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COMPACTNESS PROPERTIES OF VECTOR-VALUED INTEGRATION MAPS IN LOCALLY CONVEX SPACES

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Introduction. The importance of vector measures in modern analysis is well established. The theory is particularly well developed in the Banach space setting; see, for example, [3, 4] and the references therein. Just as important is the setting of locally convex spaces (briefly, lcs), where many classical problems have their natural formulation. For example, this is the case for summability of series, spectral theory and resolutions of the identity of normal and, more generally, spectral operators, moment problems, Stone's theorem for representation of certain groups of operators, control theory, and so on; see [7, 8, 13], for example, and the references therein. The theory in such spaces is generally not attained from the Banach space case by simply replacing norms with seminorms; genuinely new phenomena and difficulties arise.

An important aspect of the theory is the integration map. Associated with each X-valued vector measure μ, with X a lcs, is its integration map $I_\mu : \mathcal{L}^1(\mu) \to X$ given by $f \mapsto \int f\, d\mu$, for every $f \in \mathcal{L}^1(\mu)$. Here $\mathcal{L}^1(\mu)$ is the space of all scalar-valued, μ-integrable functions; it is a lcs for the mean convergence topology (see Section 1). Many classical operators, such as the Fourier transform, certain integral operators (e.g. Volterra), representations for Boolean algebras of projections (arising from normal operators or spectral operators) and so on, can be viewed as integration maps I_μ (or restrictions of such maps) for suitable measures μ and spaces X. Properties of the operator I_μ, which is always linear and continuous, are closely related to the nature of the lcs $\mathcal{L}^1(\mu)$. One of the difficulties is the lack of knowledge about the space $\mathcal{L}^1(\mu)$, even for particular measures μ (e.g. the dual space of $\mathcal{L}^1(\mu)$). Moreover, certain basic properties of $\mathcal{L}^1(\mu)$ which always hold in the Banach space setting (e.g. completeness, existence of a control

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measure) become delicate problems for more general spaces X (see [8], for example). Such difficulties concerning $\mathcal{L}^1(\mu)$ are often transferred to the operator I_μ. For example, if X is a reflexive Banach space, then I_μ is always weakly compact. For a reflexive lcs X this need no longer be the case in general, where now weak compactness is meant in the sense of Grothendieck, that is, the image of some neighbourhood of zero (in the domain space) is relatively weakly compact (in the range space). Accordingly, known compactness properties of integration maps in the Banach space setting [11] are no automatic guides as what to expect in the lcs setting. The aim of this note is to investigate certain compactness properties of integration maps with values in a lcs.

Section 1 collects together some basic facts and notions needed in the sequel. In particular, the notion of an X-valued measure μ factoring through another space Y is introduced; see Definition 1.6. This notion is fundamental to the main result of Section 2 (viz. Theorem 2.1) which states that the integration map $I_\mu : \mathcal{L}^1(\mu) \to X$ is weakly compact if and only if there exist a quasicomplete lcs Y and a continuous linear map $j : Y \to X$ such that Y is a dense subspace of the projective limit of reflexive Banach spaces, the measure μ factors through Y and the integration map $I_\nu : \mathcal{L}^1(\nu) \to Y$ (where $\mu = j \circ \nu$) is weakly compact. If X happens to be complete, then Y can be chosen to be complete. Moreover, if the lcs X is a Fréchet space (i.e. metrizable), then Y can be chosen to be a (single) reflexive *Banach* space. This shows that the corresponding (known) result for X a Banach space [11; Proposition 2.1] has an "exact analogue" for Fréchet spaces X. However, it is not always possible to choose a single reflexive Banach space Y for a general non-metrizable lcs X (see Example 3.6).

In the final section we exhibit some examples. In particular, a characterization is given of those resolutions of the identity of spectral operators (e.g. normal operators) which have compact or weakly compact integration maps; they are precisely those measures which assume only finitely many values (cf. Proposition 3.8)!

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1. Preliminaries and basic results. All vector spaces to be considered are over the scalar field, either real or complex. The continuous dual space of a locally convex Hausdorff space X (briefly, lcHs) is denoted by X'. The set of all continuous seminorms on X is denoted by $\mathcal{P}(X)$; it is directed with respect to the natural order. The space X equipped with its weak topology

$\sigma(X, X')$ is denoted by $X_{\sigma(X,X')}$. The space X' equipped with its weak-star topology $\sigma(X', X)$ is denoted by $X_{\sigma(X',X)}$.

A sequence of vectors $x_n \in X$, $n = 1, 2, \ldots$, is said to be *summable* in X if there exists a vector $x \in X$ such that $\lim_{N\to\infty} p(x - \sum_{n=1}^{N} x_n) = 0$ for every $p \in \mathcal{P}(X)$. A sequence in X is called *unconditionally summable* if each of its subsequences is summable in X.

A subset H of X' is called *total* if it separates points of X.

We adopt the notation: $\langle x', x\rangle = x'(x)$ for every $x' \in X'$ and $x \in X$. Given an X-valued set function m on a σ-algebra of sets and an element $x' \in X'$, let $\langle x', m\rangle$ denote the set function defined by $\langle x', m\rangle(E) = \langle x', m(E)\rangle$ for every set E in the domain of m.

