COLLOQUIUM MATHEMATICUM

VOL. LXVI 1993 FASC. 2

ALGEBRAS STABLY EQUIVALENT TO TRIVIAL EXTENSIONS OF HEREDITARY ALGEBRAS OF TYPE $\widetilde{A}_n$

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The study of stable equivalences of finite-dimensional algebras over an algebraically closed field seems to be far from satisfactory results. The importance of problems concerning stable equivalences grew up when derived categories appeared in representation theory of finite-dimensional algebras [8]. The Tachikawa–Wakamatsu result [17] also reveals the importance of these problems in the study of tilting equivalent algebras (compare with [1]). In fact, the result says that if A and B are tilting equivalent algebras then their trivial extensions $T(A)$ and $T(B)$ are stably equivalent. Consequently, there is a special need to describe algebras that are stably equivalent to the trivial extensions of tame hereditary algebras.

In the paper, there are studied algebras which are stably equivalent to the trivial extensions of hereditary algebras of type $\widetilde{A}_n$, that is, algebras given by quivers whose underlying graphs are of type $\widetilde{A}_n$. These algebras are isomorphic to the trivial extensions of very nice algebras (see Theorem 1). Moreover, in view of [1, 8], Theorem 2 shows that every stable equivalence of such algebras is induced in some sense by a derived equivalence of well chosen subalgebras.

In our study of stable equivalence, we shall use methods and results from [11]. We shall also use freely information on Auslander–Reiten sequences which can be found in [2].

1. Preliminaries. Let K be a fixed algebraically closed field. Throughout the paper, we shall consider finite-dimensional associative K-algebras with identity that will be assumed to be basic and connected. Such algebras are defined by their bound quivers [6]. We shall denote by Q_Λ the ordinary quiver of a finite-dimensional K-algebra Λ. A finite-dimensional algebra Λ will be called *triangular* whenever Q_Λ has no oriented cycles.

1991 *Mathematics Subject Classification*: Primary 16G20.

Supported by Polish Sci. Grant KBN 1222/2/91.

For each vertex i of Q_Λ, we shall denote by S_i the corresponding simple Λ-module. P_i (respectively E_i) will denote the projective cover (respectively, injective envelope) of S_i. For every Λ-module M, $\mathrm{rad}(M)$ will denote the radical of M, $\mathrm{soc}(M)$ the socle of M, and $\mathrm{top}(M)$ the top of M.

For every finite-dimensional algebra Λ, we shall denote by Λ-mod the category of all finite-dimensional left Λ-modules. The stable category Λ-$\underline{\mathrm{mod}}$ of the category Λ-mod is defined as follows. The objects of Λ-$\underline{\mathrm{mod}}$ are the modules from Λ-mod having no projective direct summands. For any two objects M, N in Λ-$\underline{\mathrm{mod}}$ the group of morphisms from M to N in Λ-$\underline{\mathrm{mod}}$ is the quotient

$$\underline{\mathrm{Hom}}_\Lambda(M,N) = \mathrm{Hom}_\Lambda(M,N)/\mathcal{P}(M,N),$$

where $\mathcal{P}(M,N)$ is the subspace of $\mathrm{Hom}_\Lambda(M,N)$ consisting of all Λ-homomorphisms which factor through projective Λ-modules. If $f \in \mathrm{Hom}_\Lambda(M,N)$ we shall denote by $\underline{f}$ its coset modulo $\mathcal{P}(M,N)$.

τ will always denote the Auslander–Reiten translate.

Following Drozd [4] an algebra Λ is called *tame* if for any dimension d, there is a finite number of Λ-$K[X]$-bimodules Q_i, $1 \le i \le n_d$, which are finitely generated and free as right $K[X]$-modules such that all but a finite number of isomorphism classes of indecomposable Λ-modules of dimension d are of the form $Q_i \otimes_{K[X]} K[X]/(X-\lambda)$ for some $\lambda \in K$ and some i, $1 \le i \le n_d$.

Let $\mu_\Lambda(d)$ be the least number of bimodules Q_i satisfying the condition above. Then Λ is called *of polynomial growth* [15] if there is a natural number m such that $\mu_\Lambda(d) \le d^m$ for all $d \ge 2$.

An algebra Λ is called *biserial* if the radical of any indecomposable nonuniserial left or right projective Λ-module is a sum of at most two uniserial submodules whose intersection is simple or zero. Λ is said to be *special biserial* [16] if it is isomorphic to a bound quiver algebra KQ_Λ/I_Λ, where the bound quiver (Q_Λ, I_Λ) satisfies the following conditions:

(1) The number of arrows with a given source or sink is at most two.

(2) For any arrow α of Q_Λ, there is at most one arrow β and there is at most one arrow γ such that $\alpha\beta$ and $\gamma\alpha$ do not belong to I_Λ.

2. Trivial extensions. Recall that, for a finite-dimensional algebra Λ, its *trivial extension* $T(\Lambda)$ by its minimal injective cogenerator bimodule $D\Lambda = \mathrm{Hom}_K(\Lambda, K)$ is the algebra whose additive structure is that of the group $\Lambda \oplus D\Lambda$, and whose multiplication is defined by

$$(a,f)(b,g) = (ab, ag + fb)$$

for $a, b \in \Lambda$ and $f, g \in {}_\Lambda(D\Lambda)_\Lambda$.

Throughout the paper let A be the path algebra of KQ_A of the following quiver Q_A:

$$\begin{tikzcd} \bullet \arrow[r, "\nu_q"] \arrow[d, "\lambda_p"'] & \bullet & \cdots & \bullet \arrow[r, "\nu_2"] & \bullet \arrow[d, "\nu_1"] \\ \bullet \arrow[r, "\lambda_{p-1}"] & \bullet & \cdots & \bullet \arrow[r, "\lambda_1"] & \bullet \end{tikzcd}$$

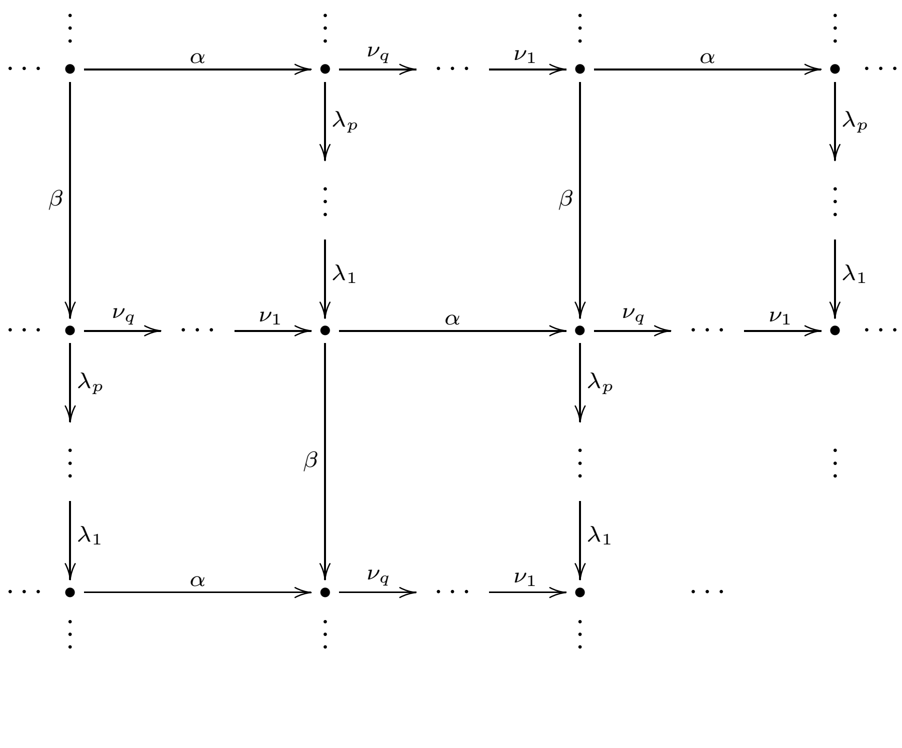
for $p, q \geq 1$. Thus the trivial extension $T(A)$ is isomorphic to the bound quiver algebra $KQ_{T(A)}/I_{T(A)}$, where $Q_{T(A)}$ is of the form

$$\begin{tikzcd} \bullet \arrow[r, "\nu_q"] \arrow[d, "\lambda_p"'] & \bullet & \cdots & \bullet \arrow[r, "\nu_2"] & \bullet \arrow[d, "\nu_1"] \\ \bullet \arrow[r, "\lambda_{p-1}"'] & \bullet & \cdots & \bullet \arrow[r, "\lambda_1"'] & \bullet \arrow[ullll, shift left, "\beta"'] \arrow[ullll, shift right, "\alpha"] \end{tikzcd}$$

and $I_{T(A)}$ is generated by $\alpha\nu_q, \nu_1\alpha, \beta\lambda_p, \lambda_1\beta, \lambda_p\lambda_{p-1}\dots\lambda_1\alpha - \nu_q\nu_{q-1}\dots\nu_1\beta$, $\alpha\lambda_p\dots\lambda_1 - \beta\nu_q\dots\nu_1, \lambda_i\dots\lambda_1\alpha\lambda_p\dots\lambda_i$, $1 \leq i \leq p$, $\nu_j\dots\nu_1\beta\nu_q\dots\nu_j$, $1 \leq j \leq q$.

