COLLOQUIUM MATHEMATICUM

VOL. LXVII 1994 FASC. 1

EQUIVALENT CHARACTERIZATIONS OF BLOCH FUNCTIONS

BY

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In this paper we obtain some equivalent characterizations of Bloch functions on general bounded strongly pseudoconvex domains with smooth boundary, which extends the known results in [1, 9, 10].

1. Introduction. Let $\mathcal{D}$ be a bounded strongly pseudoconvex domain in $\mathbb{C}^m$ with smooth boundary $\partial\mathcal{D}$, and $\varrho(z)$ be a defining function of $\mathcal{D}$. By $H(\mathcal{D})$ we denote the family of all holomorphic functions on $\mathcal{D}$, and by $K(\cdot,\cdot)$ and $\beta(\cdot,\cdot)$ the Bergman kernel and the Bergman distance on $\mathcal{D}$ respectively. For z in $\mathcal{D}$ and r positive, $E(z,r)=\{w\in\mathcal{D}:\beta(w,z)<r\}$. Let $|E(z,r)|=\int_{E(z,r)}dm$, where dm is the Lebesgue measure on $\mathbb{C}^m=\mathbb{R}^{2m}$.

In what follows, C will denote a positive constant depending only on $p,r,\ldots$, but not on $f\in H(\mathcal{D})$. Its value may change from line to line. The expression "A and B are equivalent" (denoted by $A\sim B$) means $C^{-1}A\le B\le CA$.

Following Krantz and Ma [4], we define the *Bloch space* on $\mathcal{D}$ to be

$$\mathcal{B}(\mathcal{D})=\{f\in H(\mathcal{D}):\sup|f_*(z)\cdot\xi|/F_K(z,\xi)<\infty\},$$

where the sup is taken over all $z\in\mathcal{D}$ and $0\ne\xi\in T_z(\mathcal{D})$, $f_*(z):T_z(\mathcal{D})\to T_{f(z)}(\mathbb{C})$ is induced by $f:\mathcal{D}\to\mathbb{C}$ and $F_K(z,\xi)$ is the infinitesimal form of the Kobayashi metric form. Similarly, we define the *little Bloch space* ([5]) to be

$$\mathcal{B}_0(\mathcal{D})=\{f\in H(\mathcal{D}):|f_*(z)\cdot\xi|=o(F_K(z,\xi))\text{ as }z\to\partial\mathcal{D}\text{ for all }\xi\in T_z(\mathcal{D})\}.$$

For $f\in H(\mathcal{D})$, let

$$\nabla f(z)=\left(\frac{\partial f}{\partial z_1},\ldots,\frac{\partial f}{\partial z_m}\right)\quad\text{and}\quad|\nabla f(z)|=\left(\sum_{j=1}^m\left|\frac{\partial f}{\partial z_j}\right|^2\right)^{1/2}.$$

1991 *Mathematics Subject Classification*: Primary 32A37.

This research was partially supported by a grant from NSF of P. R. China and a grant from NSF of Zhejiang Province.

We set

$$\|f\|_{\mathcal{B}(\mathcal{D})} = \sup\{|\nabla f(z)|\,|\varrho(z)| : z \in \mathcal{D}\}.$$

It is proved in [4] that $f \in \mathcal{B}(\mathcal{D})$ if and only if $\|f\|_{\mathcal{B}} < \infty$. By analogy, it is easy to prove that $f \in \mathcal{B}_0(\mathcal{D})$ if and only if $|\nabla f(z)|\,|\varrho(z)| \to 0$ as $z \to \partial\mathcal{D}$.

