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SMALL SETS WITH RESPECT TO CERTAIN CLASSES OF TOPOLOGIES

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Let E be a non-empty set and let T be a topology on E. We denote by the symbol $K(T)$ the σ-ideal of all first category sets in E with respect to the topology T. We denote by the symbol $B(T)$ the family of all subsets of E which have the Baire property with respect to the topology T.

Let Γ be some group of transformations of the set E. Let $\mathcal{T}(E,\Gamma)$ be the class of all topologies T on E which satisfy the following conditions:

1) T is a Baire space topology;
2) the Suslin number $c(T)$ is equal to ω;
3) the group Γ preserves the category and the Baire property, i.e.

$$(\forall g \in \Gamma)(\forall X \in K(T))(g(X) \in K(T)),$$
$$(\forall g \in \Gamma)(\forall X \in B(T))(g(X) \in B(T)).$$

EXAMPLE 1. Let (E,T) be a non-empty Baire topological space with $c(T) = \omega$ and let Γ be some group of homeomorphisms from E onto E. Then it is clear that $T \in \mathcal{T}(E,\Gamma)$.

Let E be a non-empty set, let Γ be some group of transformations of E and let X be a subset of E. We say that X is a *first category set with respect to the class* $\mathcal{T}(E,\Gamma)$ if

$$(\forall T \in \mathcal{T}(E,\Gamma))(\exists T' \in \mathcal{T}(E,\Gamma))(T \subseteq T' \ \&\ K(T) \subseteq K(T') \ \&\ X \in K(T')).$$

We shall denote by the symbol $K(\mathcal{T}(E,\Gamma))$ the family of all subsets of E which are first category sets with respect to the class $\mathcal{T}(E,\Gamma)$.

The following proposition is an immediate corollary from the definition of the family $K(\mathcal{T}(E,\Gamma))$.

PROPOSITION 1. *$K(\mathcal{T}(E,\Gamma))$ is a proper ideal in the power set Boolean algebra of E.*

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Further we shall consider some situations when this ideal is not σ-additive. Moreover, we shall see that in some natural situations there exists a countable family of sets from $K(\mathcal{T}(E,\Gamma))$ which is a covering of the set E.

Obviously, the following question arises: how can one characterize the sets from the ideal $K(\mathcal{T}(E,\Gamma))$ only in terms of the pair (E,Γ)? The next proposition gives one of possible characterizations of sets belonging to the ideal $K(\mathcal{T}(E,\Gamma))$.

PROPOSITION 2. *Let E be a non-empty set, let Γ be some group of transformations of E and let X be a subset of E. Then the following two sentences are equivalent*:

1) *the set X belongs to the ideal $K(\mathcal{T}(E,\Gamma))$*;

2) *for any countable family $(g_i)_{i\in I}$ of elements of Γ there exists a countable family $(h_j)_{j\in J}$ of elements of Γ such that*

$$\bigcap_{j\in J} h_j\Big(\bigcup_{i\in I} g_i(X)\Big) = \emptyset.$$

The proof of Proposition 2 is based on the following three simple lemmas.

LEMMA 1. *Let E be a non-empty set and let Γ be a group of transformations of E. Let A be a countable subset of E. Then the next two sentences are equivalent*:

1) *the set A belongs to the ideal $K(\mathcal{T}(E,\Gamma))$*;

2) *for any countable orbit $\Gamma(x)$ $(x\in E)$ the equality $\Gamma(x)\cap A=\emptyset$ holds.*

Proof. The implication 1)⇒2) is obvious. So we only need to prove the implication 2)⇒1). Suppose that a given countable set $A\subseteq E$ satisfies relation 2). Let $(g_i)_{i\in I}$ be an arbitrary countable family of elements from the group Γ. Since A is countable the family of all uncountable Γ-orbits which intersect A is at most countable. Using this fact we can find a countable family $(h_j)_{j\in J}$ of elements from Γ such that

$$\bigcap_{j\in J} h_j\Big(\bigcup_{i\in I} g_i(A)\Big) = \emptyset.$$

The last equality implies relation 1) for the set $A\subseteq E$ (compare with the proof of Lemma 3).