It is well-known that every trivial extension algebra $T(B)$, for B being an hereditary algebra of type $\widetilde{A}_n$, is stably equivalent to $T(A)$, where A has the same number of simple modules as B has. Consequently, we shall consider algebras stably equivalent to $T(A)$.

We shall fix a Galois cover [7, 3] $\widetilde{T(A)}$ of $T(A)$ given by the quiver $\widetilde{Q}_{T(A)}$:



and $\widetilde{I}_{T(A)}$ is generated in our notations by the same elements as $I_{T(A)}$ is. Moreover, the covering functor $F : K\widetilde{Q}_{T(A)}/\widetilde{I}_{T(A)} \to KQ_{T(A)}/I_{T(A)}$ is determined by setting $F(\lambda_i) = \lambda_i$, $1 \le i \le p$, $F(\nu_j) = \nu_j$, $1 \le j \le q$, $F(\alpha) = \alpha$, $F(\beta) = \beta$. F induces the push-down functor $F_\lambda : \widetilde{T(A)}\text{-mod} \to T(A)\text{-mod}$ [7, 3] whose properties we shall use freely. Following [3] we shall call $T(A)$-modules of the form $F_\lambda(M)$, for any $M \in \widetilde{T(A)}$-mod, $T(A)$-*modules of the first kind.*

For any $\widetilde{T(A)}$-module M we shall denote its support by $\operatorname{supp}(M)$. We shall use the following convention: if we denote by i the source of two different paths in $\widetilde{Q}_{T(A)}$ that do not lie in $\widetilde{I}_{T(A)}$ but their difference does, then i' will denote their sink in $\widetilde{Q}_{T(A)}$.

For the convenience of the reader we state below two lemmas that were proved in [11].

LEMMA 1. *Let M, N be two indecomposable finite-dimensional $\widetilde{T(A)}$-modules whose supports are of the form*

$$\text{---} \cdots \leftarrow r_0 \to \cdots \to x \to \cdots \to r_1 \leftarrow \cdots \text{---}$$

and

$$\text{---} \cdots \leftarrow x \to \cdots \to r_1 \to \cdots \to s_1 \leftarrow \cdots \text{---}$$

respectively. Let $f : N \to M$ be the composition of an epimorphism $f_1 : N \to X$ and a monomorphism $f_2 : X \to M$, where X is the indecomposable $\widetilde{T(A)}$-module whose support is of the form $x \to \cdots \to r_1$. Then the following implications hold:

(a) *If P_{r_0} is uniserial, then $\underline{f} \ne 0$ iff the path*

$$r_0 \to \cdots \to x \to \cdots \to r_1 \to \cdots \to s_1$$

does not contain a subpath of the form

$$r_0 \to \cdots \to x \to \cdots \to r_1 \to \cdots \to y$$

which is the support of P_{r_0}.

(b) *If P_{r_0} is not uniserial, then $\underline{f} \ne 0$ implies that either the path $r_1 \to \cdots \to s_1$ does not contain a vertex z with $S_z \cong \operatorname{soc}(P_{r_0})$, or it contains such a vertex z and then $z = s_1$,* $\operatorname{supp}(M)$ *is of the form*

$$\text{---} \cdots \to r_{-1} \leftarrow \cdots \leftarrow y \leftarrow \cdots \leftarrow r_0 \to \cdots \to x \to \cdots \to r_1 \leftarrow \cdots \text{---}$$

and $\operatorname{supp}(N)$ *is of the form*

$$\text{---} \cdots \leftarrow x \to \cdots \to r_1 \to \cdots \to r_0' \leftarrow \cdots \leftarrow r_{-1} \leftarrow \cdots \leftarrow y \to \cdots \text{---}$$

where

$$\begin{array}{ccccc}
r_0 & \to \cdots \to & x & \to \cdots \to & r_1 \\
\downarrow & & & & \\
\vdots & & & & \Big\downarrow \\
\downarrow & & & & \\
y & & & & \vdots \\
\downarrow & & & & \\
\vdots & & & & \Big\downarrow \\
\downarrow & & & & \\
r_{-1} & \longrightarrow & \cdots & \longrightarrow & r'_0
\end{array}$$

is the support of P_{r_0}.

Lemma 2. *Let* M, N *be two indecomposable finite-dimensional* $\widetilde{T(A)}$-*modules whose supports are of the form*

$$\text{—}\cdots \to r_{-1} \leftarrow \cdots \leftarrow y \leftarrow \cdots \leftarrow r_0 \to \cdots \to x \to \cdots \to r_1 \leftarrow \cdots \text{—}$$

and

$$\text{—}\cdots \leftarrow y \to \cdots \to r_{-1} \to \cdots \to r'_0 \leftarrow \cdots \leftarrow r_1 \leftarrow \cdots \leftarrow x \to \cdots \text{—}$$

respectively, where the paths $r_0 \to \cdots \to y \to \cdots \to r_{-1} \to \cdots \to r'_0$ *and* $r_0 \to \cdots \to x \to \cdots \to r_1 \to \cdots \to r'_0$ *do not belong to* $\widetilde{I}_{T(A)}$ *but their difference does. Let* $f : N \to M$ *be a morphism which is the composition of an epimorphism* $f_1 : N \to X$ *and a monomorphism* $f_2 : X \to M$, *where* X *is the indecomposable* $\widetilde{T(A)}$-*module whose support is of the form* $x \to \cdots \to r_1$. *Let* $g : N \to M$ *be a morphism which is the composition of an epimorphism* $g_1 : N \to Y$ *and a monomorphism* $g_2 : Y \to M$, *where* Y *is the indecomposable* $\widetilde{T(A)}$-*module whose support is of the form* $y \to \cdots \to r_{-1}$. *Then* $\lambda \underline{f} = \underline{g}$ *for some* $\lambda \in K^*$.

3. S-projective $T(A)$-modules. We shall recall some notions from [11] that will be the main working tools in the paper.

An indecomposable object M of $T(A)$-$\underline{\text{mod}}$ is said to be a *stable* $T(A)$-*brick* if its endomorphism ring $\underline{\text{End}}_{T(A)}(M)$ is isomorphic to K. A family $\{M_i\}_{i\in I}$ of stable $T(A)$-bricks is said to be a *maximal system of orthogonal stable* $T(A)$-*bricks* if the following conditions are satisfied:

(i) $M_i \not\cong M_j$ for $i \neq j$.

(ii) M_i is not of τ-period 1 for any $i \in I$, i.e. $\tau(M_i) \not\cong M_i$.

(iii) For any different $i, j \in I$, $\underline{\text{Hom}}_{T(A)}(M_i, M_j) = \underline{\text{Hom}}_{T(A)}(M_j, M_i) = 0$.

(iv) For any nonzero object N in $T(A)$-$\underline{\text{mod}}$ that is not of τ-period 1, $\underline{\text{Hom}}_{T(A)}(N, \bigoplus_{i\in I} M_i) \neq 0$ and $\underline{\text{Hom}}_{T(A)}(\bigoplus_{i\in I} M_i, N) \neq 0$.