The Bloch space on the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ has been studied extensively. For example, S. Axler in [1] got a series of equivalent definitions and in [9] Axler's results were generalized by Stroethoff to the following

THEOREM A. *Let* $0 < p < \infty$, $0 < r < 1$, *and* $n \in \mathbb{N}$. *Then for* $f \in H(D)$ *the following quantities are equivalent*:

(A) $\|f\|_{\mathcal{B}(D)} = \sup_{z\in D} |f'(z)|(1-|z|^2)$;

(B) $\sup_{z\in D}\Big(\frac{1}{|D(z,r)|^{1-np/2}}\int_{D(z,r)} |f^{(n)}(w)|^p\,dm(w)\Big)^{1/p} + \sum_{k=1}^{n-1}|f^{(k)}(0)|$;

(C) $\sup_{z\in D}\Big(\int_{D(z,r)} |f^{(n)}(w)|^p(1-|w|^2)^{np-2}\,dm(w)\Big)^{1/p} + \sum_{k=1}^{n-1}|f^{(k)}(0)|$;

(D) $\sup_{z\in D}\Big(\int_{D} |f^{(n)}(w)|^p(1-|w|^2)^{np-2}(1-|\varphi_z(w)|^2)^2\,dm(w)\Big)^{1/p}$
$$+ \sum_{k=1}^{n-1}|f^{(k)}(0)|.$$

THEOREM B. *Let* $0 < p < \infty$, $0 < r < 1$, *and* $n \in \mathbb{N}$. *Then for* $f \in H(D)$ *the following conditions are equivalent*:

(A) $f \in \mathcal{B}_0(D)$;

(B) $\frac{1}{|D(z,r)|^{1-np/2}}\int_{D(z,r)} |f^{(n)}(w)|^p\,dm(w) \to 0$ *as* $|z| \to 1^-$;

(C) $\int_{D(z,r)} |f^{(n)}(w)|^p(1-|w|^2)^{np-2}\,dm(w) \to 0$ *as* $|z| \to 1^-$;

(D) $\int_{D} |f^{(n)}(w)|^p(1-|w|^2)^{np-2}(1-|\varphi_z(w)|^2)^2\,dm(w) \to 0$ *as* $|z| \to 1^-$.

In Theorems A and B, $\varphi_z(w) = (w-z)/(1-\bar{z}w)$ is the Möbius function and $D(z,r)$ is the pseudo-hyperbolic disc with centre z and radius r. We have $D(z,r) = E(z,r')$ for some $r' > 0$, and vice versa. Stroethoff also obtained the analogue of Theorems A and B on the unit ball of $\mathbb{C}^m$ in [9]. What was crucial to both [1] and [9] is the transitive group of Möbius functions. The purpose of the present work is to extend the results in [9, 1] to bounded strongly pseudoconvex domains with smooth boundary. Since there is no nontrivial holomorphic automorphism for such a domain generally, our theory is more subtle.

For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\alpha_j \ge 0$ an integer, we write $|\alpha| = \sum_{j=1}^m \alpha_j$, and for $f \in H(\mathcal{D})$,

$$\frac{\partial^{|\alpha|} f}{\partial z^\alpha} = \frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \ldots \partial z_m^{\alpha_m}}.$$

Here are the main results of this paper.

THEOREM 1. *Let $\mathcal{D}$ be a bounded strongly pseudoconvex domain in $\mathbb{C}^m$ with smooth boundary, $\varrho(z)$ be its defining function, $0 < p < \infty$, $0 < r < \infty$, $n \in \mathbb{N}$ and $z_0 \in \mathcal{D}$ fixed. Then for $f \in H(\mathcal{D})$ the following quantities are equivalent*:

(A) $\|f\|_{\mathcal{B}(\mathcal{D})}$;

(B) $$\sup_{z\in\mathcal{D}} \left(\frac{1}{|E(z,r)|^{1-np/(m+1)}} \int\limits_{E(z,r)} \sum_{|\alpha|=n} \left| \frac{\partial^n f}{\partial z^\alpha}(w) \right|^p dm(w) \right)^{1/p} + \sum_{1\le|\beta|<n} \left| \frac{\partial^{|\beta|} f}{\partial z^\beta}(z_0) \right|;$$

