COLLOQUIUM MATHEMATICUM

VOL. LXVI 1993 FASC. 2

SMOOTHNESS OF DENSITIES OF SEMIGROUPS OF MEASURES ON HOMOGENEOUS GROUPS

BY

JACEK DZIUBAŃSKI AND JACEK ZIENKIEWICZ (WROCŁAW)

0. Introduction. Smoothness of densities of semigroups of measures on nilpotent Lie groups was investigated by many authors (cf. e.g. [G], [GH], [BG]). In [G] P. Głowacki proved that the densities of a stable semigroup of symmetric measures $\{\mu_t\}_{t>0}$ with smooth Lévy measure are C^∞ and belong with all their derivatives to $L^2(G)$; for a semigroup with singular Lévy measure, this is not true in general (cf. [GH]). Recently T. Byczkowski and P. Graczyk [BG] have shown that if the Lévy measure of a semigroup of symmetric measures $\{\mu_t\}_{t>0}$ is of class C^1, compactly supported and coincides on a neighborhood of 0 with a nonzero stable Lévy measure, then the μ_t have smooth densities. Their proof is based on the Malliavin Calculus for jump processes.

The purpose of the present paper is to generalize the result of Byczkowski and Graczyk. We prove, by analytic methods, that an estimate from below for the Lévy measure of a semigroup $\{\mu_t\}_{t>0}$ (cf. (1.4)) already implies smoothness of the densities of μ_t.

Acknowledgements. The authors are greatly indebted to P. Głowacki, P. Graczyk, A. Hulanicki and L. Saloff-Coste for helpful comments.

1. Statement of the result. Let $\{\mu_t\}_{t>0}$ be a semigroup of positive symmetric measures on a homogeneous group G with compactly supported Lévy measure ν. We shall assume that the generating functional $\mathcal{A}$ of $\{\mu_t\}_{t>0}$ has the form

$$(1.1)\quad \langle \mathcal{A}f, f\rangle = \sum a_{i,j}X_iX_jf(0) + \lim_{\varepsilon\to 0}\int\limits_{\|x\|>\varepsilon}(f(x)-f(0))\,d\nu(x) + cf(0)\,,$$
$$= \Delta f(0) + \langle L, f\rangle + cf(0)\,,$$

where $(a_{i,j})$ is a symmetric positive semi-definite matrix, and $X_1,\ldots,X_n$ is

---

1991 *Mathematics Subject Classification*: Primary 43A80, 22E30, 43A05.

This research was supported by grants: KBN 210449101, KBN 210429101.

a basis of the Lie algebra of G. There is no loss of generality in assuming that the constant c in (1.1) vanishes.

Theorem 5.1 of Hunt [Hu] asserts that

$$\int_G \|x\|^2\, d\nu(x) < \infty\,, \tag{1.2}$$

where $\|\cdot\|$ denotes an Euclidean norm on G.

Assume that there exist constants $\alpha \in (0,2)$, $\varrho > 0$ and a nonnegative symmetric function $\Omega \in L^1_{\mathrm{loc}}(G)$ homogeneous of degree 0 such that

$$0 < \int_{\|x\|<1} \Omega(x)\, dx \tag{1.3}$$

and

$$\frac{\Omega(x)}{|x|^{Q+\alpha}}\, dx \le d\nu(x) \quad \text{on a ball } B(0,\varrho) = \{x \in G : |x| < \varrho\}\,, \tag{1.4}$$

where Q is the homogeneous dimension of G and $|\cdot|$ is a homogeneous norm on G (cf. Section 2).

Our aim is to prove the following

THEOREM (1.5). *The measures μ_t have smooth densities p_t such that for any natural numbers η, k, and every left-invariant differential operator D on G there exist constants C and $N = N(D,k,\eta) > 0$ such that*

$$|\partial_t^k D p_t(x)| \le C t^{-N} e^{-\eta|x|} \quad \textit{for } t < 1\,. \tag{1.6}$$

Moreover, for any natural numbers s, k, η, and every left-invariant differential operator D there are constants r and C such that

$$|\partial_t^k D p_t(x)| \le C e^{-\eta|x|} t^s \quad \textit{for } t < 1 \textit{ and } |x| > r\,. \tag{1.7}$$

2. Preliminaries. A *family of dilations* on a nilpotent Lie algebra G is a one-parameter group $\{\delta_t\}_{t>0}$ of automorphisms of G determined by

$$\delta_t \mathbf{e}_j = t^{d_j} \mathbf{e}_j\,,$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is a linear basis for G, and $d_1, \ldots, d_n$ are positive real numbers called the *exponents of homogeneity*. The smallest d_j is assumed to be 1.

If we regard G as a Lie group with multiplication given by the Campbell–Hausdorff formula, then the dilations δ_t are also automorphisms of the group structure of G, and the nilpotent Lie group G equipped with these dilations is called a *homogeneous group*.

The *homogeneous dimension* of G is the number Q defined by $d(\delta_t x) = t^Q\, dx$, where dx is a right-invariant Haar measure on G.

Let

$$X_j f(x) = \frac{d}{dt}\Big|_{t=0} f(xt\mathbf{e}_j), \qquad Y_j f(x) = \frac{d}{dt}\Big|_{t=0} f(t\mathbf{e}_j x).$$

If $I = (i_1, \ldots, i_n)$ is a multi-index, $i_j \in \mathbb{N} \cup \{0\}$, we set

$$X^I f = X_1^{i_1} \ldots X_n^{i_n} f, \quad Y^I f = Y_1^{i_1} \ldots Y_n^{i_n} f, \quad |I| = i_1 d_1 + \ldots + i_n d_n,$$
$$\|I\| = i_1 + \ldots + i_n, \quad I! = i_1! \ldots i_n!, \quad x^I = x_1^{i_1} \ldots x_n^{i_n},$$

where $x = x_1 \mathbf{e}_1 + \ldots + x_n \mathbf{e}_n$.