Typical examples of maximal systems of orthogonal stable $T(A)$-bricks are obtained in the following way. Let $\Phi : B\text{-}\underline{\mathrm{mod}} \to T(A)\text{-}\underline{\mathrm{mod}}$ be an equivalence, where B is a selfinjective K-algebra. Suppose $\{S_i\}_{i=1,\ldots,n}$ is a set of representatives of all isoclasses of simple B-modules. Then $\{\Phi(S_i)\}_{i=1,\ldots,n}$ is a maximal system of orthogonal stable $T(A)$-bricks.

Let $\{M_i\}_{i\in I} = \mathcal{M}_{T(A)}$ be a maximal system of orthogonal stable $T(A)$-bricks. An indecomposable $T(A)$-module M that is not of τ-period 1 is said to be *s-projective with respect to* $\mathcal{M}_{T(A)}$ if the following conditions hold:

(i) $\underline{\mathrm{Hom}}_{T(A)}(M, \bigoplus_{i\in I} M_i) \cong K$.

(ii) If $\underline{\mathrm{Hom}}_{T(A)}(M, M_{i_0}) \neq 0$ with $M_{i_0} \in \mathcal{M}_{T(A)}$, then for every $0 \neq \underline{f} : X \to M_{i_0}$ and every $0 \neq \underline{g} : M \to M_{i_0}$ there is $\underline{h} : M \to X$ such that $\underline{f}\,\underline{h} = \underline{g}$.

Moreover, for an s-projective $T(A)$-module M with respect to $\mathcal{M}_{T(A)}$, if $\underline{\mathrm{Hom}}_{T(A)}(M, M_{i_0}) \neq 0$ with $M_{i_0} \in \mathcal{M}_{T(A)}$ then M_{i_0} is said to be an *s-top* of M and is denoted by s-top(M). For any $T(A)$-module X of the first kind we have $\dim_K \underline{\mathrm{Hom}}_{T(A)}(X, M_i) = d_i$ for all $M_i \in \mathcal{M}_{T(A)}$, and we define s-top$(X)$ to be the module $\bigoplus_{i\in I} M_i^{d_i}$.

If $\{\Phi(S_i)\}_{i=1,\ldots,n}$ is the above maximal system of orthogonal stable $T(A)$-bricks then for every indecomposable projective B-module P the module $\Phi(P/\operatorname{soc}(P))$ is an s-projective $T(A)$-module with respect to $\{\Phi(S_i)\}_{i=1,\ldots,n}$.

Let M be an s-projective $T(A)$-module with respect to $\mathcal{M}_{T(A)}$. Then a $T(A)$-module X is said to be an *s-radical* of M (it is denoted by s-rad(M)) if the following conditions are satisfied:

(i) X does not contain any projective direct summand.

(ii) There is a projective or zero $T(A)$-module P such that there exists a right minimal almost split morphism $X \oplus P \to M$ in $T(A)$-mod.

S-projective modules for selfinjective special biserial algebras were studied in [11] and their properties have been found useful. Their s-radicals are direct sums of at most two indecomposable modules of the first kind. Under the above notations we have the following proposition.

PROPOSITION 1. *Let $\mathcal{M}_{T(A)}$ be a maximal system of orthogonal stable $T(A)$-bricks. Let M be an s-projective $T(A)$-module with* s-top$(M) \cong X$ *and* s-rad$(M) \cong R_1 \oplus R_2$. *Moreover, let* s-top$(R_1) \cong Y$ *and let N be an s-projective $T(A)$-module with* s-top$(N) \cong Y$ *and* s-rad$(N) \cong L_1 \oplus L_2$. *If $0 \neq \underline{f} : N \to R_1$ is a fixed morphism such that there is $\underline{h} : R_1 \to Y$ with $\underline{hf} \neq 0$, then for irreducible maps $g_1 : L_1 \to N$ and $g_2 : L_2 \to N$ one of the composition maps $\underline{fg_1}$, $\underline{fg_2}$ is nonzero.*

Proof. Under the assumptions of the proposition we can do all calculations in $\widetilde{T(A)}\text{-}\underline{\mathrm{mod}}$ by [11, Lemma 3.1]. Suppose that $M \cong F_\lambda(\widetilde{M})$, $X \cong$

$F_\lambda(\widetilde{X})$, $R_i \cong F_\lambda(\widetilde{R}_i)$, $i=1,2$, $Y \cong F_\lambda(\widetilde{Y})$, $N \cong F_\lambda(\widetilde{N})$ and $L_i \cong F_\lambda(\widetilde{L}_i)$, $i=1,2$.

By [11, Lemma 6.4], $\operatorname{supp}(\widetilde{X})$ can be of the form

$$\overset{\lambda_t}{\leftarrow}\cdots\overset{\lambda_p}{\leftarrow} 1 \overset{\nu_q}{\rightarrow}\cdots\overset{\nu_1}{\rightarrow}\overset{\lambda_1}{\leftarrow}\cdots\overset{\lambda_p}{\leftarrow} 2 \overset{\nu_q}{\rightarrow}\cdots\overset{\nu_1}{\rightarrow}\cdots\overset{\lambda_1}{\leftarrow}\cdots\overset{\lambda_p}{\leftarrow} n \overset{\nu_q}{\rightarrow}\cdots\overset{\nu_s}{\rightarrow}$$

with $1 \le t \le p$, $1 \le s \le q$, and $\operatorname{supp}(\widetilde{M})$ of the form

$$\overset{\varrho_l}{\text{—}}\cdots\overset{\varrho_0}{\text{—}}\overset{\alpha}{\rightarrow} 1' \overset{\beta}{\leftarrow}\overset{\alpha}{\rightarrow} 2' \overset{\beta}{\leftarrow}\cdots\overset{\alpha}{\rightarrow} n' \overset{\beta}{\leftarrow}\overset{\kappa_0}{\text{—}}\cdots\overset{\kappa_r}{\text{—}}\ .$$

Moreover, if $t>1$, then $\overset{\varrho_l}{\text{—}}\cdots\overset{\varrho_0}{\text{—}}=\overset{\lambda_{t-1}}{\rightarrow}\cdots\overset{\lambda_1}{\rightarrow}$; if $t=1$, then $\overset{\varrho_l}{\text{—}}\cdots\overset{\varrho_0}{\text{—}}=\overset{\nu_2}{\leftarrow}\cdots\overset{\nu_q}{\leftarrow}\overset{\beta}{\leftarrow}$ and if $s>1$, then $\overset{\kappa_0}{\text{—}}\cdots\overset{\kappa_r}{\text{—}}=\overset{\nu_1}{\leftarrow}\cdots\overset{\nu_{s-1}}{\leftarrow}$; if $s=1$, then $\overset{\kappa_0}{\text{—}}\cdots\overset{\kappa_r}{\text{—}}=\overset{\alpha}{\rightarrow}\overset{\lambda_p}{\rightarrow}\cdots\overset{\lambda_2}{\rightarrow}$.

In this case, by [11, Lemma 6.6], $\operatorname{supp}(\widetilde{R}_1)$ is of one of the following forms:

$$\overset{\lambda_{t-2}}{\rightarrow}\cdots\overset{\lambda_1}{\rightarrow}\overset{\alpha}{\rightarrow} 1' \overset{\beta}{\leftarrow}\overset{\alpha}{\rightarrow}\cdots\overset{\beta}{\leftarrow}\overset{\alpha}{\rightarrow} n' \overset{\beta}{\leftarrow}\overset{\kappa_0}{\text{—}}\cdots\overset{\kappa_r}{\text{—}}$$

if $t>1$, where $\lambda_0=\emptyset$, or

$$1' \overset{\beta}{\leftarrow}\overset{\alpha}{\rightarrow}\cdots\overset{\beta}{\leftarrow}\overset{\alpha}{\rightarrow} n' \overset{\beta}{\leftarrow}\overset{\kappa_0}{\text{—}}\cdots\overset{\kappa_r}{\text{—}}$$

for $t=1$. Therefore, by [11, Corollary 6.9], $\operatorname{supp}(\widetilde{Y})$ has one of the following forms:

$$\overset{\lambda_{t-2}}{\rightarrow}\cdots\overset{\lambda_1}{\rightarrow}\overset{\alpha}{\rightarrow} 1' \overset{\beta}{\leftarrow}\cdots\overset{\alpha}{\rightarrow} m' \overset{\beta}{\leftarrow}\overset{\nu_1}{\leftarrow}\cdots\overset{\nu_z}{\leftarrow}$$