(C) $$\sup_{z\in\mathcal{D}} \left(\int\limits_{E(z,r)} \sum_{|\alpha|=n} \left| \frac{\partial^n f}{\partial z^\alpha}(w) \right|^p |\varrho(w)|^{np-(m+1)}\, dm(w) \right)^{1/p} + \sum_{1\le|\beta|<n} \left| \frac{\partial^{|\beta|} f}{\partial z^\beta}(z_0) \right|;$$

(D) $$\sup_{z\in\mathcal{D}} \left(\int\limits_{\mathcal{D}} \sum_{|\alpha|=n} \left| \frac{\partial^n f}{\partial z^\alpha}(w) \right|^p |\varrho(w)|^{np} \frac{|K(w,z)|^2}{K(z,z)} dm(w) \right)^{1/p} + \sum_{1\le|\beta|<n} \left| \frac{\partial^{|\beta|} f}{\partial z^\beta}(z_0) \right|;$$

(E) $$\sup_{z\in\mathcal{D}} \sum_{|\alpha|=n} \left| \frac{\partial^n f}{\partial z^\alpha}(z) \right| |\varrho(z)|^n + \sum_{1\le|\beta|<n} \left| \frac{\partial^{|\beta|} f}{\partial z^\beta}(z_0) \right|.$$

THEOREM 2. *Let $\mathcal{D}$ be a bounded strongly pseudoconvex domain in $\mathbb{C}^m$ with smooth boundary, $\varrho(z)$ be its defining function, $0 < p < \infty$, $0 < r < \infty$, $n \in \mathbb{N}$. Then for $f \in H(\mathcal{D})$ the following conditions are equivalent*:

(A) $f \in \mathcal{B}_0(\mathcal{D})$;

(B) $$\frac{1}{|E(z,r)|^{1-np/(m+1)}} \int\limits_{E(z,r)} \sum_{|\alpha|=n} \left|\frac{\partial^n f}{\partial z^\alpha}(w)\right|^p dm(w) \to 0 \quad \textit{as } z \to \partial\mathcal{D};$$

(C) $$\int\limits_{E(z,r)} \sum_{|\alpha|=n} \left|\frac{\partial^n f}{\partial z^\alpha}(w)\right|^p |\varrho(w)|^{np-(m+1)}\, dm(w) \to 0 \quad \textit{as } z \to \partial\mathcal{D};$$

(D) $$\int\limits_{\mathcal{D}} \sum_{|\alpha|=n} \left|\frac{\partial^n f}{\partial z^\alpha}(w)\right|^p |\varrho(w)|^{np} \frac{|K(w,z)|^2}{K(z,z)} dm(w) \to 0 \quad \textit{as } z \to \partial\mathcal{D};$$

(E) $$\sum_{|\alpha|=n} \left|\frac{\partial^n f}{\partial z^\alpha}(z)\right| |\varrho(z)|^n \to 0 \quad \textit{as } z \to \partial\mathcal{D}.$$

2. Proof of Theorems 1 and 2. Since two defining functions must be equivalent on $\overline{\mathcal{D}}$ [8, 5], we can fix in the sequel a defining function $\varrho(z)$ to be

$$\varrho(z) = \begin{cases} -d(z,\partial\mathcal{D}), & z \in \mathcal{D}, \\ d(z,\partial\mathcal{D}), & z \in \mathbb{C}^m \setminus \mathcal{D}, \end{cases}$$

where $d(z,\partial\mathcal{D})$ is the Euclidean distance from z to $\partial\mathcal{D}$.

For $\delta > 0$ we set $\mathcal{D}_\delta = \{z \in \mathcal{D} : \varrho(z) > -\delta\}$. It is well known [3, 8] that when δ is small enough and $z \in \mathcal{D}_\delta$, then there is a unique point $\pi(z)$ which is closest to z on $\partial\mathcal{D}$. Let n_ζ be the unit inner normal vector at $\zeta \in \partial\mathcal{D}$. Then

$$z = \pi(z) - \varrho(z) n_{\pi(z)} \quad \text{whenever } z \in \mathcal{D}_\delta \tag{2.1}$$

(see [5, p. 382]). For $z \in \mathcal{D}$, we will use $P(z, r_1, r_2)$ to denote the polydisc centered at z with radius r_1 in the complex normal direction and radius r_2 in each of $m-1$ complex tangential directions (see [3, 4] for details).