LEMMA 1.1 ([13; Proposition 0.1]). *The following statements on a total subset H of X' are equivalent.*

(i) *A sequence of vectors in X is unconditionally summable if and only if it is so with respect to the topology $\sigma(X, H)$.*

(ii) *An X-valued set function m on a σ-algebra is σ-additive if and only if the scalar-valued set function $\langle x', m\rangle$ is σ-additive for every $x' \in H$.*

A total subset H of X' is said to have the *Orlicz property* if it satisfies either (i) or (ii) of Lemma 1.1; see [13; §0]. The Orlicz–Pettis lemma (cf. [8; Theorem I.1.3]) can be rephrased to state that X' itself has the Orlicz property. For the Banach space case, the following result holds.

LEMMA 1.2 ([3; Corollary I.4.7]). *In the case when X is a Banach space, every total subset of X' has the Orlicz property if and only if X has no isomorphic copy of the Banach space ℓ^∞.*

Let $\mathcal{S}$ be a σ-algebra of subsets of a non-empty set Ω. Let $\mu : \mathcal{S} \to X$ be a vector measure, i.e. a σ-additive set function.

For every $x' \in X'$, the total variation measure of the scalar measure $\langle x', \mu\rangle$ is denoted by $|\langle x', \mu\rangle|$. Given $p \in \mathcal{P}(X)$, let $U_p^0 = \{x' \in X' : |\langle x', x\rangle| \le 1,\ x \in p^{-1}([0, 1])\}$. The *$p$-semivariation* of μ is the set function $p(\mu)$ on $\mathcal{S}$ defined by

$$p(\mu)(E) = \sup\{|\langle x', \mu\rangle|(E) : x' \in U_p^0\}, \qquad E \in \mathcal{S}.$$

For every $E \in \mathcal{S}$, let $E \cap \mathcal{S} = \{E \cap F : F \in \mathcal{S}\}$. The following result is well-known (cf. [3; Proposition I.1.11]).

LEMMA 1.3. *For every $p \in \mathcal{P}(X)$ and $E \in \mathcal{S}$,*

$$\sup_{F\in E\cap\mathcal{S}} p(\mu(F)) \le p(\mu)(E) \le 4 \sup_{F\in E\cap\mathcal{S}} p(\mu(F)).$$

A scalar-valued, $\mathcal{S}$-measurable function f on Ω is said to be *μ-integrable* if it is $\langle x', \mu\rangle$-integrable for every $x' \in X'$ and if there is a unique set function

$f\mu : \mathcal{S} \to X$ such that

$$\langle x', (f\mu)(E)\rangle = \int_E f\, d\langle x', \mu\rangle, \qquad x' \in X',\ E \in \mathcal{S}.$$

In this case, $f\mu$ is also σ-additive by the Orlicz–Pettis lemma, and will be called the *indefinite integral* of f with respect to μ. We also use the classical notation

$$\int_E f\, d\mu = (f\mu)(E), \qquad E \in \mathcal{S}.$$

The vector space of all μ-integrable functions on Ω will be denoted by $\mathcal{L}^1(\mu)$. A μ-integrable function is called *μ-null* if its indefinite integral is the zero vector measure. We denote by $\mathcal{N}(\mu)$ the subspace of $\mathcal{L}^1(\mu)$ consisting of all μ-null functions.

For every $p \in \mathcal{P}(X)$, the seminorm $f \mapsto p(f\mu)(\Omega)$, $f \in \mathcal{L}^1(\mu)$, will be denoted also by $p(\mu)$. The space $\mathcal{L}^1(\mu)$ will be equipped with the locally convex topology defined by the seminorms $p(\mu)$, $p \in \mathcal{P}(X)$. This topology is called the *mean convergence topology*. The lcHs associated with $\mathcal{L}^1(\mu)$ is the quotient space $\mathcal{L}^1(\mu)/\mathcal{N}(\mu)$.

LEMMA 1.4 ([10; Theorem 2.4]). *Suppose that the lcHs X is sequentially complete. Then a scalar-valued, $\mathcal{S}$-measurable function f on Ω is μ-integrable if and only if there exist scalar-valued $\mathcal{S}$-simple functions f_n on Ω, $n = 1, 2, \ldots,$ which are μ-almost everywhere convergent to f, such that the sequence $\{(f_n\mu)(E)\}_{n=1}^{\infty}$ is Cauchy in X for every $E \in \mathcal{S}$. In this case*

$$\int_E f\, d\mu = \lim_{n\to\infty} \int_E f_n\, d\mu, \qquad E \in \mathcal{S}.$$

The following result will be needed in Section 3.

LEMMA 1.5. *Let X be a Banach space and $\mu : \mathcal{S} \to X$ be a vector measure. Then the following statements are equivalent.*

(i) *The range $\mu(\mathcal{S})$ of μ is an infinite subset of X.*

(ii) *There exist infinitely many pairwise disjoint, non-μ-null sets in $\mathcal{S}$.*

(iii) *There exists a μ-integrable function which is not μ-essentially bounded.*

Proof. (i)$\Rightarrow$(ii). Statement (ii) follows from (i) because there exists a non-μ-null set $E \in \mathcal{S}$ which can be partitioned into infinitely many pairwise disjoint non-μ-null subsets.

(ii)$\Rightarrow$(i). The σ-additivity of μ shows that it is impossible for μ to take the same non-zero value on infinitely many pairwise disjoint, non-μ-null sets. Accordingly, $\mu(\mathcal{S})$ must be an infinite set.

