is an easy consequence of Remark 2.3.1 and (14).

LEMMA 2.3.1. *Let $b > 0$ and $R = R(\varepsilon, b)$ be as in Corollary* 2.2.1. *Then for every $a < b$,*

$$\mu_a^b(B^c(2R)) \le \varepsilon.$$

Proof. Let $f \in C_b(N)$ be a function such that $0 \le f \le 1$, $f(x) = 0$ for $x \in B(R)$ and $f(x) = 1$ for $x \in B^c(2R)$. If $F \in \mathcal{H}_a$ and $F(xa) = f(x)$ then

$$F(b) = \int_N f(y)\,\mu_a^b(dy) \le \mu_a^b(B^c(2R))$$

and thus $\mu_a^b(B^c(2R)) \le \varepsilon$. ∎

Notice that if we take arbitrary $b > a > 0$ and $1 > \varepsilon > 0$, and σ, R as in Theorem 2.2.1, then in view of Lemma 2.3.1 and Corollary 2.2.1,

$$\mu_a^b(B^c(2R\varepsilon^{-2d_k/\sigma})) \le \varepsilon,$$

hence

$$\mu_a^b(B^c(r)) \le (2R)^{\sigma/(2d_k)} r^{-\sigma/(2d_k)}.$$

Since the right hand side of the above inequality does not depend on a, the family of measures $\{\mu_a^b\}_{0<a<b}$ is tight.

Now we will show that $\{\check{\mu}_a^b\}_{b>a}$ is an approximate identity as $b \to a$, where $d\check{\mu}(x) = d\mu(x^{-1})$. Let Φ be a *Hunt function* on N, i.e. Φ, $X_j\Phi$ and $X_iX_j\Phi$ are bounded, $\Phi(x) > 0$ for $x \ne e$ and $\Phi(e) = 0$ (cf. e.g. [H]).

THEOREM 2.3.2. *Fix $M > 0$ and $f \in C_0(N)$. For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < a < b < M$ and $b - a < \delta$, then*

$$\|f * \check{\mu}_a^b - f\|_{C_0} < \varepsilon.$$

Proof. Let γ, γ_0, R and α be constants satisfying

$$0 < \gamma < 1, \quad 0 < \gamma_0 < 1, \quad \kappa + 1 - \gamma - \gamma_0 > 0, \quad \gamma + \gamma_0 < 1,$$
$$R > \sum_{j=1}^k \|X_j^2\Phi\|, \quad \alpha > (\gamma\gamma_0)^{-1}\max(M^{2d_k-\gamma}, 1).$$

Then the function $G(xb) = -\alpha(b-a)^\gamma - R^{-1}\Phi(x)$ is subharmonic in $N \times (a, M)$. Indeed,

$$\begin{aligned}(b^2\partial_b^2 - \kappa b\partial_b)G(xb) &= \alpha\gamma b(b-a)^{\gamma-2}(\kappa(b-a) + (1-\gamma)) \\ &\ge \alpha\gamma\gamma_0 b^2(b-a)^{\gamma-2}\end{aligned}$$

and

$$\Big|\sum_{j=1}^{k} b^{2d_j} X_j^2(R^{-1}\Phi(x))\Big| \le R^{-1}\max(b^{2d_1}, b^{2d_k})\sum_{j=1}^{k}\|X_j^2\Phi\|$$
$$\le \max(b^{2d_1}, b^{2d_k}).$$

Thus

$$\mathcal{L}G(xb) \ge \alpha\gamma\gamma_0 b^2(b-a)^{\gamma-2} - \max(b^{2d_1}, b^{2d_k}).$$

If $b < 1$ then $(b-a)^{\gamma-2} > 1$ and $b^{2d_1-2} \le 1$ while if $b \ge 1$ then $\alpha\gamma\gamma_0 \ge b^{2d_k}(b-a)^{2-\gamma}$ for $b < M$. This proves that $\mathcal{L}G \ge 0$ in $N \times (a, M)$.