LEMMA 1. *For each $r > 0$, there are $A, B > 0$ such that for all $z \in \mathcal{D}$,*

(1) $P(z, A|\varrho(z)|, A|\varrho(z)|^{1/2}) \subseteq E(z,r) \subseteq P(z, B|\varrho(z)|, B|\varrho(z)|^{1/2})$,

(2) $|\varrho(z)|^{m+1}/C \le |E(z,r)| \le C|\varrho(z)|^{m+1}$.

The lemma appears in [6]. All those results were proved for the Kobayashi metric in [5]. Since the Bergman metric and the Kobayashi metric are equivalent on a bounded strongly pseudoconvex domain with smooth boundary, it follows that the results are true for the Bergman metric.

LEMMA 2. *Let $z_0 \in \mathcal{D}$ and $\alpha > 0$. Then for $f \in H(\mathcal{D})$,*

$$\sup_{z\in\mathcal{D}} |\nabla f(z)|\, |\varrho(z)|^{\alpha+1} \le C \sup_{z\in\mathcal{D}} |f(z)|\, |\varrho(z)|^\alpha \tag{2.2}$$

and

$$\sup_{z\in\mathcal{D}} |f(z) - f(z_0)|\, |\varrho(z)|^\alpha \le C_1 \sup_{z\in\mathcal{D}} |\nabla f(z)|\, |\varrho(z)|^{\alpha+1}, \tag{2.3}$$

where the constant C_1 in (2.3) depends on z_0 and α.

This is a well-known fact. For completeness we sketch the proof.

The estimate (2.2) can be proved as in [7, pp. 104–105] and [3, pp. 324–325]. To prove (2.3), we let δ be so small that $z_0 \in \mathcal{D} \setminus \mathcal{D}_\delta$ and (2.1) is valid. Then we have two constants R and M such that for each $z \in \overline{\mathcal{D}\setminus\mathcal{D}_\delta}$, z and z_0 can be joined by smooth curve Γ located in the compact set $\overline{E(z_0,R)}$ with length less than M. Therefore

$$|f(z)-f(z_0)| = \Big|\int_\Gamma \nabla f(\Gamma(t))\cdot \Gamma'(t)\,dt\Big| \le M \sup_{w\in\overline{E(z_0,R)}} |\nabla f(w)|.$$

Then since $|\varrho(z)|^\alpha \le C|\varrho(w)|^{\alpha+1}$ $(z,w \in \overline{E(z_0,R)})$, we have

$$|f(z)-f(z_0)|\,|\varrho(z)|^\alpha \le CM \sup_{w\in\overline{E(z_0,R)}} |\nabla f(w)|\,|\varrho(w)|^{\alpha+1}. \tag{2.4}$$

For $z \in \mathcal{D}_\delta$, define $z' = \pi(z) + \delta n_{\pi(z)}$. Then $z' \in \partial\mathcal{D}_\delta$. Hence

$$\begin{aligned} |f(z)-f(z')| &= \Big|\int_\delta^{d(z,\partial\mathcal{D})} \nabla f(\pi(z)+tn_{\pi(z)})\cdot n_{\pi(z)}\,dt\Big| \\ &\le \sup_{w\in\mathcal{D}_\delta} |\nabla f(w)|\,|\varrho(w)|^{\alpha+1} \int_{d(z,\partial\mathcal{D})}^{\delta} t^{-(\alpha+1)}\,dt \\ &\le C|\varrho(z)|^{-\alpha} \sup_{w\in\mathcal{D}_\delta} |\nabla f(w)|\,|\varrho(w)|^{\alpha+1}. \end{aligned} \tag{2.5}$$

Now (2.3) comes from (2.4) and (2.5).