or

$$1' \overset{\beta}{\leftarrow}\overset{\alpha}{\rightarrow}\cdots\overset{\beta}{\leftarrow}\overset{\alpha}{\rightarrow} m' \overset{\beta}{\leftarrow}\overset{\nu_1}{\leftarrow}\cdots\overset{\nu_z}{\leftarrow}$$

where $0 \le z < q$ with $\nu_0=\emptyset$. Therefore $\operatorname{supp}(\widetilde{N})$ is of the form (again by [11, Lemma 6.4])

$$\overset{\lambda_t}{\leftarrow}\cdots\overset{\lambda_1}{\leftarrow} 1' \overset{\nu_q}{\rightarrow}\cdots\overset{\nu_1}{\rightarrow}\cdots\overset{\lambda_1}{\leftarrow}\cdots\overset{\lambda_p}{\leftarrow} m' \overset{\nu_q}{\rightarrow}\cdots\overset{\nu_{z+2}}{\rightarrow}$$

with $t \ge 1$, where $\nu_{q+1}=\emptyset$. Using Lemmas 1 and 2 it is not hard to see that $m \le n$ and for $\widetilde{L}_1$ whose $\operatorname{supp}(\widetilde{L}_1)$ is of the form

$$\overset{\lambda_t}{\leftarrow}\cdots\overset{\lambda_p}{\leftarrow} 1' \overset{\nu_q}{\rightarrow}\cdots\overset{\nu_1}{\rightarrow}\cdots\overset{\lambda_1}{\leftarrow}\cdots\overset{\lambda_p}{\leftarrow} m' \overset{\nu_q}{\rightarrow}\cdots\overset{\nu_{z+2}}{\rightarrow}\overset{\nu_{z+1}}{\rightarrow}, \qquad z>0$$

or

$$\overset{\lambda_t}{\leftarrow}\cdots\overset{\lambda_p}{\leftarrow} 1' \overset{\nu_q}{\rightarrow}\cdots\overset{\nu_1}{\rightarrow}\cdots\overset{\lambda_1}{\leftarrow}\cdots\overset{\lambda_p}{\leftarrow} m' \overset{\nu_q}{\rightarrow}\cdots\overset{\nu_1}{\rightarrow}\overset{\lambda_1}{\leftarrow}\cdots\overset{\lambda_p}{\leftarrow}, \qquad z=0$$

the proposition holds by the description of the procedure for constructing Auslander–Reiten sequences in $T(A)$-mod given in [18].

Other possible forms of $\operatorname{supp}(\widetilde{X})$ are considered similarly.

4. S-projective $T(A)$-modules whose s-radicals are indecomposable. A path w in $(Q_{T(A)}, I_{T(A)})$ is said to be *submaximal* if $w \notin I_{T(A)}$ and there is an arrow γ such that γw is a maximal path which does not belong to $I_{T(A)}$, or there is an arrow δ such that $w\delta$ is a maximal path which does not belong to $I_{T(A)}$.

LEMMA 3. *If M is an s-projective $T(A)$-module whose s-radical is indecomposable then either*

(a) *M is isomorphic to a simple $T(A)$-module S_i, where i is a vertex in $Q_{T(A)}$ that is a sink of exactly one arrow*, *or*

(b) *M is a uniserial nonsimple $T(A)$-module whose support is a submaximal path in $(Q_{T(A)}, I_{T(A)})$.*

Proof. The lemma is an easy consequence of the procedure for constructing Auslander–Reiten sequences in $T(A)$-mod given in [18].

LEMMA 4. *Let M be an s-projective $T(A)$-module whose s-radical is indecomposable.*

(a) *If $M \cong S_i$ is a simple $T(A)$-module then* s-top(M) *is a uniserial $T(A)$-module whose support is a submaximal path in $(Q_{T(A)}, I_{T(A)})$ ending at i.*

(b) *If M is a uniserial $T(A)$-module whose support is a submaximal path w in $(Q_{T(A)}, I_{T(A)})$ then* s-top$(M) \cong S_i$, *where i is the source of w.*

Proof. The lemma is obvious by Lemma 3 and [11, Lemma 6.4].

LEMMA 5. *Let M be an s-projective $T(A)$-module whose s-radical is indecomposable.*

(a) *If $M \cong S_i$ is a simple $T(A)$-module then* supp(s-rad(M)) *is of the form $i \xrightarrow{\varrho}\xleftarrow{\kappa_1} \cdots \xleftarrow{\kappa_t}$, where either $\varrho = \lambda_j$ or $\varrho = \nu_s$, $1 \le j < p$, $1 \le s < q$, and for $\varrho = \lambda_1$ we have $t = q$ and $\kappa_l = \nu_l$, for $\varrho = \nu_1$ we have $t = p$ and $\kappa_l = \lambda_l$, and $\kappa_l = \emptyset$ otherwise.*

(b) *If M is a uniserial nonsimple $T(A)$-module and* supp(M) *is of the form $\xrightarrow{\varrho_1} \cdots \xrightarrow{\varrho_l}$ then* supp(s-rad(M)) *is of the form $\xleftarrow{\kappa}\xrightarrow{\varrho_1} \cdots \xrightarrow{\varrho_l}$ if $\varrho_1 = \lambda_p$ or $\varrho_1 = \nu_q$, or $\xrightarrow{\lambda_{p-1}} \cdots \xrightarrow{\lambda_1}\xrightarrow{\alpha}\xleftarrow{\beta}\xrightarrow{\varrho_1} \cdots \xrightarrow{\varrho_l}$ if $\varrho_1 = \alpha$, or $\xrightarrow{\nu_{q-1}} \cdots \xrightarrow{\nu_1}\xrightarrow{\beta}\xleftarrow{\alpha}\xrightarrow{\varrho_1} \cdots \xrightarrow{\varrho_l}$ if $\varrho_1 = \beta$, or else $\xrightarrow{\varrho_2} \cdots \xrightarrow{\varrho_l}$.*

Proof. The lemma is an obvious consequence of Lemmas 3, 4 and the procedure for constructing Auslander–Reiten sequences in $T(A)$-mod given in [18].

PROPOSITION 2. *Let $\mathcal{M}_{T(A)}$ be a maximal system of orthogonal stable $T(A)$-bricks. Let M be an s-projective $T(A)$-module with* s-top$(M) \cong X$ *whose* s-rad$(M) \cong R$ *is indecomposable. Moreover*, *let* s-top$(R) \cong Y$ *and let N be the s-projective $T(A)$-module with* s-top$(N) \cong Y$.

(a) *If* s-rad$(N) \cong L_1 \oplus L_2$ *and $0 \ne \underline{f} : N \to R$ is a morphism such that there is $\underline{h} : R \to Y$ with $\underline{hf}$ nonzero*, *then for any irreducible maps $g_i : L_i \to N$, $i = 1, 2$, one of $\underline{f}\,\underline{g_1}$, $\underline{f}\,\underline{g_2}$ is nonzero.*

(b) *If* s-rad$(N) \cong L$ *is indecomposable and* $0 \neq \underline{f} : N \to R$ *is a morphism such that there is* $\underline{h} : R \to Y$ *with* $\underline{h}\underline{f} \neq 0$, *then for every irreducible map* $g : L \to N$ *the composition map* $\underline{f}\underline{g}$ *is nonzero.*

Proof. Let $M \cong F_\lambda(\widetilde{M})$ be an s-projective $T(A)$-module with s-top$(M) \cong X$, where $F_\lambda(\widetilde{X}) \cong X$. Moreover, let s-rad$(M) \cong R$ be indecomposable with $R \cong F_\lambda(\widetilde{R})$. Let s-top$(R) \cong Y$ and $F_\lambda(\widetilde{Y}) \cong Y$ and let N be the s-projective $T(A)$-module with s-top$(N) \cong Y$, where $F_\lambda(\widetilde{N}) \cong N$.