(ii)⇒(iii). Let $\|\mu\|$ denote the semivariation of μ. If $\{E_n\}_{n=1}^{\infty}$ is a sequence of pairwise disjoint, non-μ-null sets in $\mathcal{S}$, then we can choose a subsequence $\{E_{n(k)}\}_{k=1}^{\infty}$ such that $\|\mu\|(E_{n(k)}) < k^{-3}$ for each $k = 1, 2, \ldots$ Then the function $\sum_{k=1}^{\infty} k\chi_{E_{n(k)}}$ is μ-integrable, but not μ-essentially bounded.

(iii)⇒(ii). Let $f \in \mathcal{L}^1(\mu)$ be a function which is not μ-essentially bounded. Define pairwise disjoint sets $E_n = \{\omega \in \Omega : n < |f(\omega)| \le n+1\}$, for all $n = 1, 2, \ldots$ Since f is not μ-essentially bounded, there must be a subsequence $\{E_{n(k)}\}_{k=1}^{\infty}$ consisting entirely of non-μ-null sets. ■

The *integration map* $I_\mu : \mathcal{L}^1(\mu) \to X$ is defined by

$$I_\mu(f) = (f\mu)(\Omega) = \int_{\Omega} f\, d\mu, \qquad f \in \mathcal{L}^1(\mu).$$

It is clear that I_μ is linear and continuous.

Definition 1.6. The vector measure $\mu : \mathcal{S} \to X$ is said to *factor* through a lcHs Y if there exist a vector measure $\nu : \mathcal{S} \to Y$ and a continuous linear map $j : Y \to X$ such that

(F1) $\mathcal{L}^1(\mu) = \mathcal{L}^1(\nu)$ as locally convex spaces,
(F2) $\mathcal{N}(\mu) = \mathcal{N}(\nu)$ as sets, and
(F3) $I_\mu = j \circ I_\nu$.

In this case we say that μ factors through Y *via* ν and j.

Consider the following condition, which is weaker than (F1):

(F1)′ $\mathcal{L}^1(\mu) = \mathcal{L}^1(\nu)$ as vector spaces.

If X and Y are Banach spaces, then (F1) can be replaced by (F1)′ because (F1)′, (F2) and (F3) jointly imply (F1) by the open mapping theorem; see [11; Section 1]. This is not the case in general. To see this, let $\mathbb{N}$ denote the positive integers and $2^{\mathbb{N}}$ its power set (which is a σ-algebra).

Example 1.7. Let $X = \ell^1_{\sigma(\ell^1, c_0)}$ and Y be the Banach space ℓ^1. Let $\nu : 2^{\mathbb{N}} \to Y$ be the vector measure defined by $\nu(E) = \sum_{n\in E} n^{-2}\, \mathbf{e}_n$, for every $E \in 2^{\mathbb{N}}$, where $\{\mathbf{e}_n\}_{n=1}^{\infty}$ is the usual basis for Y. Let $j : Y \to X$ be the identity map and let $\mu = j \circ \nu$. Then $\mathcal{L}^1(\mu)$ and $\mathcal{L}^1(\nu)$ are the same vector space consisting of all scalar functions f on $\mathbb{N}$ satisfying $\sum_{n=1}^{\infty} |f(n)| n^{-2} < \infty$. So (F1)′ holds. Clearly $\mathcal{N}(\mu) = \mathcal{N}(\nu) = \{0\}$. The condition (F3) is obvious. However, (F1) is not valid. In fact, if (F1) were valid, then $\mathcal{L}^1(\mu)$ would be normable because $\mathcal{L}^1(\nu)$ is. Hence, there would exist a vector $\xi = (\xi_n)_{n=1}^{\infty} \in c_0$ satisfying $\xi_n > 0$ for every $n \in \mathbb{N}$ such that $\mathcal{L}^1(\mu) = \mathcal{L}^1(\langle \xi, \mu\rangle)$. This contradicts the fact that $\mathcal{L}^1(\mu)$ is strictly smaller than $\mathcal{L}^1(\langle \xi, \mu\rangle)$. ■

Let j be a continuous linear map from a lcHs Y into a lcHs X and $j' : X' \to Y'$ be its adjoint map, that is, $\langle j'(x'), y\rangle = \langle x', j(y)\rangle$, $x' \in X'$, $y' \in Y'$. If j is injective, then $j'(X')$ is a total subset of Y'.

PROPOSITION 1.8. *Let X be a sequentially complete lcHs and $\mu : \mathcal{S} \to X$ be a vector measure. Suppose that Y is a linear subspace of X containing the vector space $I_\mu(\mathcal{L}^1(\mu))$ and equipped with a locally convex topology stronger than that induced by X, with respect to which Y is sequentially complete. Let $j : Y \to X$ be the natural injection. If the total subset $j'(X')$ of Y' has the Orlicz property, then the set function $\nu : \mathcal{S} \to Y$ defined by $j \circ \nu = \mu$ is σ-additive, and* (F1)$'$, (F2) *and* (F3) *hold.*

Furthermore, if there is a neighbourhood V of 0 in $\mathcal{L}^1(\mu)$ such that $I_\nu(V)$ is a bounded subset of Y, then (F1) *holds, so that μ factors through Y via ν and j.*

Proof. By the definition of the Orlicz property, ν is σ-additive. The continuity of j implies that $\mathcal{L}^1(\nu) \subset \mathcal{L}^1(\mu)$. To prove the converse, let $f \in \mathcal{L}^1(\mu)$. We may assume that f is non-negative. Choose non-negative, $\mathcal{S}$-simple functions f_n on Ω, $n = 1, 2, \ldots$, such that $\sum_{n=1}^{\infty} f_n = f$ pointwise on Ω.

