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CONVERGENCE TO EQUILIBRIUM AND LOGARITHMIC SOBOLEV CONSTANT ON MANIFOLDS WITH RICCI CURVATURE BOUNDED BELOW

BY

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1. Introduction. Let (M,g) be a complete Riemannian manifold having finite volume $v(M)=V$, where dv denotes the Riemannian measure. Let ∇ be the Riemannian gradient. Consider the Laplace operator $\Delta=-\operatorname{div}\nabla$ on M. Let $H_t=e^{-t\Delta}$ be the corresponding heat diffusion semigroup. It will be convenient here to consider the normalized measure $d\mu=(1/V)\,dv$ and to let H_t act on $L^p=L^p(M,\mu)$, $1\le p<\infty$. The semigroup H_t admits a kernel h_t with respect to $d\mu$ and

$$H_t f(x)=\int_M h_t(x,y)f(y)\,d\mu(y),\qquad f\in L^2,$$

with $0<h_t(x,y)<\infty$ for all $t>0$, $x,y\in M$. Moreover, it is well known that $h_t(x,y)$ tends to 1 as t tends to infinity. For specific examples, it is then natural to ask for quantitative estimates on how fast h_t tends to 1. Several authors have discussed similar questions when M is a finite set. See for instance [1, 11] and the references given in these papers. The techniques used below can also be applied to certain finite Markov chains and this is developed in [12, 13].

Define the *time to equilibrium* $T=T(M,g)$ by

$$T=\inf\{t>0:\sup_{x\in M}\|h_t^x-1\|_1\le 1/e\}\tag{1}$$

where $h_t^x(y)=h_t(x,y)$ and the choice of the constant $1/e<1$ is for convenience. Roughly speaking, the present work describes upper and lower bounds on T that depend on geometric quantities such as the diameter of (M,g). Note that the inequality

$$\|h_t^x-1\|_1\le e^{-\lfloor t/T\rfloor}\tag{2}$$

follows easily from the submultiplicativity of $\sup_x\|h_t^x-1\|_1$ as a function of t. Thus, any estimate on T yields a quantitative version of the convergence

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of h_t. It is well known and easy to see that the smallest non-zero eigenvalue λ of the Laplace operator governs the asymptotic exponential rate of convergence to equilibrium. This and (2) imply that $T \geq 1/\lambda$. This does not mean, however, that equilibrium is approximately reached at time $1/\lambda$: bounding T requires further information besides estimates on λ.

In [26], which should be considered as a companion paper, very specific examples like the n-dimensional torus, the n-sphere or classical groups are studied. For these examples, the main parameter is the dimension. For instance, we have

THEOREM 1 ([26]). *The n-dimensional torus $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$ satisfies*

$$T(\mathbb{T}^n) \sim \tfrac{1}{2}\log n \qquad \text{as } n \to \infty.$$

The n-sphere $S_n \subset \mathbb{R}^{n+1}$ satisfies

$$T(S_n) \sim \frac{\log n}{2n} \qquad \text{as } n \to \infty.$$

The present paper studies families of manifolds of a fixed dimension. The main results are:

THEOREM 2. *For compact Riemannian manifolds of dimension n with non-negative Ricci curvature, there exist two dimensional constants $0 < c(n) \leq C(n) < \infty$ such that*

$$c(n)d^2 \leq T \leq C(n)d^2$$

where d is the diameter.

THEOREM 3. *Let (N, g) be a fixed compact Riemannian manifold with fundamental group π_1 and universal covering $\overline{N}$. Let $M = \overline{N}/\Gamma$ be any compact covering of M where Γ is a subgroup of π_1.*

1. *If π_1 has polynomial growth, the heat diffusion on M satisfies*

$$cd^2 \leq T(M) \leq Cd^2.$$

2. *If π_1 has Kazhdan's property, the heat diffusion on M satisfies*

$$cd \leq T(M) \leq Cd.$$

Here, d is the diameter of M and c, C depend only on (N, g).