Recall (cf. [FS, p. 26]) that for every multi-index I there exist families of polynomials $\{v_J\}_{\|J\| \le \|I\|}$, $\{w_J\}_{\|J\| \le \|I\|}$ such that

$$X^I f(x) = \sum v_J(x) Y^J f(x), \qquad Y^I f(x) = \sum w_J(x) X^J f(x). \tag{2.1}$$

For a distribution T on G and a multi-index I, we define a distribution T_I by the formula

$$\langle T_I, f \rangle = \langle T, M_{(-x)^I} f \rangle, \quad \text{where} \quad M_{(-x)^I} f(x) = (-x)^I f(x). \tag{2.2}$$

We choose and fix a *homogeneous subadditive norm* on G, that is, a continuous positive symmetric function $x \mapsto |x|$ which is, moreover, smooth on $G \setminus \{0\}$ and satisfies

$$|\delta_t x| = t|x|, \quad |x| = 0 \text{ if and only if } x = 0, \quad |xy| \le |x| + |y|.$$

The existence of such a norm was proved e.g. in [HS]. Note that if $|\cdot|_0$ is another homogeneous norm on G, not necessarily subadditive, then there is a constant C such that $C^{-1}|x| \le |x|_0 \le C|x|$.

Denote by $\|x\|$ a fixed Euclidean norm on G. Proposition (1.5) of [FS] asserts that there are constants $C_1 > 0$ and $C_2 > 0$ such that

$$C_1\|x\| \le |x| \le C_2\|x\|^{1/Q} \quad \text{for } |x| \le 1. \tag{2.3}$$

For a nonnegative constant η let us denote by $\widetilde{\eta}(\cdot)$ the weight

$$\widetilde{\eta}(x) = e^{\eta|x|}, \tag{2.4}$$

and by $L^2(\widetilde{\eta})$ the Hilbert space of functions on G with the norm

$$\|f\|_{\widetilde{\eta}}^2 = \int_G |f(x)|^2 \widetilde{\eta}(x)\, dx.$$

Let $S^\infty(G) = \{f \in C^\infty(G) : \|(X^I f)(\cdot)\widetilde{\eta}(\cdot)\|_{L^\infty} < \infty \text{ for every } I \text{ and } \eta\}$. Note that if T is a compactly supported distribution, then the operator T defined by

$$Tf(x) = f * T(x) \tag{2.5}$$

preserves $S^\infty(G)$.

For $r \ge 0$ let $\bar{r}$ be the smallest number such that $\bar{r} > r$ and $\bar{r} = |I|$ for some multi-index I.

For $f\in C_{\mathrm{c}}^\infty(G)$, $r>0$ and $x\in G$, define

$$f^{(x)}(y)=f(xy)-\sum_{|I|\le r}\frac{1}{I!}X^I f(x)y^I,\qquad y\in G. \tag{2.6}$$

Theorem (2.7) (cf. [FS], Theorem 1.37). *For $r,a>0$, there are constants C and K such that for every $f\in C^\infty(G)$,*

$$|f^{(x)}(y)|\le Cf^{\langle r\rangle}(x)|y|^{\bar r}\qquad \textit{for } |y|\le a,$$

where $f^{\langle r\rangle}(x)=\sum_{I\in W}\sup_{|z|\le K}|X^I f(xz)|$, $W=\{I: r<|I|, \|I\|\le[r]+1\}$.

A distribution T on G is said to be a *kernel of order r* if $T\in L^1_{\mathrm{loc}}(G\setminus\{0\})$ and satisfies

$$\langle T,f\circ\delta_t\rangle=t^r\langle T,f\rangle\qquad \text{for } f\in C_{\mathrm{c}}^\infty(G),\ t>0. \tag{2.8}$$

A kernel T of order r is said to be *regular* if $T\in C^\infty(G\setminus\{0\})$.

A distribution T smooth away from 0 which is supported in a compact set and coincides with a kernel of order r in a neighborhood of 0 will be called a *truncated kernel of order r*.

Note that if T is a truncated kernel of order r, then T_I is a truncated kernel of order $r-|I|$.

We shall denote by $\widetilde R$ the kernel of order α defined by

$$\langle\widetilde R,f\rangle=\lim_{\varepsilon\to0}\int_{|x|>\varepsilon}\frac{f(x)-f(0)}{|x|^{Q+\alpha}}\Omega(x)\,dx,$$

where Ω is the function from the first section.

For $\beta\in(0,2)$ denote by P_β the truncated kernel of order β defined by

$$\langle P_\beta,f\rangle=\lim_{\varepsilon\to0}\int_{\varepsilon<|x|<1}\frac{f(x)-f(0)}{|x|^{Q+\beta}}\,dx.$$

The following theorem due to P. Głowacki [G1] plays a crucial role in all what follows.