LEMMA 2. *Let E be a set and let Γ be a group of transformations of E. Let T be an arbitrary topology on E with $c(T)=\omega$ and suppose that a set $X\subseteq E$ has the Baire property with respect to the topology T. If the group Γ preserves $K(T)$ and $B(T)$ then there exists a countable family $(g_i)_{i\in I}$ of* *elements from Γ such that the set*

$$X^* = \bigcup_{i\in I} g_i(X)$$

is almost invariant under Γ, i.e.

$$(\forall g \in \Gamma)(g(X^*) \triangle X^* \text{ is in } K(T)).$$

Proof. Suppose that for every countable family $(g_i)_{i\in I}$ of elements from Γ the mentioned set X^* is not almost invariant under Γ. Then, using the method of transfinite recursion, we can construct an ω_1-sequence $(X_\xi)_{\xi<\omega_1}$ of subsets of E satisfying the following conditions:

a) $X_0 = X$;

b) the family $(X_\xi)_{\xi<\omega_1}$ is increasing with respect to inclusion;

c) for any $\xi < \omega_1$ the set X_ξ can be represented in the form

$$X_\xi = \bigcup_{j\in J} h_j(X),$$

where $(h_j)_{j\in J}$ is a countable family of elements from Γ (this family, of course, depends on ξ);

d) for any $\xi < \omega_1$ the set $X_{\xi+1} \setminus X_\xi$ is not first category with respect to the topology T.

As soon as the family of sets $(X_\xi)_{\xi<\omega_1}$ is constructed, we see that the disjoint family of sets $(X_{\xi+1} \setminus X_\xi)_{\xi<\omega_1}$ contradicts the equality $c(T) = \omega$.

LEMMA 3. *Let E be a non-empty set and let Γ be some group of transformations of E. Let a set $X \subseteq E$ satisfy relation* 2) *of Proposition* 2 *and let T be an arbitrary topology from the class $\mathcal{T}(E,\Gamma)$. Denote by $RO(T)$ the family of all regular open sets with respect to T and denote by L the smallest σ-ideal which contains the family of sets $K(T) \cup \{X\}$ and is invariant under the group Γ. Then the family*

$$T' = \{V \setminus Y : V \in RO(T) \ \&\ Y \in L\}$$

is a topology on E for which the following relations hold:

1) *T' belongs to the class $\mathcal{T}(E,\Gamma)$ and extends the topology T*;

2) $K(T') = L$;

3) $K(T) \subseteq K(T')$;

4) *X belongs to the σ-ideal $K(T')$.*

Proof. It can be directly checked that T' is a topology on the set E and extends the original topology T (cf. [1]). Also it is easy to see that relation 1) holds if no non-empty set $V \in T$ belongs to the ideal L. Suppose for a while that $V \in L$. Using Lemma 2 we can find a countable family $(g_i)_{i\in I}$ of elements from Γ such that the set

$$V^* = \bigcup_{i \in I} g_i(V)$$

is almost invariant under Γ. Of course, we also have $V^* \in L$. Since X satisfies relation 2) of Proposition 2 there exists a countable family $(h_j)_{j \in J}$ of elements from Γ such that

$$\bigcap_{j \in J} h_j(V^*) \in K(T).$$

But the last formula cannot be true because the set V^* is almost invariant and does not belong to the ideal $K(T)$. So we obtain a contradiction and thus relation 1) holds. Analogously, relations 2), 3) and 4) may be checked.

Using these three lemmas it is not difficult to prove Proposition 2. The argument of this proof is the same as in [3] or [4]. In fact, the main part of Proposition 2 is contained in Lemma 3.