Let $E \in \mathcal{S}$. By the Lebesgue dominated convergence theorem for vector measures [8; Theorem II.4.2], the sequence $\{f_n\}_{n=1}^{\infty}$ is unconditionally summable in $\mathcal{L}^1(\mu)$ and hence, so is the sequence $\{(f_n\mu)(E)\}_{n=1}^{\infty}$ in X. Since $f_n \in \mathcal{L}^1(\nu)$ and $j \circ (f_n\nu)(E) = (f_n\mu)(E)$ for every $n = 1, 2, \ldots$, and since $I_\mu(\mathcal{L}^1(\mu)) \subset Y$, the sequence $\{(f_n\nu)(E)\}_{n=1}^{\infty}$ is unconditionally summable in Y with respect to the topology $\sigma(Y, j'(X'))$. The Orlicz property of $j'(X')$ implies that $\{(f_n\nu)(E)\}_{n=1}^{\infty}$ is unconditionally summable in Y. In particular, the sequence $\{\sum_{n=1}^{N}(f_n\nu)(E)\}_{N=1}^{\infty}$ is Cauchy in Y. By Lemma 1.4, we have $f \in \mathcal{L}^1(\nu)$. Hence (F1)$'$ holds. Now (F3) is clear. The injectivity of j implies (F2).

To show the second statement, take any $p \in \mathcal{P}(X)$ such that

$$\{g \in \mathcal{L}^1(\mu) : p(\mu)(g) \le 1\} \subset V. \tag{1}$$

Denote the left-hand-side of (1) by V_p. Let $q \in \mathcal{P}(Y)$ be arbitrary. The boundedness of $I_\nu(V_p)$ in Y implies that

$$I_\nu(V_p) \subset C_q\{y \in Y : q(y) \le 1\},$$

for some constant $C_q > 0$. Let $g \in \mathcal{L}^1(\mu)$. If $p(\mu)(g) \neq 0$, then it follows easily that

$$q(I_\nu g) \le C_q p(\mu)(g). \tag{2}$$

If $p(\mu)(g) = 0$, then $\alpha g \in V_p \subset V$ and so $\alpha I_\nu g \in I_\nu(V_p)$, for all scalars α. Since $I_\nu(V_p)$ is bounded, this forces $I_\nu g = 0$ and so again (2) holds. Accordingly, (2) holds for every $g \in \mathcal{L}^1(\mu)$. It then follows from Lemma 1.3 that

$$q(\nu)(g) \le 4 \sup_{E\in\mathcal{S}} q\Big(\int_E g\, d\nu\Big) \le 4C_q p(\mu)(g), \qquad g \in \mathcal{L}^1(\mu). \tag{3}$$

This shows that the identity map $\Phi : \mathcal{L}^1(\mu) \to \mathcal{L}^1(\nu)$ is continuous. On the other hand, Φ^{-1} is continuous because j is continuous. Hence, (F1) holds. ■

2. Weakly compact integration maps. Throughout this section, let X denote a quasicomplete lcHs and let μ denote an X-valued vector measure on a σ-algebra $\mathcal{S}$ of subsets of a non-empty set Ω. Recall that the associated integration map $I_\mu : \mathcal{L}^1(\mu) \to X$ is defined by $I_\mu f = (f\mu)(\Omega) = \int_\Omega f\,d\mu$, for all $f \in \mathcal{L}^1(\mu)$.

The aim of this section is to establish the following characterization of weak compactness of the integration map I_μ.

THEOREM 2.1. *The integration map $I_\mu : \mathcal{L}^1(\mu) \to X$ is weakly compact if and only if there exist a quasicomplete lcHs Y, a vector measure $\nu : \mathcal{S} \to Y$ and a continuous linear map $j : Y \to X$ such that*

(i) *Y is a dense subspace of the projective limit of reflexive Banach spaces*;

(ii) *μ factors through Y via ν and j, and*

(iii) *the integration map $I_\nu : \mathcal{L}^1(\nu) \to Y$ is weakly compact.*

If the space X happens to be complete, *then the space Y can be chosen to be complete.*

The "if" portion of the above theorem is clear. To prove the "only if" portion, assume that I_μ is weakly compact. By definition, there is a convex and balanced neighbourhood V of 0 in $\mathcal{L}^1(\mu)$ such that its image $I_\mu(V)$ is relatively weakly compact in X. We fix the set V and let $K = I_\mu(V)$, in which case K is also convex and balanced.