(a) Let s-rad$(N) \cong L_1 \oplus L_2$ and $L_i \cong F_\lambda(\widetilde{L}_i)$, $i = 1, 2$. We conclude from Lemmas 3–5 that M is either a simple or a uniserial nonsimple $T(A)$-module. Consider the first case. Then $M \cong S_x$ and x is the sink of exactly one arrow in $Q_{T(A)}$, so in $\widetilde{Q}_{T(A)}$. Therefore supp$(\widetilde{X})$ is either of the form $x \overset{\lambda_i}{\leftarrow} \cdots \overset{\lambda_p}{\leftarrow}\overset{\alpha}{\leftarrow}\overset{\lambda_1}{\leftarrow} \cdots \overset{\lambda_{i-2}}{\leftarrow}$, $i = 2, \ldots, p$, and $\lambda_0 = \emptyset$, or $x \overset{\nu_j}{\leftarrow} \cdots \overset{\nu_q}{\leftarrow}\overset{\beta}{\leftarrow}\overset{\nu_1}{\leftarrow} \cdots \overset{\nu_{j-2}}{\leftarrow}$, $j = 2, \ldots, q$, and $\tau_0 = \emptyset$ by Lemma 4. Furthermore, supp$(\widetilde{R})$ is of one of the following forms:

(i) $x \overset{\lambda_{i-1}}{\rightarrow}$, $i \geq 3$,

(ii) $x \overset{\nu_{j-1}}{\rightarrow}$, $j \geq 3$,

(iii) $x \overset{\lambda_1}{\rightarrow}\overset{\nu_1}{\leftarrow} \cdots \overset{\nu_q}{\leftarrow}$,

(iv) $x \overset{\nu_1}{\rightarrow}\overset{\lambda_1}{\leftarrow} \cdots \overset{\lambda_p}{\leftarrow}$.

by Lemma 5. If supp$(\widetilde{R})$ is of the form (i) then supp$(\widetilde{Y})$ (by [11, Corollary 6.9]) has one of the following forms:

$$\overset{\lambda_{i-1}}{\leftarrow} \cdots \overset{\lambda_p}{\leftarrow}\overset{\alpha}{\leftarrow}\overset{\beta}{\rightarrow} \cdots \text{—},$$

$$\overset{\lambda_{i-1}}{\leftarrow} \cdots \overset{\lambda_p}{\leftarrow}\overset{\nu_q}{\rightarrow} \cdots \overset{\nu_1}{\rightarrow} \cdots \text{—},$$

$$\overset{\lambda_{i-1}}{\leftarrow} \cdots \overset{\lambda_p}{\leftarrow}\overset{\alpha}{\leftarrow}\overset{\lambda_1}{\leftarrow} \cdots \overset{\lambda_t}{\leftarrow}, \quad t < i + 2\,,$$

or any subpath of the latter path ending with λ_{i-1}.

It is not hard to verify that in each of the above cases supp$(\widetilde{N})$ is of the form $\overset{\lambda_{i-2}}{\rightarrow} \cdots$— by [11, Lemma 6.4] and there is $\widetilde{L}_1$ such that its support is of the form $\overset{\lambda_{i-2}}{\rightarrow} \cdots$— and (a) holds for L_1.

For (ii) the proof is similar.

If supp$(\widetilde{R})$ is of the form (iii), then supp$(\widetilde{Y})$ (by [11, Corollary 6.9]) is of the form $x \overset{\lambda_2}{\leftarrow} \cdots$—. Thus supp$(\widetilde{N})$ is of the form $x \overset{\lambda_1}{\rightarrow}\overset{\alpha}{\rightarrow} \cdots$—, and L_1 with supp$(\widetilde{L}_1)$ of the form $\overset{\alpha}{\rightarrow} \cdots$— satisfies (a).

Similar arguments show (a) in case (iv).

If M is uniserial nonsimple then, by Lemma 4, s-top$(M) \cong X$ is simple. Moreover, by Lemma 5, s-rad(M) is known and a similar analysis shows (a) in this case.

(b) Let $\text{s-rad}(N) \cong L$ be indecomposable with $F_\lambda(\widetilde{L}) \cong L$. Then, by Lemma 5, N is either simple or uniserial nonsimple, and one obtains (b) similarly to (a).

5. Symmetry properties. Let $\mathcal{M}_{T(A)} = \{M_i\}_{i\in I}$ be a maximal system of orthogonal stable $T(A)$-bricks. Let M be a $T(A)$-module that is not projective. M is said to have a *simple s-socle* if $\underline{\mathrm{Hom}}_{T(A)}(\bigoplus_{i\in I} M_i, M) \cong K$. Therefore there is $i_0 \in I$ such that $\underline{\mathrm{Hom}}_{T(A)}(M_{i_0}, M) \cong K$ and we write $M_{i_0} = \text{s-soc}(M)$.

PROPOSITION 3. *Let $\mathcal{M}_{T(A)}$ be a maximal system of orthogonal stable $T(A)$-bricks. Let M be an s-projective $T(A)$-module with* $\text{s-top}(M) \cong X$. *If $N \cong \tau^{-1}(M)$, then* $\text{s-soc}(N) \cong X$.

Proof. Assume that $X \cong F_\lambda(\widetilde{X})$, $M \cong F_\lambda(\widetilde{M})$ and $N \cong F_\lambda(\widetilde{N})$.

Let $\mathrm{supp}(\widetilde{X})$ be of the form

$$\xleftarrow{\lambda_t} \dots \xleftarrow{\lambda_p} 1 \xrightarrow{\nu_q} \dots \xrightarrow{\nu_1} \dots \xleftarrow{\lambda_1} \dots \xleftarrow{\lambda_p} n \xrightarrow{\nu_q} \dots \xrightarrow{\nu_s} .$$

Thus by [11, Lemma 6.4] (as in the proof of Proposition 1) $\mathrm{supp}(\widetilde{M})$ is of the form

$$\underline{\varrho_l} \dots \underline{\varrho_0} \xrightarrow{\alpha} 1' \xleftarrow{\beta} \dots \xrightarrow{\alpha} n' \xleftarrow{\beta} \underline{\kappa_0} \dots \underline{\kappa_r}$$

where

$$\underline{\varrho_l} \dots \underline{\varrho_0} = \begin{cases} \xrightarrow{\lambda_{t-1}} \dots \xrightarrow{\lambda_1} & \text{if } t > 1\,, \\ \xleftarrow{\nu_2} \dots \xleftarrow{\nu_q} \xleftarrow{\beta} & \text{if } t = 1\,, \end{cases}$$

and

$$\underline{\kappa_0} \dots \underline{\kappa_r} = \begin{cases} \xleftarrow{\nu_1} \dots \xleftarrow{\nu_{s-1}} & \text{if } s > 1\,, \\ \xrightarrow{\alpha} \xrightarrow{\lambda_p} \dots \xrightarrow{\lambda_2} & \text{if } s = 1\,. \end{cases}$$

(i_1) If $\mathrm{supp}(\widetilde{M})$ is of the form

$$\xrightarrow{\lambda_{t-1}} \dots \xrightarrow{\lambda_1} \xrightarrow{\alpha} 1' \xleftarrow{\beta} \dots \xrightarrow{\alpha} n' \xleftarrow{\beta} \xleftarrow{\nu_1} \dots \xleftarrow{\nu_{s-1}}$$

then $\mathrm{supp}(\widetilde{N})$ is of the form

$$\xrightarrow{\lambda_{t-2}} \dots \xrightarrow{\lambda_1} \xrightarrow{\alpha} 1' \xleftarrow{\beta} \dots \xrightarrow{\alpha} n' \xleftarrow{\beta} \xleftarrow{\nu_1} \dots \xleftarrow{\nu_{s-2}}$$

with $\lambda_0 = \nu_0 = \emptyset$ by [11]. Hence [11, Lemma 6.5] implies that $\text{s-soc}(N) \cong X$.

(i_2) If $\mathrm{supp}(\widetilde{M})$ is of the form

$$\xleftarrow{\nu_2} \dots \xleftarrow{\nu_q} \xleftarrow{\beta} \xrightarrow{\alpha} 1' \xleftarrow{\beta} \dots \xrightarrow{\alpha} n' \xleftarrow{\beta} \xleftarrow{\nu_1} \dots \xleftarrow{\nu_{s-1}}$$

then $\mathrm{supp}(\widetilde{N})$ is of the form $1' \xleftarrow{\beta} \dots \xrightarrow{\alpha} n' \xleftarrow{\beta} \xleftarrow{\nu_1} \dots \xleftarrow{\nu_{s-2}}$ by [18]. Hence [11, Lemma 6.5] implies that $\text{s-soc}(N) \cong X$.