Theorem (2.9). *For every regular kernel $\widetilde P$ of order β, $0\le\beta\le\alpha$, there exists a constant C such that*

$$\|\widetilde Pf\|_{L^2}\le C(\|\widetilde Rf\|_{L^2}+\|f\|_{L^2})\qquad \textit{for } f\in S^\infty(G), \tag{2.10}$$

$$\|\widetilde Pf\|_{L^2}\le C(\|P_\alpha f\|_{L^2}+\|f\|_{L^2})\qquad \textit{for } f\in S^\infty(G). \tag{2.11}$$

Using the theory of subordination and (2.9) one can prove that for a kernel $\widetilde P$ as above,

$$\langle\widetilde Pf,f\rangle\le C(-\langle\widetilde Rf,f\rangle+\|f\|_{L^2})\qquad \text{for } f\in S^\infty(G), \tag{2.12}$$

and, consequently,

$$\langle Pf, f\rangle \le C(-\langle Rf, f\rangle + \|f\|_{L^2}) \quad \text{for } f \in S^\infty(G)\,, \tag{2.13}$$

where P, R are compactly supported distributions which coincide with $\widetilde{P}$ and $\widetilde{R}$ in a neighborhood of the origin and belong to $L^1_{\mathrm{loc}}(G \setminus \{0\})$.

A subset Γ of G is said to be *uniformly discrete* if for every function $\varphi \in C_{\mathrm{c}}^\infty(G)$ the function

$$\sum_{z\in\Gamma} \lambda_z \varphi$$

is bounded, where $\lambda_z\varphi(x) = \varphi(zx)$.

The following lemma is due to B. Helffer and J. Nourrigat (cf. [HN]).

LEMMA (2.14). *For every homogeneous group G there is a uniformly discrete subset Γ of G and a function $\varphi \in C_{\mathrm{c}}^\infty(G)$ such that*

$$\sum_{a\in\Gamma} |\lambda_a \varphi(x)|^2 = 1\,.$$

LEMMA (2.15). *For a uniformly discrete subset Γ of G and every $\varepsilon > 0$ the sum*

$$\sum_{z\in\Gamma} (1 + |z|)^{-Q-\varepsilon}$$

is finite.

COROLLARY (2.16). *If $\eta > 0$, then $\int \widetilde{\eta}(x)^{-1}\,dx < \infty$, where $\widetilde{\eta}(x)$ is defined by* (2.4). *Moreover, if Γ is a uniformly discrete subset of G, then*

$$\sum_{z\in\Gamma} \widetilde{\eta}(z)^{-1} < \infty\,.$$

3. Holomorphic semigroups on weighted Hilbert spaces. The purpose of the present section is to prove the following

THEOREM (3.1). *Let $\{\mu_t\}$ be a convolution semigroup of nonnegative subprobabilistic symmetric measures on G whose generating functional has compact support. Then for every function $\widetilde{\eta}$ of the form* (2.4) *the family $T_t f = f * \mu_t$ of operators forms a C_0 semigroup on $L^2(\widetilde{\eta})$ which has an extension to a holomorphic semigroup in some sector $\Delta_\theta = \{z : |\mathrm{Arg}\, z| < \theta\}$.*

First we prove

PROPOSITION (3.2). *Let $\{T_z\}$, $\mathrm{Re}\, z > 0$, be a holomorphic semigroup of operators on $L^2(G)$ which is a C_0 semigroup on $L^2(\widetilde{\eta})$ for a fixed function $\widetilde{\eta}$. Assume that $C_{\mathrm{c}}^\infty(G)$ is contained in the domain of the infinitesimal generator A of $\{T_t\}$ considered on $L^2(\widetilde{\eta})$. Then for every $\theta \in [0, 1)$ the semigroup $\{T_z\}$ is holomorphic on $L^2(\widetilde{\eta}^{1-\theta})$ in the sector $\Delta_\theta = \{z : |\mathrm{Arg}\, z| < \theta\}$.*

Proof. The idea of our proof comes from [S]. Without restriction of generality we can assume that there are constants M_0 and M_1 such that

$$\|T_t f\|_{\tilde\eta} \le M_0\|f\|_{\tilde\eta} \quad \text{and} \quad \|T_z f\|_{L^2(G)} \le M_1\|f\|_{L^2(G)}\,. \tag{3.3}$$

Step 1: *The family* $\{T_z\}_{z\in\Delta_\theta}$ *is uniformly bounded on* $L^2(\widetilde\eta^{1-\theta})$.

Proof. For $f,g\in C_{\mathrm c}^\infty(G)$ such that $\|f\|_{L^2}=\|g\|_{L^2}=1$ define a holomorphic function $F_{f,g}$ in the strip $0\le \operatorname{Re} z\le 1$ by

$$F_{f,g}=\int_G T_{e^{iz}}(f\cdot\widetilde\eta^{-(1-z)/2})(x)(g\cdot\widetilde\eta^{-(1-z)/2})(x)\widetilde\eta^{1-z}(x)\,dx\,. \tag{3.4}$$

Since $f,g\in C_{\mathrm c}^\infty(G)$ the function $F_{f,g}$ is bounded. Obviously, by (3.3) and the fact that $\|f\cdot\widetilde\eta^{-(1-z)/2}\|_{\tilde\eta^{1-\operatorname{Re} z}}=\|g\cdot\widetilde\eta^{-(1-z)/2}\|_{\tilde\eta^{1-\operatorname{Re} z}}=1$, we get

$$|F_{f,g}(it)|\le M_0,\qquad |F_{f,g}(1+it)|\le M_1\,. \tag{3.5}$$

In view of the Phragmén–Lindelöf theorem, we have

$$|F_{f,g}(z)|\le \max(M_0,M_1)=M\,. \tag{3.6}$$

The definition of $F_{f,g}$ and (3.6) imply that for $t\in\mathbb R$ and $\theta\in[0,1]$ the operator $T_{e^{i\theta-t}}$ is bounded on $L^2(\widetilde\eta^{1-\theta})$ and

$$\|T_{e^{i\theta-t}}f\|_{\tilde\eta^{1-\theta}}\le M\|f\|_{\tilde\eta^{1-\theta}}\,. \tag{3.7}$$