Theorem 2 is proved in Section 3. The first part of Theorem 3 is proved in Section 4. Part 2 of Theorem 3 is proved in Section 5. These results follow from known spectral estimates and Harnack inequalities.

In Section 6, we use a lower bound on $T(M)$ to estimate the logarithmic Sobolev constant from above when M has Ricci curvature bounded below.

THEOREM 4. *Fix $n \geq 2$. For any $\varepsilon > 0$, there exist compact Riemannian manifolds of dimension n with constant sectional curvature equal to -1 and such that $\alpha/\lambda \leq \varepsilon$. Here, α is the log-Sobolev constant defined in* (18).

For contrast, note that compact manifolds of dimension n with non-negative Ricci curvature satisfy $c(n)\lambda \le \alpha \le \lambda/2$; see Rothaus [23, 22].

The results proved in the paper are somewhat more precise than those stated in this introduction. In particular, L^2 and uniform convergence are also considered.

2. Basics. In order to obtain quantitative estimates for the convergence of h_t to equilibrium, some notion of "distance" must be chosen. We will mainly work with L^1 and L^2 distances. Other L^p norms yield a scale of different choices. Set

$$N_p(h_t - 1) = \sup_x \|h_t^x - 1\|_p.$$

For $p \ge 2$,

$$N_2(h_t - 1) \le N_p(h_t - 1) \le N_\infty(h_t - 1) = (N_2(h_{t/2} - 1))^2.$$

For $1 < p \le 2$, define $m_p = 1 + \lceil (2-p)/(2(p-1)) \rceil$ to be the smallest integer greater than or equal to $1 + (2-p)/(2(p-1))$. We claim that, for $1 < p \le 2$,

$$N_2(h_t - 1) \le (N_p(h_{t/m_p} - 1))^{m_p}.$$

To begin with, recall that

$$N_q(h_t - 1) \le N_r(h_{t_1} - 1) N_s(h_{t_2} - 1)$$

for all $t = t_1 + t_2 > 0$ and $1 \le q, r, s \le \infty$ with $1 + 1/q = 1/r + 1/s$. To prove the claim, apply the last inequality successively with $q = p_{i+1}$, $r = p_i$, $s = p$ and $p_1 = p$. This gives

$$N_{p_{i+1}}(h_t - 1) \le N_{p_i}(h_{it/(i+1)} - 1) N_p(h_{t/(i+1)} - 1) \le N_p(h_{t/(i+1)} - 1)^{i+1}.$$

Clearly, $1/p_i = 1/p - (i-1)(1 - 1/p)$ and thus, $p_i \ge 2$ if and only if

$$i \ge 1 + \frac{2-p}{2(p-1)}.$$

This proves the claim. We conclude that $N_p(h_t - 1)$ does not depend too much on p when p is bounded away from 1. Note that, for $p > 1$, $N_p(h_t - 1)$ tends to infinity when t tends to zero whereas $N_1(h_t - 1)$ is always bounded by 2. Consider, however, the first time T_p for which $N_p(h_t - 1) \le 1/e$. For several classes of examples, the different T_p's, including T_1, behave all the same. For instance, this is the case for manifolds of a fixed dimension with non-negative Ricci curvature. See also the examples in [26].

The reasons for considering N_1 is that it yields the weakest notion of convergence (among the N_p's), and that it does not depend on the normalization $\mu(M) = 1$. Note indeed that

$$N_p(h_t - 1) = V^{1-1/p} \sup_{x \in M} \Big(\int_M |\underline{h}_t(x,y) - 1|^p \, dv(y) \Big)^{1/p}$$

where $\underline{h}_t = V^{-1}h_t$ is the canonical heat kernel on M, i.e., the kernel of $e^{-t\Delta}$ with respect to the Riemannian measure dv. Whether we work with N_2 or N_∞ does not really matter since $N_\infty(h_t - 1) = (N_2(h_{t/2} - 1))^2$.