(i$_3$) If $\operatorname{supp}(\widetilde{M})$ is of the form

$$\overset{\nu_2}{\leftarrow} \cdots \overset{\nu_q}{\leftarrow}\overset{\beta}{\leftarrow}\overset{\alpha}{\rightarrow} 1' \overset{\beta}{\leftarrow} \cdots \overset{\alpha}{\rightarrow} n' \overset{\beta}{\leftarrow}\overset{\alpha}{\rightarrow}\overset{\lambda_p}{\rightarrow} \cdots \overset{\lambda_2}{\rightarrow}$$

then $\operatorname{supp}(\widetilde{N})$ is of the form $1' \overset{\beta}{\leftarrow} \cdots \overset{\alpha}{\rightarrow} n'$ by [18]. Thus [11, Lemma 6.5] implies that $\text{s-soc}(N) \cong X$.

(i$_4$) If $\operatorname{supp}(\widetilde{M})$ is of the form

$$\overset{\lambda_{t-1}}{\rightarrow} \cdots \overset{\lambda_1}{\rightarrow}\overset{\alpha}{\rightarrow} 1' \overset{\beta}{\leftarrow} \cdots \overset{\alpha}{\rightarrow} n' \overset{\beta}{\leftarrow}\overset{\alpha}{\rightarrow}\overset{\lambda_p}{\rightarrow} \cdots \overset{\lambda_2}{\rightarrow}$$

then we get $\text{s-soc}(N) \cong X$ similarly to (i$_2$).

Other forms of $\operatorname{supp}(\widetilde{X})$ are considered similarly.

Proposition 4. *Let $\mathcal{M}_{T(A)}$ be a maximal system of orthogonal stable $T(A)$-bricks. Let M be an s-projective $T(A)$-module with* $\text{s-top}(M) \cong X$. *If Z is an indecomposable $T(A)$-module of the first kind such that $Z \not\cong X$ and $X \cong \text{s-soc}(Z)$, then $\underline{\operatorname{Hom}}_{T(A)}(M, Z) = 0$.*

Proof. Let $M \cong F_\lambda(\widetilde{M})$, $X \cong F_\lambda(\widetilde{X})$, $Z \cong F_\lambda(\widetilde{Z})$.

Let $\operatorname{supp}(\widetilde{X})$ be of the form

$$\overset{\lambda_t}{\leftarrow} \cdots \overset{\lambda_p}{\leftarrow} 1 \overset{\nu_q}{\rightarrow} \cdots \overset{\nu_1}{\rightarrow} \cdots \overset{\lambda_1}{\leftarrow} \cdots \overset{\lambda_p}{\leftarrow} n \overset{\nu_q}{\rightarrow} \cdots \overset{\nu_s}{\rightarrow}\ .$$

Then by [11, Lemma 6.4], $\operatorname{supp}(\widetilde{M})$ is of the form

$$\overset{\lambda_{t-1}}{\rightarrow} \cdots \overset{\lambda_1}{\rightarrow}\overset{\alpha}{\rightarrow} 1' \overset{\beta}{\leftarrow} \cdots \overset{\alpha}{\rightarrow} n' \overset{\beta}{\leftarrow}\overset{\nu_1}{\leftarrow} \cdots \overset{\nu_{s-1}}{\leftarrow}\ .$$

Furthermore, $\operatorname{supp}(\widetilde{Z})$ is of one of the following forms (by Lemmas 1, 2): if $t, s > 1$, then

$$\overset{\lambda_a}{\rightarrow} \cdots \overset{\lambda_1}{\rightarrow}\overset{\alpha}{\rightarrow} 1 \overset{\beta}{\leftarrow} \cdots \overset{\alpha}{\rightarrow} n \overset{\beta}{\leftarrow}\overset{\nu_1}{\leftarrow} \cdots \overset{\nu_b}{\leftarrow}$$

with $a \le t-2$, $b \le s-2$; if $t = 1, s > 1$, then

$$1 \overset{\beta}{\leftarrow} \cdots \overset{\alpha}{\rightarrow} n \overset{\beta}{\leftarrow}\overset{\nu_1}{\leftarrow} \cdots \overset{\nu_b}{\leftarrow}$$

with $b \le t-2$; if $t > 1, s = 1$, then

$$\overset{\lambda_a}{\rightarrow} \cdots \overset{\lambda_1}{\rightarrow}\overset{\alpha}{\rightarrow} 1 \overset{\beta}{\leftarrow} \cdots \overset{\alpha}{\rightarrow} n$$

with $a \le t-2$; if $t = s = 1$, then $1 \overset{\beta}{\leftarrow} \cdots \overset{\alpha}{\rightarrow} n$. An easy analysis shows that in each case the proposition holds.

Other forms of $\operatorname{supp}(\widetilde{X})$ are considered similarly.

6. Algebras stably equivalent to $T(A)$. An algebra B is stably equivalent to $T(A)$ iff B-$\underline{\text{mod}}$ and $T(A)$-$\underline{\text{mod}}$ are equivalent categories. As a consequence of the above propositions we get the following fact.

Corollary 1. *If B is stably equivalent to $T(A)$, then B is a self injective K-algebra that satisfies the following conditions*:

(a) *B is of polynomial growth.*

(b) *$B \cong KQ_B/I_B$, where the bound quiver (Q_B, I_B) satisfies:*

(i) *Q_B has the same number of vertices as $Q_{T(A)}$.*

(ii) *The number of arrows with a given source or sink is at most two.*

(iii) *For any arrow α in Q_B, there is at most one arrow β and there is at most one arrow γ such that $\alpha\beta$ and $\gamma\alpha$ do not belong to I_B.*

(iv) *For any arrow α in Q_B, there is at most one arrow δ and there is at most one arrow η such that $\alpha\delta$ and $\eta\alpha$ belong to I_B.*

(v) *If $\alpha_1 \ldots \alpha_n$ is an oriented cycle of pairwise different arrows in Q_B that does not belong to I_B then $\alpha_1 \ldots \alpha_n \alpha_1$ belongs to I_B.*

(vi) *For any arrow α in Q_B, there is an oriented cycle $\alpha\alpha_2 \ldots \alpha_n$ of pairwise different arrows that does not belong to I_B.*

Proof. (a) It is well-known that an algebra stably equivalent to a selfinjective algebra is also selfinjective. Let B be a selfinjective algebra stably equivalent to $T(A)$. Since $T(A)$ is special biserial, so is B by [11, Theorem 7.3]. Thus by [5, Theorems 2.1, 2.2], B is of polynomial growth since $T(A)$ is. Consequently, (a) is proved.

(b) By [11, Corollary 5.1] two selfinjective special biserial algebras which are stably equivalent have the same number of isoclasses of simple modules; hence (i) follows. Moreover, B is special biserial, so (ii) and (iii) hold. Suppose that $\Phi : B\text{-mod} \to T(A)\text{-mod}$ yields an equivalence of categories. Suppose that $\{S_i\}_{i=1,\ldots,n}$ is a set of representatives of all isoclasses of simple B-modules. Consider the maximal system of orthogonal stable $T(A)$-bricks $\mathcal{M}_{T(A)} = \{\Phi(S_i)\}_{i=1,\ldots,n}$ and consider the set of indecomposable s-projective $T(A)$-modules $\{\Phi(P_i/\operatorname{soc}(P_i))\}_{i=1,\ldots,n}$ with respect to $\mathcal{M}_{T(A)}$. Then Propositions 1, 2 imply that (iv) holds. Proposition 4 implies (v) and Proposition 3 implies (vi). This finishes the proof of our corollary.

We shall call an algebra B *regular* if it shares all properties of Corollary 1. If B is regular and we write $B \cong KQ_B/I_B$ then we mean that the bound quiver (Q_B, I_B) satisfies (b) of Corollary 1.