By the same argument, we get

$$\|T_{e^{-i\theta-t}}f\|_{\tilde\eta^{1-\theta}}\le M\|f\|_{\tilde\eta^{1-\theta}}\,. \tag{3.8}$$

Fix $x\in\mathbb R$ with $0\le|x|\le\theta$. By (3.7), (3.8), and (3.3), we have

$$\|T_{e^{ix-t}}\|_{L^2(\tilde\eta^{1-|x|})\to L^2(\tilde\eta^{1-|x|})}\le M,\qquad \|T_{e^{ix-t}}\|_{L^2(G)\to L^2(G)}\le M\,.$$

An interpolation argument gives

$$\|T_{e^{ix-t}}f\|_{\tilde\eta^{1-\theta}}\le M\|f\|_{\tilde\eta^{1-\theta}}.$$

Step 2: *The function* $\Delta_\theta\ni z\mapsto T_z\in\mathcal L(L^2(\widetilde\eta^{1-\theta}))$ *is holomorphic.*

Proof. This follows from Step 1 and from the fact that for $f,g\in C_{\mathrm c}^\infty(G)$ the function

$$\Delta_\theta\ni z\mapsto \int_G (T_zf)(x)g(x)\widetilde\eta^{1-\theta}(x)\,dx$$

is holomorphic.

Step 3: *If* $f\in L^2(\widetilde\eta^{1-\theta})$, *then*

$$\lim_{z\to0,\,z\in\Delta_{\theta-\varepsilon}}\|T_zf-f\|_{\tilde\eta^{1-\theta}}=0\,. \tag{3.9}$$

Proof. This follows from Steps 1 and 2 and from the fact that $\bigcup_{t>0}\operatorname{Ran}(T_t)$ is dense in $L^2(\widetilde\eta^{1-\theta})$ (cf. [Da, p. 63, Problem 2.35]). ∎

Proof of Theorem (3.1). Since $T_t f = f * \mu_t$ form a semigroup of selfadjoint contractions on $L^2(G)$, by the spectral theorem, we conclude that $\{T_t\}$ has an extension to a holomorphic semigroup $\{T_z\}_{\mathrm{Re}\, z>0}$ on $L^2(G)$. Now fix η sufficiently large. Theorem (4.1) of Hulanicki [H] asserts that for every $s > 0$ there is a constant C_s such that

$$\langle \mu_t, \widetilde{\eta}\rangle \le C_s < \infty \quad \text{for } t \in (0, s)\,. \tag{3.10}$$

It follows from [H, Proposition (4.2)] that $\{T_t\}$ is a C_0 semigroup of operators on $L^1(\widetilde{\eta})$, and C_c^∞ is contained in the domain of the infinitesimal generator A of $\{T_t\}$ considered on $L^1(\widetilde{\eta})$. Hence by (3.10) for $f \in C_c^\infty(G)$,

$$\|f * \mu_t\|_{\tilde{\eta}} \le C\|f\|_{\tilde{\eta}}\langle \mu_t, \widetilde{\eta}^{1/2}\rangle \le C_s\|f\|_{\tilde{\eta}} \quad \text{for } t \in (0, s)\,, \tag{3.11}$$

and

$$\begin{aligned}\overline{\lim_{t\to 0}}\, &\|t^{-1}(T_t f - f) - Af\|^2_{\tilde{\eta}} \\ &\le \overline{\lim_{t\to 0}}\, \|t^{-1}(T_t f - f) - Af\|_{L^\infty} \|t^{-1}(T_t f - f) - Af\|_{L^1(\tilde{\eta})} = 0.\end{aligned} \tag{3.12}$$

Now, (3.11) and (3.12) imply that the family $\{T_t\}$ is a C_0 semigroup on $L^2(\widetilde{\eta})$, and $C_c^\infty(G)$ is contained in the domain of the infinitesimal generator of $\{T_t\}$ considered on $L^2(\widetilde{\eta})$. Our proof is finished by applying Proposition (3.2). ∎

4. Weighted subelliptic estimates. In this section we prove some subelliptic estimates associated with the operator $\mathcal{A}$. Our aim is the following

THEOREM (4.1). *For any weights $\widetilde{\eta}$, $\widetilde{\eta'}$ of the form* (2.4) *such that $\eta > \eta'$ and for every multi-index I there are constants N and C such that*

$$\|X^I f\|^2_{\widetilde{\eta'}} \le C\sum_{j=1}^{N} \|\mathcal{A}^j f\|^2_{\tilde{\eta}} + C\|f\|^2_{\tilde{\eta}} \quad \textit{for } f \in S^\infty(G)\,. \tag{4.2}$$

First we prove some lemmas.

LEMMA (4.3). *For every multi-index I with $\|I\| = 1$ there is a constant C such that*

$$\|\mathcal{A}_I f\|^2_{L^2} \le -C\langle \mathcal{A}f, f\rangle \quad \textit{for } f \in S^\infty(G)\,. \tag{4.4}$$

Moreover, if $\|I\| > 1$ then $\mathcal{A}_I$ is bounded on $L^2(G)$, and

$$\|L_I f\|^2_{L^2} \le C\|f\|^2_{L^2}\,, \quad \|\Delta_I f\|^2 \le C\|f\|^2_{L^2}\,. \tag{4.5}$$

Proof. Note that if $\|I\| > 1$, then the estimate $\|L_I f\|^2_{L^2} \le C\|f\|^2_{L^2}$ follows from (1.2) and the definition of L_I. It is obvious that Δ_I is bounded in this case. So (4.5) is proved.