From Proposition 2 we immediately obtain the following

PROPOSITION 3. *Let E be a non-empty set and let Γ be a countable group of transformations of E. Then $K(\mathcal{T}(E, \Gamma)) = \{\emptyset\}$.*

Proposition 2 sometimes yields nice geometrical properties of certain sets belonging to the ideal $K(\mathcal{T}(E, \Gamma))$. For instance, we have the following

PROPOSITION 4. *Let E be a normed vector space and let Γ be a non-separable subgroup of the additive group of E. Then every ball in E belongs to the ideal $K(\mathcal{T}(E, \Gamma))$.*

Proof. Take any ball B in the space E and consider an arbitrary countable family $(g_i)_{i \in I}$ of elements from Γ. Denote by G the closed vector subspace of E generated by the family $(g_i)_{i \in I}$. The space G is separable, hence there exists an element $g \in \Gamma \setminus G$. Without loss of generality we may assume that the center of the ball B coincides with the zero element of the additive group of E. Let κ be a natural number such that

$$\mathrm{dist}(\kappa g, G) > \mathrm{diam}(B).$$

Let us put $h = \kappa g$. Then it is easy to see that $h \in \Gamma$ and

$$(h + (G + B)) \cap (G + B) = \emptyset.$$

This finishes the proof of Proposition 4.

In fact, we have proved that if E is a normed vector space and Γ is a non-separable subgroup of the additive group of E then the ideal $K(\mathcal{T}(E, \Gamma))$ contains the ideal of all bounded subsets of E.

Proposition 5. *If E is a normed vector space* (*with norm* $\|\cdot\|$) *and Γ is a non-separable subgroup of the additive group of E then there exists a countable covering of E by sets belonging to the ideal $K(\mathcal{T}(E,\Gamma))$.*

Proof. For any natural number $n \geq 1$ let us put $B_n = \{x \in E : \|x\| \leq n\}$. Then, by Proposition 4, every ball B_n is a first category set with respect to the class $\mathcal{T}(E,\Gamma)$. But it is obvious that $\bigcup_{n\geq 1} B_n = E$, so we have found a countable covering of E by sets belonging to the ideal $K(\mathcal{T}(E,\Gamma))$.

Lemma 4. *Let $\mathbb{R}$ be the real line and let Γ be any uncountable subgroup of $\mathbb{R}$. Let $l^2(\omega_1)$ be the Hilbert space with Hilbert dimension ω_1. Consider $\mathbb{R}$ and $l^2(\omega_1)$ as abstract groups. Then there exists an isomorphism $\varphi : \mathbb{R} \to l^2(\omega_1)$ such that $\varphi(\Gamma)$ is an everywhere dense subgroup of $l^2(\omega_1)$.*

Proof. The topological weight of the space $l^2(\omega_1)$ is equal to ω_1. Hence there exists a family $(e_\xi)_{\xi<\omega_1}$ of elements from $l^2(\omega_1)$ such that

a) $(e_\xi)_{\xi<\omega_1}$ is everywhere dense in the space $l^2(\omega_1)$;
b) $(e_\xi)_{\xi<\omega_1}$ is independent over the field $\mathbb{Q}$ of all rational numbers;
c) if E is the vector space (over $\mathbb{Q}$) generated by the family $(e_\xi)_{\xi<\omega_1}$ then the algebraic codimension of E in $l^2(\omega_1)$ is equal to the cardinality continuum, i.e. the algebraic dimension of the quotient vector space $l^2(\omega_1)/E$ is equal to the cardinality continuum.

Notice that a family $(e_\xi)_{\xi<\omega_1}$ with the above properties can be easily constructed by the method of transfinite recursion.

Now, using the fact that $\mathrm{card}(\Gamma) \geq \omega_1$ we can also construct by transfinite recursion a family $(g_\xi)_{\xi<\omega_1}$ of elements from Γ which are also independent over $\mathbb{Q}$. Let Γ^* be the vector space (over $\mathbb{Q}$) generated by the family $(g_\xi)_{\xi<\omega_1}$. Without loss of generality we may assume that the algebraic codimension of the vector space Γ^* in $\mathbb{R}$ is also equal to the cardinality continuum. Let

$$\psi : (g_\xi)_{\xi<\omega_1} \to (e_\xi)_{\xi<\omega_1}$$

be a bijection such that $\psi(g_\xi) = e_\xi$ ($\xi < \omega_1$). Then it is easy to see that the mapping ψ can be extended to the required isomorphism φ between the abstract groups $\mathbb{R}$ and $l^2(\omega_1)$.