Let $p \in \mathcal{P}(X)$. Let X_p denote the *completion* of the quotient normed space $X/p^{-1}(\{0\})$. The natural projection from X into X_p is denoted by π_p. We denote by B_p the closed unit ball of X_p. Given $q \in \mathcal{P}(X)$ satisfying $q \geq p$, let $\alpha_{pq} : X_q \to X_p$ be the continuous linear map satisfying $\alpha_{pq} \circ \pi_q = \pi_p$. For every $n = 1, 2, \ldots$, let p_n denote the gauge of the subset $2^n\pi_p(K) + 2^{-n}B_p$ of X_p. Let W_p denote the space of elements w_p of X_p such that

$$\Big(\sum_{n=1}^{\infty} p_n(w_p)^2\Big)^{1/2} < \infty. \tag{4}$$

For every $w_p \in W_p$, let $\|w_p\|$ denote the left-hand side of (4). Then $\|\cdot\|_p$ is a norm on W_p.

LEMMA 2.2 ([2; Lemma 1]). *Let $p \in \mathcal{P}(X)$. Then*

(i) *the normed space W_p is a reflexive Banach space*;

(ii) *the set $\pi_p(K)$ is contained in the closed unit ball of W_p*;

(iii) *the natural injection $j_p : W_p \to X_p$ is continuous.*

Let p and q be continuous seminorms on X such that $p \le q$. For every $n = 1, 2, \ldots$, we have

$$\alpha_{pq}(2^n \pi_q(K) + 2^{-n} B_q) \subset 2^n \pi_p(K) + 2^{-n} B_p,$$

and hence

$$\|\alpha_{pq}(w_q)\|_p \le \|w_q\|_q, \quad w_q \in W_q. \tag{5}$$

In other words, $\alpha_{pq}(W_q) \subset W_p$. Let $\beta_{pq} : W_q \to W_p$ denote the restriction of α_{pq} to W_q. Then β_{pq} is continuous by (5), and $\alpha_{pq} \circ j_q = j_p \circ \beta_{pq}$.

The definition of projective limits can be found, for example, in [12; p. 52]. Let W be the projective limit of the family of reflexive Banach spaces $\{W_p : p \in \mathcal{P}(X)\}$ with respect to the maps β_{pq} ($p, q \in \mathcal{P}(X)$, $p \le q$). That is, W is the space of elements $(w_p)_{p\in\mathcal{P}(X)}$ of the product space $\prod_{p\in\mathcal{P}(X)} W_p$ such that $w_p = \beta_{pq}(w_q)$ whenever $p, q \in \mathcal{P}(X)$ satisfy $p \le q$, and the topology of W is induced by $\prod_{p\in\mathcal{P}(X)} W_p$. Then W is complete because so is W_p for every $p \in \mathcal{P}(X)$. By $k_p : W \to W_p$ is denoted the natural projection, for every $p \in \mathcal{P}(X)$.

The space X will be identified with a dense subspace of its completion $\widehat{X}$ which is realized as the projective limit of the family of Banach spaces $\{X_p : p \in \mathcal{P}(X)\}$ with respect to the maps α_{pq} ($p, q \in \mathcal{P}(X)$, $p \le q$). With this identification, the set K will be identified with the subset

$$\{(\pi_p(x))_{p\in\mathcal{P}(X)} : x \in K\}$$

of $X \subset \widehat{X} \subset \prod_{p\in\mathcal{P}(X)} X_p$. Then K is contained in W because $\pi_p(K) \subset W_p$ for every $p \in \mathcal{P}(X)$.

Let the linear space $Y = X \cap W$ be equipped with the topology induced by W. Clearly $K \subset Y$. The space Y may not be dense in W. So, consider the closure Z of Y in W and the closure Z_p of $k_p(Y)$ in W_p, for every $p \in \mathcal{P}(X)$. For seminorms $p, q \in \mathcal{P}(X)$ such that $p \le q$, let $\gamma_{pq} : Z_q \to Z_p$ be the restriction of β_{pq} to Z_q. Equip Z and Z_p, $p \in \mathcal{P}(X)$, with the topologies induced by W and W_p, $p \in \mathcal{P}(X)$, respectively. Then Z_p is a reflexive Banach space for every $p \in \mathcal{P}(X)$. Furthermore, Z is the projective limit of the family $\{Z_p : p \in \mathcal{P}(X)\}$ with respect to the maps γ_{pq} ($p, q \in \mathcal{P}(X)$, $p \le q$). Then Y is a dense subspace of the complete space Z, and $Y = X \cap Z$.

LEMMA 2.3. *The lcHs Y is quasicomplete and K is a relatively weakly compact subset of Y.*

Proof. An arbitrary bounded Cauchy net in Y has a limit z in the complete space Z and a limit x in the quasicomplete space X. Since the topologies on X and Z are stronger than or equal to those induced by the Hausdorff space $\widehat{X}$, we conclude that $x = z$, which is a member of Y. Hence, Y is quasicomplete.

The subset K of Y is homeomorphic to a subset of $\prod_{p\in\mathcal{P}(X)}\pi_p(K)$ which is relatively weakly compact in the product space $\prod_{p\in\mathcal{P}(X)}Z_p$ (cf. [12; Theorem IV.4.3]); hence, K is relatively weakly compact in the quasicomplete space Y. ∎

Let $j : Y \to X$ denote the natural injection; then $j'(X')$ is a total subset of Y'.

LEMMA 2.4. *The subset $j'(X')$ of Y' has the Orlicz property.*

Proof. Suppose that m is a Y-valued set function defined on a σ-algebra of subsets of a non-empty set such that $\langle j'(x'), m\rangle$ is σ-additive for every $x' \in X'$. Then the X-valued set function $j \circ m$ is σ-additive by the Orlicz–Pettis lemma.