Let

$$F(xb) = R^{-1}\int_N \Phi(xy^{-1})\,\check{\mu}_a^b(dy), \qquad F(xa) = R^{-1}\Phi(x).$$

Then $\mathcal{L}(G+F) \ge 0$ in $N \times (a, M)$ and $(G+F)(xa) = 0$ for $x \in N$. In view of the maximum principle,

$$\int_N \Phi(y^{-1})\,\check{\mu}_a^b(dy) \le R\alpha(b-a)^\gamma \quad \text{for } 0 < a < b < M. \tag{15}$$

Since $\Phi(x) > 0$ for $x \ne e$, (15) implies that for every $\varepsilon > 0$ and every neighborhood U of the identity there is $\delta > 0$ such that

$$\check{\mu}_a^b(U) \ge 1 - \varepsilon \quad \text{for } b - a < \delta,\ 0 < a < b < M.$$

The rest of the proof is trivial. ∎

THEOREM 2.3.3. *There is a probability measure μ^b on N such that*

$$\mu_a^b \Rightarrow \mu^b \quad \textit{as } a \to 0.$$

Proof. Since the family $\{\mu_a^b\}_{0<a<b}$ is tight, it is enough to show that if $\mu_{a_n}^b \Rightarrow \mu_1^b$ and $\mu_{b_n}^b \Rightarrow \mu_2^b$, where $\{a_n\}$ and $\{b_n\}$ are arbitrary sequences tending to 0, then $\mu_1^b = \mu_2^b$. Without lost of generality we may assume that $a_n > b_n$ for $n = 1, 2, \ldots$ We have

$$\check{\mu}_{b_n}^b = \check{\mu}_{b_n}^{a_n} * \check{\mu}_{a_n}^b,$$

which in view of Theorem 2.3.2 shows that $\check{\mu}_{b_n}^b \Rightarrow \check{\mu}_1^b$. ∎

2.4. *Pointwise estimates for the Poisson kernel.* In this section we assume that $d_1 = \ldots = d_k = 1$. Let $p_t(x)dx$ be the semigroup of measures generated by $L = X_1^2 + \ldots + X_k^2 - \partial_t$. Then p_1 is a bounded function and

$$p_t(x) = t^{-Q/2} p_1(\delta_{t^{-1/2}}(x)) \tag{16}$$

(cf. e.g. [FS]). Moreover, in [H] it is proved that there is $c > 0$ such that

$$\int_N p_t(x)\exp(s\tau(x))\,dx \le \exp(tc(s+s^2)) \tag{17}$$

for all $s > 0$. In fact, (17) gives a better integral estimate, which is included in the following lemma.

LEMMA 2.4.1. *Let $c > 0$ be a constant such that*

$$\int_N p_1(x)\exp(s\tau(x))\,dx \le \exp(c(s+s^2)).$$

Then for $0 < \varepsilon < 1/(4c)$,

$$\int_N p_1(x)\exp(\varepsilon\tau^2(x))\,dx < \infty. \tag{18}$$

Proof. By assumption,

$$\int_N p_1(x)\exp(s\tau(x) - cs^2 - (c+1)s)\,dx \le \exp(-s).$$

Integrating both sides with respect to s, we obtain

$$\int_0^\infty\int_N p_1 \exp(s(\tau(x)-c-1) - cs^2)\,dx\,ds \le 1. \tag{19}$$

Since

$$-cs^2 + (\tau(x)-c-1)s = -\left(sc^{1/2} - \frac{1}{2c^{1/2}}(\tau(x)-c-1)\right)^2 + \frac{1}{4c}(\tau(x)-c-1)^2,$$

the left hand side of (19) is equivalent to

$$\int_N p_1(x)\exp\left(\frac{1}{4c}(\tau(x)-c-1)^2\right) \times \int_0^\infty \exp\left(-\left(sc^{1/2} - \frac{1}{2c^{1/2}}(\tau(x)-c-1)\right)^2\right)ds\,dx.$$

Now we calculate the inner integral to get

$$\int_{\{x:\tau>c+1\}} p_1(x)\exp(\varepsilon\tau(x)^2) \times \exp\left[\left(\frac{1}{4c}-\varepsilon\right)\tau(x)^2 - \frac{c+1}{2c}\tau(x) + \frac{(c+1)^2}{4c}\right]dx \le c'. \tag{20}$$