Let $\|I\| = 1$. Since ν is symmetric,

$$\int \|f_x - f\|^2_{L^2}\, d\nu(x) = -2\langle Lf, f\rangle\,,$$

where $f_x(y)=f(yx)$. Applying the Schwarz inequality and (1.2), we have

$$\begin{aligned}\|L_I f\|^2_{L^2(G)} &= \int \Big|\int (f(yx)-f(y))x^I\,d\nu(x)\Big|^2 dy\\ &\le \int \Big(\int |f(yx)-f(y)|^2\,d\nu(x)\Big)\Big(\int (x^I)^2\,d\nu(x)\Big)\,dy\\ &\le -2C\langle Lf,f\rangle\,.\end{aligned}$$

Change the coordinates in such a way that $\Delta=\sum_j Z_j^2$, where the Z_j are left-invariant vector fields (not necessarily homogeneous). Then $\Delta_I f=(\sum_j Z_j^2)_I f=\sum_j \alpha_{j,I}Z_jf$, and

$$\|\Delta_I f\|^2_{L^2}\le C\sum_j \alpha^2_{j,I}\|Z_jf\|^2_{L^2}\le -C\sum_j\langle Z_j^2 f,f\rangle\le -C\langle\Delta f,f\rangle. \ \blacksquare$$

COROLLARY (4.6). *For every $\varepsilon>0$ there exists a constant C_ε such that*

$$\|\mathcal{A}_I f\|^2_{L^2}\le\varepsilon\|\mathcal{A}f\|^2_{L^2}+C_\varepsilon\|f\|^2_{L^2},\qquad f\in S^\infty(G),\ \|I\|=1\,.$$

LEMMA (4.7). *Assume that $\varphi\in C_c^\infty(G)$. Then*

$$[M_\varphi,\mathcal{A}]f(y)=\sum_{0<|I|\le Q}\frac{1}{I!}X^I\varphi(y)\mathcal{A}_I f(y)+K_\varphi f(y)\,,$$

where $K_\varphi f(y)=\int\varphi^{(y)}(x)f(yx)\,d\nu(x)$ (cf. (2.6)). *Moreover, the operator K_φ is bounded on $L^2(G)$.*

Proof. Using the Taylor expansion (cf. (2.6)) to the function φ at the point y, we get the required equalities. $\blacksquare$

The following lemma is a weighted version of Corollary (4.6).

LEMMA (4.8). *For every function $\widetilde{\eta}$ of the form* (2.4) *and every $\varepsilon>0$ there exists a constant C_ε such that if $\|I\|=1$, then*

$$\|\mathcal{A}_I f\|^2_{\tilde\eta}\le\varepsilon\|\mathcal{A}f\|^2_{\tilde\eta}+C_\varepsilon\|f\|^2_{\tilde\eta}\,. \tag{4.9}$$

Moreover, if $\|I\|>1$, then

$$\|\mathcal{A}_I f\|^2_{\tilde\eta}\le C_{I,\eta}\|f\|^2_{\tilde\eta}\quad \textit{for } f\in S^\infty(G)\,. \tag{4.10}$$

Proof. (4.10) is obvious since $\mathcal{A}_I$ is bounded on $L^2(G)$ and has compact support (cf. [Dz, Lemma (4.6)]).

Fix I_0 with $\|I_0\|=1$. Let φ and Γ be as in Lemma (2.14). Let $\psi\in C_c^\infty(G)$ with $\psi=1$ on $\operatorname{supp}\varphi\cdot\operatorname{supp}\nu$. Since Γ is uniformly discrete, by Corollary (4.6), we get

$$\begin{aligned}\|\mathcal{A}_{I_0}f\|^2_{\tilde\eta}&=\int_G\Big|\sum_{a\in\Gamma}\mathcal{A}_{I_0}((\lambda_a\varphi)f)(y)\Big|^2\widetilde{\eta}(y)\,dy \\ &\le C\int_G\sum_{a\in\Gamma}|\mathcal{A}_{I_0}((\lambda_a\varphi)f)(y)|^2\widetilde{\eta}(y)\,dy\end{aligned}\tag{4.11}$$

$$\leq C_\varepsilon \sum_{a\in\Gamma} \|(\lambda_a\varphi)f\|^2_{L^2}\widetilde{\eta}(a) + C\sum_{a\in\Gamma} \varepsilon\|\mathcal{A}((\lambda_a\varphi)f)\|^2_{L^2}\widetilde{\eta}(a)$$
$$\leq C_\varepsilon \sum_{a\in\Gamma} \|(\lambda_a\varphi)f\|^2_{L^2}\widetilde{\eta}(a) + C\varepsilon\sum_{a\in\Gamma} \|(\lambda_a\varphi)(\mathcal{A}f)\|^2_{L^2}\widetilde{\eta}(a)$$
$$+ C\varepsilon\sum_{a\in\Gamma} \|(\lambda_a\psi)[M_{\lambda_a\varphi},\mathcal{A}]f\|^2_{L^2}\widetilde{\eta}(a) = I_1 + I_2 + I_3\,.$$