Let $p \in \mathcal{P}(X)$. Recall that $k_p : W \to W_p$ is the natural projection. Since $\pi_p \circ j \circ m = j_p \circ k_p \circ m$, the X_p-valued set function $j_p \circ k_p \circ m$ is σ-additive. In other words, $\langle j_p'(x_p'), k_p \circ m\rangle$ is σ-additive for every $x_p' \in X_p'$. Since the reflexive Banach space Z_p has no isomorphic copy of ℓ^∞, it follows from Lemma 1.2 that $j_p'(X_p')$ has the Orlicz property, so that $k_p \circ m$ is σ-additive. Since p is arbitrary, we deduce the σ-additivity of m and the statement of the lemma follows. ∎

Proof of Theorem 2.1. To show the "only if" portion, we have fixed a convex, balanced neighbourhood V of 0 in $\mathcal{L}^1(\mu)$ whose image $K = I_\mu(V)$ is relatively weakly compact in X. The space Y clearly satisfies statement (i).

Since V is absorbing and since $I_\mu(V) = K \subset Y$, we have $I_\mu(\mathcal{L}^1(\mu)) \subset Y$.

Let $\nu : \mathcal{S} \to Y$ be the set function defined by $j \circ \nu = \mu$; then ν is σ-additive by Lemma 2.4. Statement (ii) follows from Proposition 1.8.

Statement (iii) is a consequence of (ii) and Lemma 2.3.

Finally, suppose that X is complete. Then $Y = X \cap W = \widehat{X} \cap W = W$. So, the second half of Theorem 2.1 holds. This completes the proof of the theorem. ∎

Remark 2.5. If the space X happens to be a Fréchet space, then there exists a reflexive Banach space Y such that the statement of Theorem 2.1 is valid. This can be proved as in [11; Proposition 2.1] by applying [2; Remark 2]. In the general case, there exists a vector measure μ which does not factor through any reflexive Banach space while its associated integration map I_μ is weakly compact (see Example 3.6). ∎

Remark 2.6. The statement of Theorem 2.1 is still valid if we replace "weakly compact" by "compact" there. For, suppose that I_μ is compact. Take a convex, balanced neighbourhood V of 0 in $\mathcal{L}^1(\mu)$ whose image $K = I_\mu(V)$ is a relatively compact subset of X. Then $\pi_p(K)$ is a relatively compact subset of W_p (cf. [1; Theorem 17.19]) and hence of Z_p for every

$p \in \mathcal{P}(X)$. Therefore K is relatively compact in Y because it is homeomorphic to a subset of the relatively compact set $\prod_{p\in\mathcal{P}(X)} \pi_p(K)$ in $\prod_{p\in\mathcal{P}(X)} Z_p$ and because Y is quasicomplete. The converse statement is clear. ■

3. Examples. Given a vector measure μ, the space $\mathcal{L}^1(\mu)$ of all μ-integrable functions will be identified with its associated Hausdorff space $\mathcal{L}^1(\mu)/\mathcal{N}(\mu)$ in order to make the presentation simpler.

We begin with two "concrete" examples; in both $\mathcal{S}$ denotes the σ-algebra of Borel subsets of the unit interval $\Omega = [0,1]$, λ is Lebesgue measure on $\mathcal{S}$ and $\|\cdot\|_1$ is the usual norm in $\mathcal{L}^1(\lambda)$.

Example 3.1. Let $X = \ell^1_{\sigma(\ell^1,c_0)}$, in which case X is quasicomplete and semireflexive. Let $\mu : \mathcal{S} \to X$ be the vector measure given by

$$\mu(E) = \Big(n^{-2} \int_E t^{n-1}\,dt\Big)_{n=1}^{\infty}, \qquad E \in \mathcal{S}.$$

Then an $\mathcal{S}$-measurable function $f : \Omega \to \mathbb{C}$ belongs to $\mathcal{L}^1(\mu)$ if and only if $\sum_{n=1}^{\infty} n^{-2} \int_0^1 t^{n-1}|f(t)|\,dt$ is finite, in which case

$$\int_E f\,d\mu = \Big(n^{-2} \int_E t^{n-1} f(t)\,dt\Big)_{n=1}^{\infty}, \qquad E \in \mathcal{S}.$$

Each element $\xi = (\xi_n)_{n=1}^{\infty}$ of $c_0 = X'$ determines a seminorm q_ξ for the topology of X given by

$$q_\xi(x) = |\langle \xi, x\rangle|, \qquad x \in X.$$

This seminorm then induces a continuous seminorm $q_\xi(\mu)$ on $\mathcal{L}^1(\mu)$ (for the definition, see Section 1).