Starting from Lemma 4 it is not difficult to prove the following

Proposition 6. *Let Γ be an arbitrary uncountable subgroup of the additive group of $\mathbb{R}$. Then there exists a countable family $(X_n)_{n\geq 1}$ of subsets of $\mathbb{R}$ such that*

1) $(\forall n \geq 1)$(*the set X_n belongs to the ideal $K(\mathcal{T}(\mathbb{R},\Gamma))$*);
2) $\mathbb{R} = \bigcup_{n\geq 1} X_n$.

Proof. Let φ be an isomorphism between abstract groups $\mathbb{R}$ and $l^2(\omega_1)$ such that the set $\varphi(\Gamma)$ is everywhere dense in the space $l^2(\omega_1)$. Then it is clear that the group $\varphi(\Gamma)$ is non-separable. So we can apply Proposition 5. Let $(B_n)_{n\geq 1}$ be a countable covering of the space $l^2(\omega_1)$ by sets belonging to the ideal $K\left(\mathcal{T}(l^2(\omega_1),\ \varphi(\Gamma))\right)$. Let us put

$$X_n = \varphi^{-1}(B_n) \quad (n \geq 1).$$

Then it is easy to check that the family $(X_n)_{n\geq 1}$ satisfies conditions 1) and 2) of Proposition 6.

We remark here that a quite different proof of Proposition 6 may be obtained by the method used in [2].

EXAMPLE 2. Let E be a non-empty set and Γ be a group of transformations of E. Let μ be a non-zero complete σ-finite Γ-quasi-invariant measure defined on some σ-algebra of subsets of E. We denote by $T(\mu)$ von Neumann's topology associated with μ. Recall that (see e.g. [1]) the topology $T(\mu)$ satisfies the following conditions:

a) $B(T(\mu)) = \mathrm{dom}(\mu)$;
b) $(\forall X \subseteq E)(X \in K(T(\mu)) \Leftrightarrow \mu(X) = 0)$;
c) $T(\mu)$ is a Baire space topology and $c(T(\mu)) = \omega$.

Moreover, since the measure μ is Γ-quasi-invariant, we conclude that the group Γ preserves both families of sets $B(T(\mu))$ and $K(T(\mu))$. Hence, we have

$$T(\mu) \in \mathcal{T}(E, \Gamma).$$

If there exists a countable covering of E by sets belonging to the ideal $K(\mathcal{T}(E, \Gamma))$, then we can construct a measure $\bar{\mu}$ with the following properties:

1) $\bar{\mu}$ is a proper extension of μ;
2) $\bar{\mu}$ is a complete Γ-quasi-invariant measure;
3) $T(\mu) \subseteq T(\bar{\mu})$.

For details of construction of such an extension $\bar{\mu}$ of the initial measure μ see, for instance, [3].

Using the arguments from [3] or [4] one can also prove the following

PROPOSITION 7. *Let E be a set and let Γ be some group of transformations of E such that*

1) $\mathrm{card}(\Gamma) = \omega_1$;
2) *Γ acts freely on E, i.e. for any $g \in \Gamma$ we have*

$$(\exists x \in E)(g(x) = x) \Rightarrow (g = \mathrm{id}_E).$$

Then there exists a countable covering of E by sets belonging to the ideal $K(\mathcal{T}(E, \Gamma))$. In particular, every topology from the class $\mathcal{T}(E, \Gamma)$ has a proper extension in the same class.

The following problem remains open.

PROBLEM. Let E be a non-empty set and let Γ be some group of transformations of E. In terms of the pair (E, Γ) give a necessary and sufficient condition for the existence of a countable covering of E by sets belonging to the ideal $K(\mathcal{T}(E, \Gamma))$.

Finally, let us notice that if we have two groups Γ_1 and Γ_2 of transformations of a non-empty set E, then, in general, the inclusions $\Gamma_1 \subseteq \Gamma_2$ or $\Gamma_2 \subseteq \Gamma_1$ do not imply any inclusion relation between the ideals $K(\mathcal{T}(E, \Gamma_1))$ and $K(\mathcal{T}(E, \Gamma_2))$ (in this connection see [3]).

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