LEMMA 6. *Let $B \cong KQ_B/I_B$ be regular. If the difference $\alpha_1 \ldots \alpha_n - \beta_1 \ldots \beta_m$ of two nonzero paths in (Q_B, I_B) belongs to I_B then $\alpha_1 \ldots \alpha_n$ is an oriented cycle of pairwise different arrows.*

Proof. Suppose that there is a commutativity relation $\alpha_1 \ldots \alpha_n - \beta_1 \ldots \beta_m$ in I_B. It is not hard to check that there is a nonzero oriented cycle $\alpha_1 \ldots \alpha_n v$ by regularity of B. If $\alpha_1 \ldots \alpha_n$ is not an oriented cycle then v is nontrivial, hence $v = \kappa v'$ and $\alpha_1 \ldots \alpha_n \kappa = \beta_1 \ldots \beta_m \kappa \notin I_B$. Consequently, by regularity of B, $\alpha_1 \ldots \alpha_n$ is a subpath of $\beta_1 \ldots \beta_m$ or $\beta_1 \ldots \beta_m$ is a subpath of $\alpha_1 \ldots \alpha_n$. Consider the case $\beta_1 \ldots \beta_m = \beta_1 \ldots \beta_t \alpha_1 \ldots \alpha_n$. Then we have the following equality in $B \cong KQ_B/I_B$: $\beta_1 \ldots \beta_t \alpha_1 \ldots \alpha_n =$

$(\beta_1 \dots \beta_t)^2 \alpha_1 \dots \alpha_n = 0$, which shows that $\alpha_1 \dots \alpha_n$ is an oriented cycle and the lemma is proved, because the other case is similar.

LEMMA 7. *Let $B \cong KQ_B/I_B$ be regular. If w is a path in Q_B which belongs to I_B then w is of one of the following forms*:

(a) *$w = w_1\alpha\beta w_2$ and $\alpha\beta \in I_B$, where α, β are arrows in Q_B.*

(b) *$w = w_1\alpha_1 \dots \alpha_n\alpha_1 w_2$, where $\alpha_1 \dots \alpha_n$ is a nonzero oriented cycle.*

(c) *$w = w_1\alpha_1 \dots \alpha_n\alpha_{n+1}w_2$ and there is a commutativity relation $\alpha_1 \dots \alpha_n - \alpha_{n+1}v$ in I_B.*

Proof. Suppose that w is a zero path in (Q_B, I_B). Let $w = \alpha_1 \dots \alpha_n\alpha_{n+1}v$ where $\alpha_1 \dots \alpha_n \notin I_B$ and $\alpha_1 \dots \alpha_n\alpha_{n+1} \in I_B$. If $n = 1$ then $\alpha_1\alpha_2 \in I_B$ and (a) holds. If $n > 1$ then by regularity of B, there is a nonzero oriented cycle $\alpha_1 \dots \alpha_n\beta_1 \dots \beta_t$. If $t = 0$, then $w = \alpha_1 \dots \alpha_n\alpha_{n+1}w'$ and either $\alpha_{n+1} = \alpha_1$, so that (b) holds, or $\alpha_{n+1} \neq \alpha_1$ and if $\alpha_n\alpha_1 \in I_B$ then there is a commutativity relation $\alpha_1 \dots \alpha_n - \alpha_{n+1}v$ in I_B, hence (c) holds. If $\alpha_n\alpha_1 \notin I_B$ then $\alpha_n\alpha_{n+1} \in I_B$ and (a) holds. If $t > 1$, then $\alpha_n\alpha_{n+1} \in I_B$ and hence (a) holds. Consequently, our lemma is proved.

An algebra Λ is said to be *gentle* (see [1]) if $\Lambda \cong KQ_\Lambda/I_\Lambda$, where the bound quiver (Q_Λ, I_Λ) satisfies (1), (2) of Section 1 and the following conditions:

(3) For any arrow α in Q_Λ there is at most one arrow δ and there is at most one arrow η such that $\alpha\delta$ and $\eta\alpha$ belong to I_Λ.

(4) I_Λ is generated by paths of length two.

LEMMA 8. *Every regular algebra B is isomorphic to the trivial extension $T(B_1)$ of a gentle algebra B_1.*

Proof. We start the proof with constructing a bound quiver (Q_1, I_1) from (Q_B, I_B), where $B \cong KQ_B/I_B$. Let Q_1 be the quiver obtained from Q_B by removing exactly one arrow that is not a loop in each nonzero oriented cycle in (Q_B, I_B). I_1 is obtained from I_B by removing all relations involving the removed arrows. We define $B_1 = KQ_1/I_1$. By regularity of B and Lemmas 6, 7, B_1 is gentle. If one applies the construction of a quiver for trivial extensions of gentle algebras given in [12], then it is not hard to verify that $B \cong T(B_1)$.

PROPOSITION 5. *Every representation-infinite regular algebra B is isomorphic to the trivial extension $T(B_1)$ of a gentle algebra B_1 whose bound quiver (Q_1, I_1) contains exactly one nonoriented cycle.*

Proof. By Lemma 8, $B \cong T(B_1)$, where B_1 is a gentle algebra. But B_1 must be such that its bound quiver (Q_1, I_1) contains at least one (oriented or not) cycle, otherwise $T(B_1)$ is representation-finite. Consequently, by [10, Theorem], $B \cong T(B_2)$, where B_2 is a gentle factor of an hereditary algebra whose bound quiver (Q_2, I_2) contains exactly one nonoriented cycle.

7. Main results. We start this section with two useful lemmas.

LEMMA 9. *Let $B \cong KQ_B/I_B$ be a gentle algebra whose bound quiver (Q_B, I_B) contains exactly one nonoriented cycle $\mathcal{C}$.*

(a) *If the number of nonzero maximal subpaths in $\mathcal{C}$ is even, then there is a gentle algebra B_1 whose only nonoriented cycle $\mathcal{C}'$ in its bound quiver (Q_{B_1}, I_{B_1}) is relation-free and $T(B) \cong T(B_1)$.*

(b) *If the number of nonzero maximal subpaths in $\mathcal{C}$ is odd, then there is a gentle algebra B_1 whose only nonoriented cycle $\mathcal{C}'$ in its bound quiver (Q_{B_1}, I_{B_1}) is bound by exactly one zero-relation of length two and $T(B) \cong T(B_1)$.*

Proof. Let $\mathcal{S}$ be the set of nonzero maximal subpaths in $\mathcal{C}$. It is easily seen that these subpaths in $(Q_{T(B)}, I_{T(B)})$ are contained in nonzero cycles $c_1, \ldots, c_n$ that satisfy the following conditions:

(i) different elements of $\mathcal{S}$ are contained in different cycles,

(ii) every c_i, $i = 1, \ldots, n$, contains an element of $\mathcal{S}$,

(iii) if $n > 2$, then c_i has exactly one common vertex with c_{i+1} for $i = 1, \ldots, n$, where $c_{n+1} = c_1$,

(iv) different cycles contain different arrows,

(v) if the composition $\alpha\beta$ makes sense and α, β are contained in different cycles, then $\alpha\beta \in I_{T(B)}$,

(vi) if $c_i = \alpha_{i_0} \ldots \alpha_{i_{t_i}}$, then $\alpha_{i_0} \ldots \alpha_{i_{t_i}} \alpha_{i_0} \in I_{T(B)}$ for each $i = 1, \ldots, n$.

Let the common vertex for c_i, c_{i+1}, $i = 1, \ldots, n$, be the sink of α_{i,z_i} and the source of $\alpha_{i+1,s_{i+1}}$, where $\alpha_{n+1,s_{n+1}} = \alpha_{1,s_1}$. Consider the following two walks w_1, w_2 in $(Q_{T(B)}, I_{T(B)})$:

$$\begin{aligned} w_1 = {}& \alpha^{-1}_{2,s_2-1}\alpha^{-1}_{2,s_2-2} \ldots \alpha^{-1}_{2,z_2+1}\alpha_{3,s_3}\alpha_{3,s_3+1} \ldots \alpha_{3,z_3}\alpha^{-1}_{4,s_4-1} \ldots \alpha^{-1}_{4,z_4+1} \ldots \\ & \ldots \alpha_{n-1,s_{n-1}} \ldots \alpha_{n-1,z_{n-1}}\alpha^{-1}_{n,s_n-1} \ldots \alpha^{-1}_{n,z_n+1}\alpha_{1,s_1} \ldots \alpha_{1,z_1} \end{aligned}$$

for n even, and

$$\begin{aligned} w_2 = {}& \alpha^{-1}_{2,s_2-1} \ldots \alpha^{-1}_{2,z_2+1}\alpha_{3,s_3} \ldots \alpha_{3,z_3} \\ & \cdot \alpha^{-1}_{4,s_4-1} \ldots \alpha^{-1}_{4,z_4+1} \ldots \alpha_{n,s_n} \ldots \alpha_{n,z_n} \cdot \alpha^{-1}_{1,s_1-1} \ldots \alpha^{-1}_{1,z_1+1} \end{aligned}$$

for n odd, where the addition of the second indices for arrows contained in c_i is modulo $t_i + 1$. These two walks of pairwise different arrows and formal inverses of arrows determine a nonoriented cycle $\mathcal{C}'$ in $(Q_{T(B)}, I_{T(B)})$. If n is even then $\mathcal{C}'$ is relation-free. If n is odd then $\mathcal{C}'$ is bound by the relation $\alpha_{2,s_2-1}\alpha_{1,z_1+1}$. Consequently, if we remove from $Q_{T(B)}$ exactly one arrow in each nonzero oriented cycle in $(Q_{T(B)}, I_{T(B)})$ in such a way that we do not remove any arrow from $\mathcal{C}'$, then we obtain a quiver Q_1. I_1 is obtained by removing from $I_{T(B)}$ all elements involving the removed arrows. Thus $B_1 \cong KQ_1/I_1$ satisfies (a) or (b) depending on whether n is even or odd. Therefore our lemma is proved.