LEMMA 3. *For $\alpha > 0$ and $f(z) \in H(\mathcal{D})$,*

$$|f(z)|\,|\varrho(z)|^\alpha \to 0 \quad \textit{as } z \to \partial\mathcal{D} \tag{2.6}$$

if and only if

$$|\nabla f(z)|\,|\varrho(z)|^{\alpha+1} \to 0 \quad \textit{as } z \to \partial\mathcal{D}. \tag{2.7}$$

Proof. If $f(z)$ satisfies (2.6), it is trivial that $f(z)$ must satisfy (2.7) [3, 7]. Now if f satisfies (2.7), for $\varepsilon > 0$ we fix δ small enough such that for $z \in \mathcal{D}_\delta$,

$$|\nabla f(z)|\,|\varrho(z)|^{\alpha+1} < \alpha\varepsilon.$$

Then by (2.5), $|f(z)-f(z')|\,|\varrho(z)|^\alpha < \varepsilon$. Hence

$$|f(z)|\,|\varrho(z)|^\alpha < \varepsilon + \big(\sup_{z\in\mathcal{D}\setminus\mathcal{D}_\delta} |f(z)|\big)|\varrho(z)|^\alpha.$$

This implies $|f(z)|\,|\varrho(z)|^\alpha < 2\varepsilon$ if $|\varrho(z)| < \delta_1$. The lemma is proved.

We are now ready to prove the theorems.

Proof of Theorem 1. The proof is divided into five steps.

1) (A)∼(E). For $n = 1$, there is nothing to prove. If $n > 1$, by Lemma 2 we have

$$\left|\frac{\partial^n f}{\partial z^\alpha}(z)\right| |\varrho(z)|^n \le C \sup_{z\in\mathcal{D}} \sum_{|\beta|=n-1} \left|\frac{\partial^{n-1} f}{\partial z^\beta}(z)\right| |\varrho(z)|^{n-1}.$$

This yields

$$\sup_{z\in\mathcal{D}} \sum_{|\alpha|=n} \left|\frac{\partial^n f}{\partial z^\alpha}(z)\right| |\varrho(z)|^n \le C \sup_{z\in\mathcal{D}} \sum_{|\beta|=n-1} \left|\frac{\partial^{n-1} f}{\partial z^\beta}(z)\right| |\varrho(z)|^{n-1}. \tag{2.8}$$

On the other hand, another application of Lemma 2 gives

$$\left|\frac{\partial^{n-1} f}{\partial z^\beta}(z) - \frac{\partial^{n-1} f}{\partial z^\beta}(z_0)\right| |\varrho(z)|^{n-1} \le C \sup_{z\in\mathcal{D}} \sum_{|\alpha|=n} \left|\frac{\partial^n f}{\partial z^\alpha}(z)\right| |\varrho(z)|^n.$$

Then

$$\left|\frac{\partial^{n-1} f}{\partial z^\beta}(z)\right| |\varrho(z)|^{n-1} \le C\left\{\sup_{z\in\mathcal{D}} \sum_{|\alpha|=n} \left|\frac{\partial^n f}{\partial z^\alpha}(z)\right| |\varrho(z)|^n + \left|\frac{\partial^{n-1} f}{\partial z^\beta}(z_0)\right|\right\}.$$

Hence

$$\begin{aligned}\sup_{z\in\mathcal{D}} \sum_{|\beta|=n-1} &\left|\frac{\partial^{n-1} f}{\partial z^\beta}(z)\right| |\varrho(z)|^{n-1} \\ &\le C\left\{\sup_{z\in\mathcal{D}} \sum_{|\alpha|=n} \left|\frac{\partial^n f}{\partial z^\alpha}(z)\right| |\varrho(z)|^n + \sum_{|\beta|=n-1} \left|\frac{\partial^{n-1} f}{\partial z^\beta}(z_0)\right|\right\}.\end{aligned} \tag{2.9}$$