Since the expression in square brackets, as a function of $\tau(x)$, has a minimum when $0 < \varepsilon < 1/(4c)$, we put this value to (20) and we obtain (18). ■

LEMMA 2.4.2. *Let c be the constant in Lemma* 2.4.1. *Then for* $0 \le \varepsilon \le 1/(4c)$,

$$p_1(x) \le c' \exp\bigg(-\frac{\varepsilon}{4}\tau(x)^2\bigg).$$

P r o o f. In view of the semigroup property,

$$\begin{aligned} p_1(x)\exp\left(\frac{\varepsilon}{4}\tau(x)^2\right) &= \exp\left(\frac{\varepsilon}{4}\tau(x)^2\right)p_{1/2} * p_{1/2}(x) \\ &= \int \exp\left(\frac{\varepsilon}{4}\tau(x)^2\right)p_{1/2}(xy^{-1})p_{1/2}(y)\,dy. \end{aligned}$$

Notice that $\tau(x)^2 \le 2\tau(xy^{-1})^2 + 2\tau(y)^2$. Thus

$$\begin{aligned} &p_1(x)\exp\left(\frac{\varepsilon}{4}\tau(x)^2\right) \\ &\quad\le \int \exp(\varepsilon/(2\tau(xy^{-1})^2))p_{1/2}(xy^{-1})\exp(\varepsilon/(2\tau(y)^2))p_{1/2}(y)\,dy \\ &\quad\le \Big\{\int \exp(\varepsilon\tau(xy^{-1})^2)p_{1/2}(xy^{-1})^2\,dy\Big\}^{1/2}\Big\{\int \exp(\varepsilon\tau(y)^2)p_{1/2}(y)^2\,dy\Big\}^{1/2} \\ &\quad\le \|p_{1/2}\|_{L^\infty(N)}\int \exp(\varepsilon\tau(y)^2)p_{1/2}(y)\,dy. \ \blacksquare \end{aligned}$$

The following theorem comes from [Br] (where it is proved for more general operators defined on $\mathbb{R}^n \times \mathbb{R}_+$).

THEOREM 2.4.1 ([Br]). *Let $w(x,t)$ be a bounded function on NA such that*

$$\Big(\sum_{j=1}^k X_j^2 - \partial_t\Big)w(x,t) = 0 \quad \text{for } x \in N,\ t > 0,$$

on the boundary $w(x,0) = \phi(x)$, and

$$u(x,t) = \frac{1}{\Gamma(1/(2\kappa)+1/2)}\int_0^\infty e^{-\sigma}\sigma^{1/(2\kappa)-1/2}w(x,t^2/(4\sigma))\,d\sigma$$

where $\kappa > -1$. Then the function $u(x,t)$ satisfies the equation

$$\bigg(\sum_{j=1}^k X_j^2 + \partial_t^2 - \frac{\kappa}{t}\partial_t\bigg)u(x,t) = 0 \quad \text{for } x \in N,\ t > 0,$$

and $u(x,0) = \phi(x)$.

Let $P_t(x)\,dx = \mu^t(x)$. Theorem 2.4.1 implies

$$P_t(x) = \frac{1}{\Gamma(1/(2\kappa)+1/2)}\int_0^\infty e^{-\sigma}\sigma^{1/(2\kappa)-1/2}p_{t^2/(4\sigma)}(x)\,d\sigma.$$

Using (16) and our estimation for p_t included in Lemma 2.4.2 we have

$$P_t(x) \le c' \int_0^\infty e^{-\sigma} \sigma^{1/(2\kappa)-1/2} t^{-Q} (4\sigma)^{Q/2} e^{-4\varepsilon\tau(x)^2\sigma/t^2}\, d\sigma.$$

This yields

$$P_t(x) \le c' \frac{t^{\kappa+1}}{(t^2 + 4\varepsilon\tau(x)^2)^{(Q+\kappa+1)/2}}$$

for sufficiently small ε.

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INSTITUTE OF MATHEMATICS
WROCŁAW UNIVERSITY
PL. GRUNWALDZKI 2/4
50-384 WROCŁAW, POLAND
E-mail: SOLOWIEJ@MATH.UNI.WROC.PL

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