Now let $\xi = (1,0,0,\ldots)$. Then

$$q_\xi(\mu)(f) = \int_0^1 |f(t)|\,dt = \|f\|_1, \qquad f \in \mathcal{L}^1(\mu),$$

so that $V_\xi = \{f \in \mathcal{L}^1(\mu) : \|f\|_1 \le 1\}$ is a neighbourhood of 0 in $\mathcal{L}^1(\mu)$. Accordingly, the integration map $I_\mu : \mathcal{L}^1(\mu) \to X$ satisfies

$$I_\mu(V_\xi) = \Big\{ \int_0^1 f\,d\mu : f \in \mathcal{L}^1(\mu),\ \|f\|_1 \le 1\Big\}.$$

Let Y denote the Banach space ℓ^1 with the usual norm $\|\cdot\|_Y$. Then, with $C = \sum_{n=1}^{\infty} n^{-2}$, it follows that

$$\Big\| \int_0^1 f\,d\mu\Big\|_Y = \sum_{n=1}^{\infty} n^{-2} \int_0^1 t^{n-1}|f(t)|\,dt \le C\|f\|_1, \qquad f \in \mathcal{L}^1(\mu),$$

and hence, $I_\mu(V_\xi) \subset C\{y \in Y : \|y\|_Y \le 1\}$ is a norm bounded set in Y (and, of course, a bounded set in X). The Banach–Alaoglu theorem implies that $I_\mu(V_\xi)$ is relatively compact (= weakly compact) in X. That is, I_μ is compact (= weakly compact). ∎

EXAMPLE 3.2. Let $X = \mathbb{C}^{\mathbb{N}}$, equipped with the seminorms given by

$$q_n : x \mapsto \max_{1\le r\le n} |x_r|, \qquad x = (x_i)_{i=1}^\infty \in X,$$

for all $n = 1, 2, \ldots$ Then X is a separable, reflexive Fréchet space. The set function $\mu : \mathcal{S} \to X$ given by

$$\mu(E) = \Big(\int_E t^{n-1}\, dt\Big)_{n=1}^\infty, \qquad E \in \mathcal{S},$$

is a vector measure. Since $\int_E f\, d\mu = (\int_E t^{n-1} f(t)\, dt)_{n=1}^\infty$, for every $E \in \mathcal{S}$ and $f \in \mathcal{L}^1(\mu)$, it is clear that $\mathcal{L}^1(\mu)$ and $\mathcal{L}^1(\lambda)$ coincide as vector spaces. A direct computation shows that

$$q_n(\mu)(f) \le 4\|f\|_1 = 4q_1(\mu)(f), \qquad f \in \mathcal{L}^1(\mu),$$

for each $n = 1, 2, \ldots$ Accordingly, $\mathcal{L}^1(\mu)$ and $\mathcal{L}^1(\lambda)$ are isomorphic.

Now let $V = q_1(\mu)^{-1}([0,1])$, which is a bounded neighbourhood of 0 in $\mathcal{L}^1(\mu)$. Its image $I_\mu(V)$ under the integration map $I_\mu : \mathcal{L}^1(\mu) \to X$ is a bounded subset of the Montel space X, so that $I_\mu(V)$ is relatively compact in X. Therefore, I_μ is compact (= weakly compact). ∎

We wish to exhibit a vector measure μ for which I_μ is weakly compact, but μ does not factor through any reflexive Banach space. First a preliminary result is needed.

DEFINITION 3.3. Let T be a continuous linear map from a lcHs U into a lcHs W. We say that T *factors* through a lcHs Z if there exist continuous linear maps $R : U \to Z$ and $S : Z \to W$ such that $T = S \circ R$.

Remark 3.4. If a vector measure $\mu : \mathcal{S} \to X$ factors through a lcHs Y (cf. Definition 1.6), then the associated integration map $I_\mu : \mathcal{L}^1(\mu) \to X$ also factors through Y. Indeed, let $\nu : \mathcal{S} \to Y$ and $j : Y \to X$ be as in Definition 1.6. By (F1) there is a bicontinuous isomorphism $\Phi : \mathcal{L}^1(\mu) \to \mathcal{L}^1(\nu)$ and hence, $R = I_\nu \circ \Phi$ is continuous from $\mathcal{L}^1(\mu)$ into Y. Then $I_\mu = S \circ R$ (where $S = j$) is a factorization of I_μ through Y. ∎

LEMMA 3.5. *Let the lcHs X be a non-reflexive Pták space. Let T be a bijective, continuous linear map from X onto a lcHs Y. Then T does not factor through any reflexive Banach space.*

Proof. Suppose that there is a reflexive Banach space Z and continuous linear maps $R : X \to Z$ and $S : Z \to Y$ such that $T = S \circ R$. Without loss of generality we may assume that S is an injection. Otherwise, if $\pi : Z \to Z/S^{-1}(\{0\})$ is the quotient map and $\underline{S} : Z/S^{-1}(\{0\}) \to Y$ is the continuous linear map such that $\underline{S} \circ \pi = S$, then $T = \underline{S} \circ \underline{R}$ with $\underline{S} : \underline{Z} \to Y$ injective and $\underline{R} = \pi \circ R : X \to \underline{Z}$, where $\underline{Z} = Z/S^{-1}(\{0\})$ is again a reflexive Banach space. Since T is surjective, so is S and hence, S is a bijection. Accordingly, $R = S^{-1} \circ T$ is a bijection of the Pták space X onto Z. Since Z is barrelled, R is a nearly open map and hence, R^{-1} is also continuous [12; Corollary 1, p. 164], that is, R is a bicontinuous isomorphism of X onto Z, which contradicts the fact that X is non-reflexive. ∎

We note that every Fréchet lcs is a Pták space and hence, in particular, Banach spaces are Pták spaces.