Obviously $I_1 + I_2 \leq C_\varepsilon\|f\|^2_{\tilde\eta} + \varepsilon\|\mathcal{A}f\|^2_{\tilde\eta}$. Similarly, using Lemmas (4.3) and (4.7), we obtain

$$I_3 \leq C\varepsilon \sum_{\|I\|=1} \|\mathcal{A}_I f\|^2_{\tilde\eta} + C_\varepsilon\|f\|^2_{\tilde\eta}\,.$$

Finally, we have

$$\|\mathcal{A}_{I_0} f\|^2_{\tilde\eta} \leq C_\varepsilon\|f\|^2_{\tilde\eta} + \varepsilon\|\mathcal{A}f\|^2_{\tilde\eta} + C\varepsilon\sum_{\|I\|=1}\|\mathcal{A}_I f\|^2_{\tilde\eta}\,. \tag{4.12}$$

Now taking ε sufficiently small and summing (4.12) over all I_0 with $\|I_0\| = 1$, we get (4.9). ∎

LEMMA (4.13). *For a fixed function $\widetilde{\eta}$ of the form* (2.4) *there is a constant C such that*

$$\|P_{\alpha/2}f\|^2_{\tilde\eta} \leq C(\|\mathcal{A}f\|^2_{\tilde\eta} + \|f\|^2_{\tilde\eta}) \quad \textit{for } f\in S^\infty(G)\,. \tag{4.14}$$

Proof. By (2.9), we get

$$\|P_{\alpha/2}f\|^2_{L^2} \leq C(-\langle\widetilde{R}f,f\rangle + \|f\|^2_{L^2}) \leq C(\|\mathcal{A}f\|^2_{L^2} + \|f\|^2_{L^2})\,.$$

The last inequality holds because

$$-2\langle\widetilde{R}f,f\rangle = \int \|f_x - f\|^2_{L^2}\frac{\Omega(x)}{|x|^{Q+\alpha}}\,dx \tag{4.15}$$
$$\leq \int_{|x|<\varrho} \|f_x - f\|^2_{L^2}\,d\nu(x) + \int_{|x|\geq\varrho} \|f_x - f\|^2\frac{\Omega(x)}{|x|^{Q+\alpha}}\,dx$$
$$\leq -2\langle Lf,f\rangle + C\|f\|^2_{L^2} \leq -2\langle\mathcal{A}f,f\rangle + C\|f\|^2_{L^2}\,.$$

Analogously to the proof of the previous lemma, we have

$$\|P_{\alpha/2}f\|^2_{\tilde\eta} \leq C\sum_{a\in\Gamma}\|P_{\alpha/2}((\lambda_a\varphi)f)\|^2_{L^2}\widetilde{\eta}(a)$$
$$\leq C\sum_{a\in\Gamma}(\|\mathcal{A}((\lambda_a\varphi)f)\|^2_{L^2}\widetilde{\eta}(a) + \|(\lambda_a\varphi)f\|^2_{L^2}\widetilde{\eta}(a))$$
$$\leq C\Big(\|\mathcal{A}f\|^2_{\tilde\eta} + \sum_{\|I\|=1}\|\mathcal{A}_I f\|^2_{\tilde\eta} + \|f\|^2_{\tilde\eta}\Big)\,.$$

Using Lemma (4.8) we obtain (4.14). ∎

LEMMA (4.16). *For every $\alpha \in (0,1)$ there is $\beta > 0$ such that for any functions $\widetilde{\eta} > \widetilde{\eta}'$ of the form* (2.4) *there is a constant C such that*

$$\|P_\beta * f\|_{\widetilde{\eta}'}^2 \le C(\|P_\alpha f\|_{\widetilde{\eta}}^2 + \|f\|_{\widetilde{\eta}}^2), \qquad f \in S^\infty(G)\,. \tag{4.17}$$

P r o o f. Assume that $\phi \in C_c^\infty(G)$, $0 \le \phi$, $\int \phi = 1$, $\phi(x) = \phi(x^{-1})$. Let $\phi_t(x) = t^{-Q}\phi(\delta_{t^{-1}}x)$.

S t e p 1: *There is a constant C such that*

$$\|f * \phi_t - f\|_{L^2}^2 \le -C_\alpha t^\alpha \langle P_\alpha f, f\rangle \quad \textit{for } f \in C_c^\infty(B(0,1))\,. \tag{4.18}$$

P r o o f. By the definition of P_α, we have

$$\lim_{\varepsilon \to 0} \int\limits_{\varepsilon<|x|<1} \int |f(yx) - f(y)|^2\, dy\, \frac{1}{|x|^{Q+\alpha}}\, dx = -2\langle P_\alpha f, f\rangle\,.$$

Hence, for $j > 0$,

$$\int\limits_{2^{-j-1} \le |x| \le 2^{-j}} \|f_x - f\|_{L^2}^2\, dx \le -2^{-j(Q+\alpha)+1}\langle P_\alpha f, f\rangle\,,$$

and, consequently,

$$\begin{aligned}
\|f * \phi_t - f\|_{L^2}^2 &\le \int\limits_G \|f_x - f\|_{L^2}^2 \phi_t(x)\, dx \\
&\le Ct^{-Q} \int\limits_{|x|<ct} \|f_x - f\|_{L^2}^2\, dx \\
&\le Ct^{-Q} \sum_{j \ge 0,\, 2^{-j} < ct} 2^{-j(Q+\alpha)}\langle P_\alpha f, f\rangle \\
&\le C_1 \sum_{j \ge 0,\, 2^{-j} < ct} 2^{-j\alpha}\langle P_\alpha f, f\rangle \le C_\alpha t^\alpha \langle P_\alpha f, f\rangle\,.
\end{aligned}$$