Recall that the *spectral gap* of (M, g) is defined by

$$\lambda = \min\{\|\nabla f\|_2^2/\|f\|_2^2 : f \in \mathcal{C}^\infty \cap L^2,\ \|f\|_2 \neq 0,\ Uf = 0\} \tag{3}$$

where $Uf = \int_M f\,d\mu$. Because $\partial_t\|H_t f\|_2^2 = -2\|\nabla H_t f\|_2^2$, it follows easily from the definition that

$$\|H_t - U\|_{2\to 2} \le e^{-\lambda t}. \tag{4}$$

For compact manifolds, we have $\lambda > 0$ but, in general, it may well happen that $\lambda = 0$. For instance, [3] shows that $\lambda = 0$ for manifolds of finite volume such that

$$\limsup_{r\to\infty} \frac{-1}{r}\log(1 - \mu(B(x, r))) = 0.$$

Bounding $\|h_t^x - 1\|_2$ in terms of λ is, in theory, very easy. Write $t = \varepsilon + s$, $h_t^x(y) = H_s h_\varepsilon^x(y)$ and

$$\|h_t^x - 1\|_2 = \|(H_s - U)h_\varepsilon^x\|_2 \le \|h_\varepsilon^x\|_2\|H_s - U\|_{2\to 2} \le \|h_\varepsilon^x\|_2 e^{-\lambda s}. \tag{5}$$

This leaves us with the task of bounding $\|h_\varepsilon^x\|_2 = (h_{2\varepsilon}(x, x))^{1/2}$. This can be done by different means. Sobolev's type inequalities can for instance be used for that purpose. See [29, 30] and, more specifically, [26]. In this paper, we will use Harnack inequalities instead. Examples where N_2 is bounded by using the full description of the spectrum of Δ are given in [26].

3. Manifolds with non-negative Ricci curvature. In this section, we prove Theorem 2 and, more precisely, the following result:

THEOREM 5. *Let (M, g) be an n-dimensional compact Riemannian manifold with non-negative Ricci curvature and diameter d. The heat diffusion associated with the Laplace–Beltrami operator satisfies*

$$N_2(h_{t+s} - 1) \le \exp\left(\frac{3n^{1/2}d}{4t^{1/2}} - \lambda s\right) \quad \textit{for all } s, t > 0$$

and

$$N_2(h_t - 1) \le e^{-c} \quad \textit{for } t = \left(\left(1 + \frac{3}{\pi^2}\right)n^{1/3} + \frac{4}{\pi^2}c\right)d^2 \textit{ with } c \ge 0.$$

Moreover,

$$N_1(h_t - 1) \ge e^{-c} \quad \textit{for } t = cd^2/(\pi^2 n).$$

P r o o f. Start with the lower bound. Recall that

$$\|h_t^x - 1\|_1 = \sup_{\|f\|_\infty \le 1} |(H_t - U)f(x)|.$$

As a test function, choose an eigenfunction ϕ associated with the eigenvalue λ, normalized by $\|\phi\|_\infty = \phi(x_0) = 1$, for some $x_0 \in M$. Since $U\phi = 0$, we get

$$\|h_t^x - 1\|_1 \geq H_t\phi(x_0) = e^{-\lambda t}$$

and the desired result then follows from Cheng's estimate $\lambda \leq n\pi^2/d^2$. See [8].