Now (2.8) and (2.9) give

$$\begin{aligned}\sup_{z\in\mathcal{D}} \sum_{|\beta|=n-1} &\left|\frac{\partial^{n-1} f}{\partial z^\beta}(z)\right| |\varrho(z)|^{n-1} \\ &\sim \sup_{z\in\mathcal{D}} \sum_{|\alpha|=n} \left|\frac{\partial^n f}{\partial z^\alpha}(z)\right| |\varrho(z)|^n + \sum_{|\beta|=n-1} \left|\frac{\partial^{n-1} f}{\partial z^\beta}(z_0)\right|.\end{aligned}$$

By induction, we have

$$\begin{aligned}\sup_{z\in\mathcal{D}} \sum_{j=1}^{m} &\left|\frac{\partial f}{\partial z_j}(z)\right| |\varrho(z)| \\ &\sim \sup_{z\in\mathcal{D}} \sum_{|\alpha|=n} \left|\frac{\partial^n f}{\partial z^\alpha}(z)\right| |\varrho(z)|^n + \sum_{1\le|\beta|\le n-1} \left|\frac{\partial^{|\beta|} f}{\partial z^\beta}(z_0)\right|.\end{aligned}$$

This is the desired result.

2) (C) $\le C\cdot$ (D). For $r > 0$, we have $\delta > 0$ such that if $z \in \mathcal{D}_\delta$ then

$$|\varrho(z)|^{-(m+1)}/C \le |K(z,w)| \le C|\varrho(z)|^{-(m+1)} \quad \text{for } w \in E(z,r)$$

(for details see [5, Theorem 12]). Then for $z \in \mathcal{D}_\delta$,

$$\int_{E(z,r)} \sum_{|\alpha|=n} \left|\frac{\partial^n f}{\partial z^\alpha}(w)\right|^p |\varrho(w)|^{np-(m+1)} dm(w) \tag{2.10}$$
$$\le C \int_{E(z,r)} \sum_{|\alpha|=n} \left|\frac{\partial^n f}{\partial z^\alpha}(w)\right|^p |\varrho(w)|^{np} \frac{|K(z,w)|^2}{K(z,z)}\, dm(w)$$
$$\le C \int_{\mathcal{D}} \sum_{|\alpha|=n} \left|\frac{\partial^n f}{\partial z^\alpha}(w)\right|^p |\varrho(w)|^{np} \frac{|K(z,w)|^2}{K(z,z)}\, dm(w).$$

To get the estimate for $z \in \mathcal{D} \setminus \mathcal{D}_\delta$, we set $E^* = \bigcup_{z\in\mathcal{D}\setminus\mathcal{D}_\delta} E(z,r)$. Then $\overline{E^*} \subset \mathcal{D}$ is compact, and for $z \in \overline{E^*}$ we have $r(z) > 0$ such that