S t e p 2: *There are constants C and $d > 0$ such that for $|x| < 1$,*

$$\|\lambda_x(f * \phi_t) - f * \phi_t\|_{L^2}^2 \le C|x|^2 t^{-2d}\|f\|_{L^2}^2, \qquad f \in C_c^\infty(B(0,1)),\ t < 1\,. \tag{4.19}$$

P r o o f. Indeed, let $x = \|x\|Y$, $\|Y\| = 1$. By (2.1) we get

$$\int\limits_G |f * \phi_t(xy) - f * \phi_t(y)|^2\, dy = \int\limits_G \bigg| \int\limits_0^{\|x\|} \frac{d}{ds}(f * \phi_t)(sY \cdot y)\, ds \bigg|^2 dy$$

$$= \int_G \Big| \int_0^{\|x\|} Y(f * \phi_t)(sY \cdot y)\, ds\Big|^2 dy$$

$$= \int_G \Big| \int_0^{\|x\|} \sum_{\|I\|=1} w_I(sY \cdot y)(f * X^I \phi_t)(sY \cdot y)\, ds\Big|^2 dy$$

$$\le C \sum_{\|I\|=1} \Big(\int_0^{\|x\|} \Big(\int_G |f * X^I \phi_t(sY \cdot y)|^2\, dy\Big)^{1/2} ds\Big)^2,$$

which combined with the Schwarz inequality and (2.3) implies (4.19).

Step 3: *There are* $\gamma > 0$ *and* $C > 0$ *such that for* $f \in C_c^\infty(B(0,1))$,

$$\|\lambda_x f - f\| \le C|x|^\gamma (\|P_\alpha f\|_{L^2} + \|f\|_{L^2})\,. \tag{4.20}$$

Proof. It suffices to consider $|x| < 1$. By (4.17) and (4.19), we have

$$\begin{aligned}\|\lambda_x f - f\|_{L^2}^2 &\le C(\|\lambda_x f - \lambda_x(f * \phi_t)\|_{L^2}^2 \\ &\quad + \|\lambda_x(f * \phi_t) - f * \phi_t\|_{L^2}^2 + \|f * \phi_t - f\|_{L^2}^2) \\ &\le -2C_\alpha t^\alpha \langle P_\alpha f, f\rangle + C|x| t^{-2d} \|f\|_{L^2}^2\,.\end{aligned}$$

Putting $t = |x|^\sigma$ with sufficiently small $\sigma > 0$, we get (4.20).

Step 4: *There are* m *and* $\gamma > 0$ *such that for* $f \in C_c^\infty(B(0,r))$,

$$\|\lambda_x f - f\|_{L^2} \le C|x|^\gamma (1+r)^m (\|P_\alpha f\|_{L^2} + \|f\|_{L^2})\,. \tag{4.21}$$

Proof. This follows by applying dilations to the function f and using Step 3.

Step 5: *There is* $\gamma > 0$ *such that for every* $\eta > \eta'$ *there is a constant* C *such that*

$$\|\lambda_x f - f\|_{\widetilde{\eta'}}^2 \le C|x|^\gamma (\|P_\alpha f\|_{\widetilde{\eta}}^2 + \|f\|_{\widetilde{\eta}}^2) \quad \textit{for } f \in S^\infty(G),\ |x| < 1\,. \tag{4.22}$$

Proof. Let Γ and φ be as in Lemma (2.14). Then

$$\begin{aligned}\|\lambda_x f - f\|_{\widetilde{\eta'}}^2 &= \sum_{a\in\Gamma} \int |f(xy) - f(y)|^2 (\lambda_a\varphi)(y)^2 \widetilde{\eta'}(y)\, dy \\ &\le C\Big(\sum_{a\in\Gamma} \int |f(xy)(\lambda_a\varphi)(xy) - f(y)(\lambda_a\varphi)(y)|^2 \widetilde{\eta'}(a)\, dy \\ &\quad + \sum_{a\in\Gamma} \int |f(xy)|^2 |(\lambda_a\varphi)(xy)^2 - (\lambda_a\varphi)(y)^2| \widetilde{\eta'}(a)\Big) dy \\ &= I_1 + I_2\,.\end{aligned}$$

Using Step 4 we conclude that there is a polynomial w such that

$$\begin{aligned} I_1 &= C\sum_{a\in\Gamma}\|\lambda_x(f\lambda_a\varphi)-(f\lambda_a\varphi)\|_{L^2}^2\widetilde{\eta}'(a)\widetilde{\eta}(a)^{-1}\widetilde{\eta}(a) \\ &\le C|x|^\gamma\sum_{a\in\Gamma}w(a)\widetilde{\eta}'(a)\widetilde{\eta}(a)^{-1}(\|P_\alpha((\lambda_a\varphi)f)\|_{L^2}^2+\|(\lambda_a\varphi)f\|_{L^2}^2)\widetilde{\eta}(a)\,. \end{aligned}$$