The upper bound begins with the Harnack inequality

$$h_t(x,x) \leq h_{(1+\varepsilon)t}(x,y)(1+\varepsilon)^{n/2}\exp(d^2/(4\varepsilon t))$$

obtained by P. Li and S.-T. Yau in [19]. Integrating over M yields

$$h_t(x,x) \leq (1+\varepsilon)^{n/2}\exp(d^2/(4\varepsilon t)).$$

In particular, for $\varepsilon = d/\sqrt{nt}$ we get

$$h_t(x,x) \leq \exp\left(\frac{3n^{1/2}d}{4t^{1/2}}\right).$$

Together with (5), this gives

$$\|h_{t+s}^x - 1\|_2 \leq \exp\left(\frac{3n^{1/2}d}{4t^{1/2}} - \lambda s\right).$$

For the second upper bound in Theorem 5, we take

$$t = n^{1/3}d^2 \quad \text{and} \quad s = \left(\frac{3}{\pi^2}n^{1/3} + \frac{4}{\pi^2}\right)d^2$$

and use the estimate

$$\lambda \geq \frac{\pi^2}{4d^2}, \tag{6}$$

which is taken from [18]. ■

Remarks. 1. Theorem 5 is stated for compact manifolds without boundary but it also holds for the heat diffusion with Neumann boundary condition on compact manifolds of non-negative Ricci curvature with convex boundary. See [19]. In particular, this theorem applies to convex bounded domains in $\mathbb{R}^n$. In this case, (6) holds without the factor 4 and is due to Payne and Weinberger [21]. In all cases, Theorem 5 shows that t of order $d^2n^{1/3}$ suffices for the heat diffusion to be close to equilibrium whereas d^2/n is necessary. This should be compared with the result stated for the torus $\mathbb{T}^n$ in the introduction. In this case, the diameter is $d = 2\pi n^{1/2}$, $\lambda = 1$, and equilibrium is approximately achieved at time $t = \frac{\log n}{8\pi^2 n}d^2$. This shows that the lower bound in Theorem 5 is sharp up to a logarithmic dimensional factor.

2. Theorem 5 and the above remark lead us to the following question:

For the heat diffusion on a convex bounded domain in $\mathbb{R}^n$ *with Neumann boundary condition, what is a good upper bound* (*depending on* n) *on the equilibrium time* T *defined in* (1)?

In the same spirit, working out the details in the case of the euclidean ball in $\mathbb{R}^n$ with Neumann boundary condition seems to be a worthwhile project.

4. Coverings under polynomial volume growth. As a second example, we consider towers of compact coverings under polynomial growth. Recall that a finitely generated group G has *polynomial volume growth* if there exist two constants A and d such that

$$\#(E^k) \leq Ak^d, \quad k = 1, 2, \ldots,$$

where E is one (or any) fixed finite generating set of G containing the identity. The following result is based on Gromov's theorem [14] and contains the first statement of Theorem 3.

THEOREM 6. *Let* $\overline{N}$ *be a normal covering of a compact Riemannian manifold* (N, g) *with deck transformation group* G *having polynomial volume growth. There exist constants* A, b, B, C, C' *depending only on* G, N, g *such that*, *for any subgroup* $\Gamma \subset G$ *with finite index*, *the heat diffusion on* $M = \overline{N}/\Gamma$ *satisfies*

$$N_2(h_t - 1) \leq C' e^{-s} \quad \textit{for } t = Cd^2(1+s),\ s > 0,$$

and

$$e^{-At/d^2} \leq N_1(h_t - 1) \leq Be^{-bt/d^2} \quad \textit{for } t > 0$$

where d *is the diameter of* M.

Proof. It follows from the results in [25] that the diffusion on M satisfies a parabolic Harnack inequality, uniformly over all possible choices of Γ. Namely, there exists a constant C_1 depending only on G, N, g but not on Γ such that, for any $x \in M = \overline{N}/\Gamma$, any $r > 0$, and any positive solution u of $(\partial_t + \Delta)u = 0$ in $]0, r^2[\, \times B(x, r)$, we have

$$\sup_{Q_-} u \leq C_1 \inf_{Q_+} u \tag{7}$$

where $Q_- = \,]r^2/4, r^2/2[\, \times B(x, r/2)$ and $Q_+ = \,]3r^2/4, r^2[\, \times B(x, r/2)$. In fact, [25] shows that (7) holds on $\overline{N}$ and this easily implies that it holds with the same constant C_1 on any quotient $M = \overline{N}/\Gamma$.