LEMMA 10. *Let $B \cong KQ_B/I_B$ be a gentle algebra whose bound quiver (Q_B, I_B) contains exactly one nonoriented cycle $\mathcal{C}$ that is bound by exactly one zero-relation of length two. Then $T(B)$ is not stably equivalent to any $T(A)$.*

Proof. Suppose that $Q_1 = Q_B$ and I_1 is constructed from I_B in the following way: if $\alpha\beta$ is the only zero-path in $\mathcal{C}$ then we remove from I_B the generator $\alpha\beta$. If there is an arrow γ in Q_B such that $\gamma\beta$ makes sense then $\gamma\beta$ is a generator in I_1. If moreover $\gamma\delta$ was a generator in I_B then we remove it passing to I_1. Let $B_1 = KQ_1/I_1$. Then B_1 is a gentle algebra whose bound quiver (Q_1, I_1) contains exactly one nonoriented cycle $\mathcal{C}$ free of relations. By [1] and [17], $T(B_1)$ is stably equivalent to some $T(A)$. By [11, Corollary 5.1], if $T(B)$ is stably equivalent to some $T(A)$ then $T(B_1)$ is stably equivalent to $T(B)$. On the other hand, as in the proof of Lemma 9, we have a sequence $c_1, \ldots, c_n$ of nonzero oriented cycles in $(Q_{T(B)}, I_{T(B)})$ that satisfies (i)–(vi). Consider the following walk in $(Q_{T(B)}, I_{T(B)})$ under the notations from the proof of Lemma 7:

$$\begin{aligned} w = {} & \alpha_{2,s_2-1}^{-1}\alpha_{2,s_2-2}^{-1}\ldots\alpha_{2,z_2+1}^{-1}\alpha_{3,s_3}\ldots\alpha_{3,z_3} \\ & \cdot\alpha_{4,s_4-1}^{-1}\ldots\alpha_{4,z_4+1}^{-1}\ldots\alpha_{n,s_n}\ldots\alpha_{n,z_n} \\ & \cdot\alpha_{1,s_1-1}^{-1}\ldots\alpha_{1,z_1+1}^{-1}\alpha_{2,s_2}\ldots\alpha_{2,z_2}\alpha_{3,s_3-1}^{-1}\ldots\alpha_{3,z_3+1}^{-1} \\ & \cdot\alpha_{4,s_4}\ldots\alpha_{4,z_4}\ldots\alpha_{n,s_n-1}^{-1}\ldots\alpha_{n,z_n+1}^{-1}\alpha_{1,s_1}\ldots\alpha_{1,z_1}\,, \end{aligned}$$

where n is the odd number of nonzero maximal subpaths in $\mathcal{C}$. Since w involves all arrows of $\mathcal{C}$ twice, its length is greater than the number of arrows of $\mathcal{C}$. By the procedure for constructing Auslander–Reiten sequences from [18] we infer that w produces a τ-periodic module whose τ-period is greater than the τ-period of any τ-periodic module produced by $\mathcal{C}$, and each τ-periodic $T(B)$-module of τ-period greater than 1 is produced by $\mathcal{C}$. This contradicts the fact that stable equivalences preserve τ-periods. Therefore $T(B)$ is not stably equivalent to any $T(A)$.

Now we can state the main results of the paper.

THEOREM 1. *Let Λ be an hereditary algebra of type $\widetilde{A}_n$. Then B is stably equivalent to $T(\Lambda)$ if and only if B is isomorphic to the trivial extension of a gentle factor $B_1 \cong KQ_1/I_1$ of an hereditary algebra whose bound quiver (Q_1, I_1) contains exactly one nonoriented relation-free cycle. Moreover, Λ and B_1 have the same number of simple modules.*

P r o o f. By the main results of [1, 17], if $B_1 \cong KQ_1/I_1$ is a gentle factor of an hereditary algebra and the bound quiver (Q_1, I_1) contains exactly one nonoriented relation-free cycle, then $B \cong T(B_1)$ is stably equivalent to the trivial extension of an hereditary algebra of type $\widetilde{A}_n$.

Suppose now that B is stably equivalent to the trivial extension $T(\Lambda)$ of an hereditary algebra Λ of type $\widetilde{A}_n$. Then B is stably equivalent to some $T(A)$. Therefore by Proposition 5, B is isomorphic to the trivial extension $T(B_0)$ of a gentle algebra B_0 whose bound quiver contains exactly one nonoriented cycle $\mathcal{C}$. Consequently, by Lemmas 9, 10, B is isomorphic to the trivial extension of a gentle factor $B_1 \cong KQ_1/I_1$ of an hereditary algebra and (Q_1, I_1) contains exactly one nonoriented relation-free cycle $\mathcal{C}$. This finishes the proof of our theorem.

Recall from [1, 9, 14] that a module ${}_\Lambda T$ is said to be a *tilting* (respectively, *cotilting*) Λ-module provided $\mathrm{Ext}^2_\Lambda(T, -) = 0$ (respectively, $\mathrm{Ext}^2_\Lambda(-, T) = 0$), $\mathrm{Ext}^1_\Lambda(T, T) = 0$ and the number of nonisomorphic indecomposable direct summands of ${}_\Lambda T$ equals the rank of the Grothendieck group of Λ. Two algebras Λ and Γ are called *tilting-cotilting equivalent* (see [1]) if there is a sequence of finite-dimensional algebras $\Lambda = \Lambda_0, \Lambda_1, \ldots, \Lambda_m, \Lambda_{m+1} = \Gamma$ and a sequence of modules ${}_{\Lambda_i}T_i$, $0 \le i \le m$, such that $\Lambda_{i+1} = \mathrm{End}_{\Lambda_i}(T_i)$ and T_i is either a tilting or a cotilting module.

We have the following important consequence of Theorem 1.

THEOREM 2. *Let Λ be an hereditary algebra of type $\widetilde{A}_n$. Then B is stably equivalent to $T(\Lambda)$ if and only if B is isomorphic to the trivial extension $T(\Gamma)$ of an algebra Γ tilting-cotilting equivalent to Λ.*

P r o o f. If B is isomorphic to $T(\Gamma)$ and Γ is tilting-cotilting equivalent to Λ, then by the main results of [1, 17], B is stably equivalent to $T(\Lambda)$.

On the other hand, if B is stably equivalent to $T(\Lambda)$, then $B \cong T(\Gamma)$ and Γ is a gentle factor of an hereditary algebra whose bound quiver contains exactly one nonoriented relation-free cycle, and Γ, Λ have the same number of nonisomorphic simple modules. Therefore by [1, Theorem A], Γ is tilting-cotilting equivalent to Λ.

In [8] Happel showed that the stable module category of a selfinjective algebra has a natural structure of a triangulated category. Thus [13, Theorem 3.1] implies that Theorem 2 can be interpreted in the following way.

R e m a r k 1. *Let Λ be an hereditary algebra of type $\widetilde{A}_n$. Let B be a stably equivalent algebra to $T(\Lambda)$. Then there is a triangular equivalence Φ : $\underline{T(\Lambda)\text{-mod}} \to \underline{B\text{-mod}}$.*

After the paper had been written the author obtained a preprint by Peng Liangang and Xiao Jie in which a more general result was proved.

Acknowledgements. The author is indebted to the referee for his suggestions.

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Reçu par la Rédaction le 14.4.1992;
en version modifiée le 15.4.1993