$$|K(z,w)| \ge \tfrac{1}{2} K(z,z) > 0 \quad \text{for } w \in E(z, r(z)).$$

Select finitely many points $z_1, \ldots, z_k$ (k depends only on δ) such that

$$\overline{E^*} \subseteq \bigcup_{j=1}^{k} E(z_j, r(z_j)).$$

Then, for $z \in \mathcal{D} \setminus \mathcal{D}_\delta$,

$$\int_{E(z,r)} \sum_{|\alpha|=n} \left|\frac{\partial^n f}{\partial z^\alpha}(w)\right|^p |\varrho(w)|^{np-(m+1)} dm(w) \tag{2.11}$$
$$\le \int_{E^*} \left|\frac{\partial^n f}{\partial z^\alpha}(w)\right|^p |\varrho(w)|^{np-(m+1)} dm(w)$$
$$\le \sum_{j=1}^{k} \int_{E(z_j,r(z_j))} \sum_{|\alpha|=n} \left|\frac{\partial^n f}{\partial z^\alpha}(w)\right|^p |\varrho(w)|^{np-(m+1)} dm(w)$$
$$\le C \sum_{j=1}^{k} \int_{E(z_j,r(z_j))} \sum_{|\alpha|=n} \left|\frac{\partial^n f}{\partial z^\alpha}(w)\right|^p |\varrho(w)|^{np} \frac{|K(z_j,w)|^2}{K(z_j,z_j)}\, dm(w)$$
$$\le C \sum_{j=1}^{k} \int_{\mathcal{D}} \sum_{|\alpha|=n} \left|\frac{\partial^n f}{\partial z^\alpha}(w)\right|^p |\varrho(w)|^{np} \frac{|K(z_j,w)|^2}{K(z_j,z_j)}\, dm(w)$$
$$\le C \sup_{z\in\mathcal{D}} \int_{\mathcal{D}} \sum_{|\alpha|=n} \left|\frac{\partial^n f}{\partial z^\alpha}(w)\right|^p |\varrho(w)|^{np} \frac{|K(z,w)|^2}{K(z,z)}\, dm(w).$$

From (2.10) and (2.11) we see that the quantity (C) is less than or equal to C times the quantity (D).

3) (C) $\sim$ (B). For $r>0$ fixed and $\beta(z,w)<r$, by Lemma 1,

$$|\varrho(z)|^{m+1} \le C|E(z,r)| \le C|E(w,2r)| \le C|\varrho(w)|^{m+1}.$$

Hence

$$|\varrho(z)|/C \le |\varrho(w)| \le C|\varrho(z)| \quad \text{if } \beta(z,w)<r. \tag{2.12}$$

Now by Lemma 1 again, together with (2.12),

$$\begin{aligned}
&\frac{1}{|E(z,r)|^{1-np/(m+1)}} \int\limits_{E(z,r)} \sum_{|\alpha|=n} \left|\frac{\partial^n f}{\partial z^\alpha}(w)\right|^p dm(w) \\
&\qquad \sim C|\varrho(z)|^{np-(m+1)} \int\limits_{E(z,r)} \sum_{|\alpha|=n} \left|\frac{\partial^n f}{\partial z^\alpha}(w)\right|^p dm(w) \\
&\qquad \sim C \int\limits_{E(z,r)} \sum_{|\alpha|=n} \left|\frac{\partial^n f}{\partial z^\alpha}(w)\right|^p |\varrho(w)|^{np-(m+1)} dm(w).
\end{aligned}$$

4) (E) $\le C\cdot$(B). By Lemma 1 and the plurisubharmonicity of $\left|\frac{\partial^n f}{\partial z^\alpha}(w)\right|^p$ we get

$$\sum_{|\alpha|=n} \left|\frac{\partial^n f}{\partial z^\alpha}(z)\right| \le C\left\{\frac{1}{|E(z,r)|} \int\limits_{E(z,r)} \sum_{|\alpha|=n} \left|\frac{\partial^n f}{\partial z^\alpha}(w)\right|^p dm(w)\right\}^{1/p}.$$

Hence

$$\begin{aligned}
&\sum_{|\alpha|=n} \left|\frac{\partial^n f}{\partial z^\alpha}(z)\right| |\varrho(z)|^n \\
&\qquad \le C\left\{\frac{1}{|E(z,r)|^{1-np/(m+1)}} \int\limits_{E(z,r)} \sum_{|\alpha|=n} \left|\frac{\partial^n f}{\partial z^\alpha}(w)\right|^p dm(w)\right\}^{1/p}.
\end{aligned}$$

5) (D) $\le C\cdot$ (E). By the reproducing property of the Bergman kernel $K(z,w)$, we have

$$\begin{aligned}
&\int\limits_{\mathcal{D}} \sum_{|\alpha|=n} \left|\frac{\partial^n f}{\partial z^\alpha}(w)\right|^p |\varrho(w)|^{np} \frac{|K(z,w)|^2}{K(z,z)}\, dm(w) \\
&\qquad \le \left(\sup_{z\in\mathcal{D}} \sum_{|\alpha|=n} \left|\frac{\partial^n f}{\partial z^\alpha}(z)\right|^p |\varrho(z)|^{np}\right) \int\limits_{\mathcal{D}} \frac{K(z,w)K(w,z)}{K(z,z)} dm(w) \\
&\qquad \le C\left(\sup_{z\in\mathcal{D}} \sum_{|\alpha|=n} \left|\frac{\partial^n f}{\partial z^\alpha}(z)\right| |\varrho(z)|^n\right)^p.
\end{aligned}$$

This implies that (D) can be dominated by (E).