It also follows from [25] that there exist C_2, C_3 depending only on G, N, g such that, for any $x \in M$ and $r > 0$, we have

$$\int_B |f - f_B|^2 \, d\mu \leq C_2 r^2 \int_B |\nabla f|^2 \, d\mu, \quad f \in \mathcal{C}^\infty(B), \tag{8}$$

(9) $$\mu(B(x,2r)) \le C_3\mu(B(x,r))$$

where $B = B(x,r)$ and f_B is the mean of f over the ball B. For more details on this matter, see [9].

Now, applying (7) to the heat kernel h_t on M, we obtain

$$h_{d^2}(x,x) \le C_1$$

where d is the diameter of M. Moreover, for $r = d$, the balls of radius r are equal to M and (8) shows that the spectral gap $\lambda = \lambda(M)$ satisfies

$$\lambda \ge \frac{1}{C_2 d^2}.$$

The last two inequalities and (5) give the desired upper bound on $N_2(h_t - 1)$ and $N_1(h_t - 1)$.

To estimate $N_1(h_t - 1)$ from below, it suffices to have an upper bound on λ. Consider two points $x_1, x_2 \in M$ at distance d apart. For $i = 1, 2$, let $\phi_i(x) = (d/2 - d(x_i, x))_+$ where $d(x,y)$ is the Riemannian distance and $(z)_+ = \max\{0, z\}$. Set $\psi = \phi_1 - a\phi_2$ where $a > 0$ is chosen so that $\int \psi\, d\mu = 0$. Obviously, we have

$$\begin{aligned}\int |\psi|^2 &\ge \int |\phi_1|^2 + a^2 \int |\phi_2|^2 \\ &\ge (1+a^2)\frac{d^2}{8}\inf_M\{\mu(B(x,d/4))\} \ge \frac{(1+a^2)d^2}{8C_3^2}\end{aligned}$$

and

$$\int |\nabla\psi|^2\, d\mu \le 1 + a^2.$$

Using ψ as a test function in (3), we get

$$\lambda \le \frac{8C_3^2}{d^2}.$$

This and the argument of Section 3 end the proof of Theorem 6. ∎

Remarks. 1. Theorem 6 shows that a time of order d^2 is necessary and suffices for approximate equilibrium of the heat diffusion on $M = \overline{N}/\Gamma$ uniformly over all possible choices of Γ. This proves the first assertion in Theorem 3.

2. In Theorem 6, the Laplace–Beltrami operator can be replaced by any uniformly subelliptic operator. See [25].

5. A lower bound on T. This section presents a lower bound on the time to equilibrium T under the condition that $\mathrm{Ric} \ge -Kg$. It also contains the proof of the second statement of Theorem 3 concerning towers of compact coverings, where π_1 has Kazhdan's property.

THEOREM 7. *Let (M, g) be a compact Riemannian manifold of dimension n. Assume that* $\mathrm{Ric} \ge -Kg$ *for some* $K \ge 0$. *Then there exist* $0 < c, C < \infty$

depending only on n and K such that if the diameter d of M satisfies $d \ge C$, then $N_1(h_t - 1) \ge 1/2$ for all $0 < t \le cd$. In particular, $T(M) \ge cd$.

Proof. We need to introduce some notation. Let $V(x,r) = v(B(x,r))$ be the Riemannian volume of the ball $B(x,r)$ and set $W(x) = 1/\mu(B(x,1)) = V/V(x,1)$. Using classical bounds on the volume of balls in constant curvature spaces and a refinement of Bishop's comparison theorem (see [7], Proposition 4.1), one gets the well known inequality

$$\frac{V(x,r)}{V(x,s)} \le (r/s)^n \exp(\sqrt{(n-1)K}r) \quad \text{for } 0 < s \le r,\ x \in M. \tag{10}$$

Let $\underline{h}_t(x,y)$ be the kernel of the heat diffusion semigroup $e^{-t\Delta}$ with respect to the Riemannian measure dv. We have $\underline{h}_t = V^{-1}h_t$.