Example 3.6. The duality between the spaces c and ℓ^1 is given by

$$\langle \xi, x\rangle = (x_1 \lim_{n\to\infty} \xi_n) + \sum_{n=2}^{\infty} x_n \xi_{n-1},$$

for every $\xi = (\xi_n)_{n=1}^\infty \in c$ and $x = (x_n)_{n=1}^\infty \in \ell^1$ (cf. [9; §14.7]). Let $X = \ell^1_{\sigma(\ell^1,c)}$, in which case $X' = c$. Let $\mu : 2^{\mathbb{N}} \to X$ be the vector measure defined by

$$\mu(E) = \sum_{n\in E} n^{-2}\, \mathbf{e}_n, \qquad E \in 2^{\mathbb{N}}.$$

The associated integration map $I_\mu : \mathcal{L}^1(\mu) \to X$ is then bijective. Let $\mathbf{1}$ denote the member of c in which every coordinate is 1. Then the space $\mathcal{L}^1(\mu)$ is isomorphic to the non-reflexive Pták space $\mathcal{L}^1(\langle \mathbf{1}, \mu\rangle)$. In particular, $\mathcal{L}^1(\mu)$ is normable, so that I_μ is compact because every bounded subset of X is relatively compact by the Banach–Alaoglu theorem. However, it follows from Lemma 3.5 that I_μ cannot factor through any reflexive Banach space. In other words, neither does μ. ∎

We conclude this section by considering an important class of measures arising in operator theory. Let X be a Banach space and $L(X)$ be the space of all continuous linear operators of X into X. With respect to the topology of pointwise convergence in X (i.e. the strong operator topology), $L(X)$ is a quasicomplete lcHs; it is denoted by $L_s(X)$. An *operator-valued measure* is a set function $P : \mathcal{S} \to L(X)$, where $\mathcal{S}$ is a σ-algebra of subsets of some set Ω, which is σ-additive in $L_s(X)$. If, in addition, $P(\Omega) = I$ (the identity operator in X) and P is multiplicative (i.e. $P(E \cap F) = P(E)P(F)$, for all $E, F \in \mathcal{S}$), then P is called a *spectral measure*. Examples of such measures are provided by the resolution of the identity of any normal operator T in a Hilbert space, where $\Omega = \sigma(T)$ is the spectrum of T and $\mathcal{S}$ is the σ-algebra of Borel subsets of $\sigma(T)$. Or, if $X = \mathcal{L}^p(\lambda)$, $1 \le p < \infty$, where $\lambda : \mathcal{S} \to [0,\infty]$ is some σ-additive measure, then the projectionvalued set function given by $E \mapsto P(E)$, $E \in \mathcal{S}$, where $P(E)f = \chi_E f$, for every $f \in \mathcal{L}^p(\lambda)$, is a spectral measure. The following result is needed later.

LEMMA 3.7. *Let X be a Banach space and $P : \mathcal{S} \to L_s(X)$ be a spectral measure.*

(i) *Every P-integrable function is P-essentially bounded.*

(ii) *If the range $P(\mathcal{S})$, of P, is a closed subset of $L_s(X)$, then the lcHs $\mathcal{L}^1(P)$ is complete and the integration map $I_P : \mathcal{L}^1(P) \to L_s(X)$ is a bicontinuous isomorphism onto its range $I_P(\mathcal{L}^1(P)) \subset L_s(X)$, equipped with the relative topology from $L_s(X)$.*

Proof. For part (i) see [6; Theorem XVIII.2.1]. Part (ii) follows from [5; Propositions 1.4 and 1.5]. ■

We can now provide a complete characterization of weakly compact integration maps associated with spectral measures.

PROPOSITION 3.8. *Let X be a Banach space and $P : \mathcal{S} \to L_s(X)$ be a spectral measure with range $P(\mathcal{S})$ a closed subset of $L_s(X)$. Then the integration map $I_P : \mathcal{L}^1(P) \to L_s(X)$ is weakly compact if and only if its range $P(\mathcal{S})$ is a finite subset of $L(X)$.*

Proof. Suppose that $P(\mathcal{S})$ is a finite subset of $L_s(X)$. Then $I_P(\mathcal{L}^1(P))$ coincides with the closed linear span of $P(\mathcal{S})$ because $\mathcal{S}$-simple functions are dense in $\mathcal{L}^1(P)$ (cf. Lemma 1.4). In particular, $I_P(\mathcal{L}^1(P))$ is finite-dimensional, so that I_P is weakly compact.

Conversely, suppose that I_P is weakly compact. Choose a neighbourhood V of 0 in $\mathcal{L}^1(P)$ such that $I_P(V)$ is relatively weakly compact in $L_s(X)$. By Lemma 3.7(ii) the range $W = I_P(\mathcal{L}^1(P))$, equipped with the relative topology from $L_s(X)$, is a complete lcHs with a weakly compact neighbourhood of 0. Accordingly, W is normable and hence, by completeness, is a Banach space. So, we may consider $P : \mathcal{S} \to W$ as a Banach space-valued measure. By Lemma 3.7(i) the only P-integrable functions are the P-essentially bounded ones. Therefore, Lemma 1.5 implies that $P(\mathcal{S})$ is a finite set. ■

Remark 3.9. (i) The condition that the range $P(\mathcal{S})$ be closed in $L_s(X)$ is automatically satisfied in separable Banach spaces X.

(ii) A simple consequence of Proposition 3.8 is that the integration map of a spectral measure is weakly compact if and only if it is compact, which happens if and only if it is nuclear. ■

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