The proof is complete.

Proof of Theorem 2. Applying the method used in the proof of Theorem 1, we can easily prove that (A) and (E) are equivalent and that (D) implies (C), (C) implies (B) and (B) implies (E). To complete the proof we need only show that (E) implies (D).

Suppose f satisfies (E). Then

$$M(f) := \sup\Big\{ \sum_{|\alpha|=n} \Big|\frac{\partial^n f}{\partial z^\alpha}(z)\Big|^p |\varrho(z)|^{np} : z \in \mathcal{D}\Big\} < \infty,$$

and given $\varepsilon > 0$, there is $\delta > 0$ such that

$$\sum_{|\alpha|=n} \Big|\frac{\partial^n f}{\partial z^\alpha}(z)\Big|^p |\varrho(z)|^{np} < \varepsilon \quad \text{for } z \in \mathcal{D}_\delta.$$

Hence

$$\begin{aligned}\int_{\mathcal{D}} \sum_{|\alpha|=n} &\Big|\frac{\partial^n f}{\partial z^\alpha}(w)\Big|^p |\varrho(w)|^{np} \frac{|K(z,w)|^2}{K(z,z)}\, dm(w) \\ &= \Big(\int_{\mathcal{D}\setminus\mathcal{D}_\delta} + \int_{\mathcal{D}_\delta}\Big) \sum_{|\alpha|=n} \Big|\frac{\partial^n f}{\partial z^\alpha}(w)\Big|^p |\varrho(w)|^{np} \frac{|K(z,w)|^2}{K(z,z)}\, dm(w) \\ &\le M(f) \int_{\mathcal{D}\setminus\mathcal{D}_\delta} \frac{|K(z,w)|^2}{K(z,z)}\, dm(w) + \varepsilon \int_{\mathcal{D}_\delta} \frac{|K(z,w)|^2}{K(z,z)}\, dm(w) \\ &\le M(f) \int_{\mathcal{D}\setminus\mathcal{D}_\delta} \frac{|K(z,w)|^2}{K(z,z)}\, dm(w) + \varepsilon.\end{aligned}$$

Since $\mathcal{D} \setminus \mathcal{D}_\delta$ is compact, we know from [2] that $K(z,w)$ is bounded for $(z,w) \in \mathcal{D} \times (\mathcal{D} \setminus \mathcal{D}_\delta)$. Then by [5, Theorem 12] we have

$$\begin{aligned}\int_{\mathcal{D}\setminus\mathcal{D}_\delta} \frac{|K(z,w)|^2}{K(z,z)}\, dm(w) &\le C|\varrho(z)|^{n+1} \int_{\mathcal{D}\setminus\mathcal{D}_\delta} |K(z,w)|^2\, dm(w) \\ &\to 0 \quad \text{as } z \to \partial\mathcal{D}.\end{aligned}$$

Therefore if $|\varrho(z)| < \delta_1$ then

$$\int_{\mathcal{D}} \sum_{|\alpha|=n} \Big|\frac{\partial^n f}{\partial z^\alpha}(w)\Big|^p |\varrho(w)|^{np} \frac{|K(z,w)|^2}{K(z,z)}\, dm(w) < 2\varepsilon.$$

This ends the proof.

Acknowledgements. I would like to thank Dr. Kehe Zhu and Dr. Huiping Li for sending me their preprints. I am also very grateful to the referee for helpful suggestions.

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Reçu par la Rédaction le 16.6.1993;
en version modifiée le 15.10.1993