Using Theorem 3.1 of Li–Yau's paper [19] (see also Theorem 6.3 in [24]), we find that

$$h_t(x,y) \le C_1 W(x) \exp\left(-\frac{d(x,y)^2}{5t}\right) \quad \text{for } t \ge 1,\ x,y \in M \tag{11}$$

where $d(x,y)$ is the Riemannian distance between x and y. Here and in the sequel the different constants c_i, C_i depend only on n and K. For all $t \ge 1$ and $x \in M$, $r > 0$, (11) gives

$$\begin{aligned} \|h_t^x - 1\|_1 &\ge \int\limits_{d(x,y)\ge r} |1 - h_t(x,y)|\, d\mu(y) \\ &\ge \left(1 - \frac{V(x,r)}{V}\right)(1 - C_1 W(x) e^{-r^2/(5t)}). \end{aligned}$$

Define $R(x)$ by requiring that

$$V(x, R(x)) = \tfrac{1}{4}V, \tag{12}$$

and note that (10) and (12) imply

$$W(x) = 4\frac{V(x,R(x))}{V(x,1)} \le \exp(C_2 R(x)) \tag{13}$$

whenever $R(x) \ge 1$. Using (12), we obtain

LEMMA 8. *Let (M,g) be a complete manifold of dimension n, having finite volume and satisfying* Ric $\ge -Kg$ *for some constant $K \ge 0$. Then, for $t \ge 1$, we have*

$$\|h_t^x - 1\|_1 \ge \tfrac{3}{4}(1 - C_1 W(x) e^{-R(x)^2/(5t)})$$

and, in particular,

$$\|h_t^x - 1\|_1 \ge 1/2 \quad \text{for } 1 \le t \le \frac{R(x)^2}{5\log(2C_1 W(x))}. \tag{14}$$

Inequalities (13) and (14) show that

$$\|h_t^x - 1\|_1 \geq 1/2 \quad \text{for } 1 \leq t \leq c_1 R(x).$$

To finish the proof of Theorem 7 it suffices to prove that, when M is compact with diameter d,

$$\max_M R(x) \geq d/6. \tag{15}$$

To see this, consider two points x_1, x_4 at distance d apart and two other points x_2, x_3 on a geodesic joining x_1 to x_4 and such that $d(x_1, x_2) = d(x_2, x_3) = d(x_3, x_4) = d/3$. The balls $B(x_i, d/6)$ are disjoint and $\sum_{i=1}^4 V(x_i, d/6) \leq V$. This implies that at least one of the x_i's satisfies $R(x) \geq d/6$. Together with Lemma 8, this shows that if $d \geq \max\{1, 6/c_1\}$, then $N_1(h_t - 1) \geq 1/2$ for $t \leq c_1 d/6$. ■

We can now give a proof of the second statement of Theorem 3. Assume that (N, g) is a compact manifold with universal covering $\overline{N}$ and fundamental group π_1 having Kazhdan's property (i.e., property T, see [17, 5, 20]). We want to prove that there exist two constants $0 < c, C < \infty$ such that any compact covering of N satisfies

$$cd \leq T(M, g) \leq Cd.$$

Because N is compact, there exists $K \geq 0$ such that any covering of N satisfies $\mathrm{Ric} \geq -Kg$. In particular, Theorem 7 applies uniformly to any compact covering of N and yields the desired lower bound on T. This part does not use the fact that π_1 has Kazhdan's property.

For the upper bound, we need the fact that there exists $\varepsilon > 0$ such that the smallest non-zero eigenvalue $\lambda = \lambda(M)$ of any compact covering M of (N, g) satisfies

$$\lambda \geq \varepsilon. \tag{16}$$

This is a consequence of π_1 having Kazhdan's property (see [5, 20]). Note also that there exists $a > 0$ such that $\sup_M V(x, 1) \geq a$ for all coverings M of N. Hence, (11) gives

$$\sup_{x,y \in M} h_1(x, y) \leq AV \tag{17}$$

where A depends only on (N, g).

Inserting (16) and (17) in (5) shows that

$$N_1(h_t - 1) \leq N_2(h_t - 1) \leq AVe^{-\varepsilon(t-2)} \quad \text{for } t \geq 2.$$

Together with (10), this yields a constant C depending only on (N, g) such that $T \leq Cd$. ■

6. An upper bound on the log-Sobolev constant. Consider the *log-Sobolev constant* $\alpha = \alpha(M)$ associated with the Laplace operator on M.

It is defined by

$$\alpha = \min\{\|\nabla f\|_2^2/\mathcal{L}(f) : \mathcal{L}(f) \neq 0,\ f \in \mathcal{C}^\infty \cap L^2\}$$

where

$$\mathcal{L}(f) = \int_M |f|^2 \log(|f|^2/\|f\|_2^2)\, d\mu.$$

In other words, α is the largest constant c such that

$$c\mathcal{L}(f) \le \|\nabla f\|_2^2, \qquad f \in \mathcal{C}^\infty \cap L^2. \tag{18}$$

L. Gross introduced this type of logarithmic Sobolev inequality in [15] where he proved that (18) is equivalent to

$$\|e^{-t\Delta}\|_{2\to q} \le 1 \qquad \text{for all } t > 0 \text{ and } q \ge 2 \text{ such that } q \le 1 + e^{4ct}. \tag{19}$$

This property is called *hypercontractivity*. We refer the reader to [2, 10, 15] for detailed discussions of the log-Sobolev inequality and hypercontractivity. For the n-dimensional torus or the n-sphere the log-Sobolev constant α is known explicitly and satisfies $\alpha = \lambda/2$, whereas in general, $\alpha \le \lambda/2$. Moreover, O. Rothaus has shown in [23] that for compact manifolds of non-negative Ricci curvature, $2n\lambda/(n+1)^2 \le \alpha \le \lambda/2$. This contrasts with the following result:

THEOREM 9. *Fix* $n \ge 2$ *and* $K \ge 0$. *Let* (M, g) *be a complete Riemannian manifold of dimension* n *such that* $\mathrm{Ric} \ge -Kg$. *Then*:

1. *The log-Sobolev constant* α *is positive if and only if* M *is compact.*
2. *Moreover, if* M *is compact with diameter* d, *there exist* $C_1, C_2 > 0$, *depending only on* n *and* K, *such that*

$$\alpha \le C_1 \frac{\log d}{d} \qquad \text{if } d \ge C_2. \tag{20}$$

Proof. To relate α to the time to equilibrium T, write $t = \varepsilon + \theta + s$, $h_t^x(y) = H_{s+\theta} h_\varepsilon^x(y)$ and

$$\|h_t^x - 1\|_2 = \|(H_s - U) H_\theta h_\varepsilon^x\|_2 \le \|h_\varepsilon^x\|_{q'} \|H_\theta\|_{q'\to 2} \|H_s - U\|_{2\to 2}$$

for any $q' \le 2$. Now, choose $q' = q'(\theta)$ by setting $1/q + 1/q' = 1$ and $q = q(\theta) = 1 + e^{4\alpha\theta}$ so that (19) yields

$$\|H_\theta\|_{q'\to 2} = \|H_\theta\|_{2\to q} \le 1.$$

Also, note that

$$\|h_\varepsilon^x\|_{q'} \le \|h_\varepsilon^x\|_2^{2/q}.$$

These inequalities, together with (4), give

$$\|h_t^x - 1\|_2 \le \|h_\varepsilon^x\|_2^{2/q(\theta)} e^{-\lambda s} \qquad \text{for } t = \varepsilon + \theta + s \text{ and } q(\theta) = 1 + e^{4\alpha\theta}. \tag{21}$$

Similar manipulations can be found in [16, 28] where they are used to obtain qualitative results on uniform convergence for Ising models. In [26], (21) is used to derive an upper bound on T for semisimple compact Lie groups.

Now, using (11) and (21) with $\varepsilon = 1$, we find that

$$\|h_t^x - 1\|_1 \le \|h_t^x - 1\|_2 \le (C_1 W(x))^{1/q(\theta)} e^{-\lambda s}$$

for $t = 1 + \theta + s$ and $q(\theta) = 1 + e^{4\alpha\theta}$. Here, as in Section 5, $W(x) = 1/\mu(B(x,1)) = V/V(x,1)$. Choosing

$$\theta = \frac{1}{4\alpha} \log\log(C_1 W(x)), \qquad s = \frac{1}{\alpha} \ge \frac{2}{\lambda},$$

we get

$$\|h_t^x - \mu\|_1 \le \frac{1}{e} \qquad \text{for } t \ge 1 + \frac{1}{4\alpha}(4 + \log\log(C_1 W(x))).$$

Comparing this with (14) yields

$$\alpha \le (4 + \log\log(C_1 W(x))) \frac{5\log(2C_1 W(x))}{4(R(x)^2 - 5\log(2C_1 W(x)))}$$

if $R(x)$ is large enough. Recall that (13) says that $\log W(x) \le C_2 R(x)$ when $R(x) \ge 1$. Therefore, there exist C, C'' depending only on n, K such that

$$\alpha \le C \frac{\log R(x)}{R(x)} \qquad \text{if } R(x) \ge C'. \tag{22}$$

Now, if (M, g) has finite volume but is not compact, then $R(x)$ tends to infinity when x tends to infinity and this proves the first claim in Theorem 9. The second claim follows from (15) and (22). ■

Remarks. 1. Set $W_* = \max_{x \in M} W(x)$. The above proof shows in fact that

$$\alpha \le C \log\log W_* \frac{\log W_*}{d^2} \qquad \text{if } d \ge C'.$$

The volume estimate (13) shows that this is stronger than (20). One also has, for W_* large enough,

$$\alpha \le C \log\log W_* / \log W_*,$$

which is weaker than (20).

2. Another variant of the above argument yields

$$\lambda \le C \left(\frac{\log W_*}{d} \right)^2 \qquad \text{if } d \ge C'.$$

3. Let (M, g) be as in Theorem 9 and such that $\lambda \ge \varepsilon$ for some fixed $\varepsilon > 0$. Then (11), (13) and classical arguments from the theory of hypercontractivity show that, in this case, the log-Sobolev constant satisfies

$$\alpha \ge c(n, K, \varepsilon)/d.$$

4. It is well known that, in any given dimension, there exist compact manifolds of constant negative sectional curvature with large diameter and λ uniformly bounded away from zero. For instance, if N is compact with fundamental group $\pi_1(N)$ having Kazhdan's property, there exists $\varepsilon > 0$ such that any finite covering of N satisfies $\lambda \geq \varepsilon$. See [4, 5, 20] for details and further results. This yields examples where α/λ can be made arbitrarily small and proves Theorem 4.

It is also well known that there are examples of non-compact complete Riemannian manifolds of finite volume with a positive spectral gap. In fact, any n-dimensional manifold with sectional curvature bounded above by -1 has essential spectrum bounded below by $(n-1)^2/4$. See [6]. If such a manifold has finite volume, it follows that $\lambda > 0$. A quantitative (but difficult!) result is given by Selberg's theorem which asserts that $\lambda(\mathbb{H}/\Gamma(m)) \geq 3/16$ where $\mathbb{H}$ is the hyperbolic plane and $\Gamma(m) = \{A \in SL_2(\mathbb{Z}) : A = I \bmod m\}$. See [20, 27, 5].

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