Since $\eta>\eta'$ we obtain

$$I_1\le C_0C|x|^\gamma\sum_{a\in\Gamma}(\|(\lambda_a\varphi)(P_\alpha f)\|_{L^2}^2+\|(\lambda_a\varphi)f\|_{L^2}^2+\|[M_{\lambda_a\varphi},P_\alpha]f\|_{L^2}^2)\widetilde{\eta}(a),$$

where $C_0=\sup_{a\in\Gamma}\{w(a)\widetilde{\eta}'(a)\widetilde{\eta}(a)^{-1}\}$. One can easily check using the Taylor expansion of φ that

$$\sum_{a\in\Gamma}\|[M_{\lambda_a\varphi},P_\alpha]f\|_{L^2}^2\widetilde{\eta}(a)\le C\|f\|_{\tilde{\eta}}^2\,.$$

Hence,

$$I_1\le C|x|^\gamma(\|P_\alpha f\|_{\tilde{\eta}}^2+\|f\|_{\tilde{\eta}}^2)\,.$$

Let us remark that there is a polynomial v and positive $\omega,\ \delta$ such that

$$|\lambda_a\varphi(xy)-\lambda_a\varphi(y)|\le C|axa^{-1}|^\omega\le Cv(a)|x|^\delta\quad\text{ for } |x|<1\,. \tag{4.23}$$

Moreover, there is $r>0$ such that

$$\lambda_a\varphi(xy)-\lambda_a\varphi(y)=0\quad\text{ for } y\notin\{z\in G:|a|-r<|z|<|a|+r\}\,. \tag{4.24}$$

We are now in a position to estimate I_2. By (4.24) we get

$$\begin{aligned} I_2 &\le C\sum_{a\in\Gamma}|f(xy)|^2|(\lambda_a\varphi)(xy)^2-(\lambda_a\varphi)(y)^2|\,\widetilde{\eta}'(a)\,dy \\ &\le C\sum_{k=0}^\infty\sum_{a\in\Gamma_k}\int\limits_{k-r<|y|<k+r}|f(xy)|^2v(a)|x|^\delta\,\widetilde{\eta}'(a)\,dy\,, \end{aligned}$$

where $\Gamma_k=\{a\in\Gamma:k-r<|a|<k+r\}$. Since card Γ_k increases polynomially with respect to k and $|x|<1$, we get $I_2\le C|x|^\delta\|f\|_{\tilde{\eta}}^2$.

Step 6: $\|P_\beta*f\|_{\tilde{\eta}'}^2\le C(\|P_\alpha f\|_{\tilde{\eta}}^2+\|f\|_{\tilde{\eta}}^2)$.

Proof. By the Schwarz inequality

$$\|P_\beta*f\|_{\tilde{\eta}'}^2=\int\limits_G\Big|\int\limits_{|x|<1}(f(xy)-f(y))\frac{\phi(|x|)}{|x|^{Q+\beta}}\,dx\Big|^2\widetilde{\eta}'(y)\,dy$$

$$\le \int_G \Bigg(\Bigg(\int_{|x|<1} \frac{1}{|x|^{Q-\varepsilon}}\, dx \Bigg) \Bigg(\int_{|x|<1} |f(xy) - f(y)|^2 \frac{dx}{|x|^{Q+\varepsilon+2\beta}} \Bigg) \Bigg) \eta'(y)\, dy$$

$$\le C \int_{|x|<1} \|\lambda_x f - f\|^2_{\tilde{\eta}'} \frac{dx}{|x|^{Q+\varepsilon+2\beta}}\,.$$

Taking ε and β sufficiently small and using (4.22), we obtain the required estimate. ∎

Note that the operator $f \mapsto P_\beta * f$ commutes with the operator $f \mapsto \mathcal{A}f = f * \mathcal{A}$ on $S^\infty(G)$. Hence by Lemmas (4.16) and (4.13), we get

COROLLARY (4.25). *For every natural k there exist constants N and $C = C_{k,N,\eta,\eta'}$ such that*

$$\|P_\beta^k * f\|_{\tilde{\eta}'} \le C \sum_{j=0}^{N} \|\mathcal{A}^j f\|_{\tilde{\eta}} \quad \text{for } f \in S^\infty(G)\,.$$

LEMMA (4.26). *For every multi-index I there are constants N and C such that*

$$\|Y^I f\|_{\tilde{\eta}'} \le C(\|f\|_{\tilde{\eta}'} + \|P_\beta^N * f\|_{\tilde{\eta}'})\,.$$

Proof. This lemma is a consequence of [Dz, Theorem (4.3)]. ∎

Proof of Theorem (4.1). Since $\|X^I f\|_{\tilde{\eta}'} \le C \sum_{\|J\| \le \|I\|} \|Y^J f\|_{\tilde{\eta}}$, using Lemma (4.26) and Corollary (4.25), we get (4.2). ∎

5. Smoothness and pointwise estimates. In the present section we give the proof of Theorem (1.5).

LEMMA (5.1). *For any weights $\widetilde{\eta} > \widetilde{\eta'}$, every multi-index I and every relatively compact neighborhood U of the origin there are constants C and N such that for every $a \in G$,*

$$\|X^I f\|^2_{L^\infty(aU)} \le C\widetilde{\eta'}(a)^{-1} \sum_{j=0}^{N} \|\mathcal{A}^j f\|^2_{\tilde{\eta}}\,. \tag{5.2}$$

Proof. Let V be relatively compact such that $\overline{U} \subset V$. The Sobolev inequality implies that there is a constant C such that

$$\|X^I f\|^2_{L^\infty(aU)} \le C \sum_{\|J\| \le M(I)} \|X^J f\|^2_{L^2(aV)}\,, \tag{5.3}$$

which combined with Theorem (4.1) gives

$$\|X^I f\|^2_{L^\infty(aU)} \le C\widetilde{\eta'}(a)^{-1} \sum_{\|J\| \le M} \|X^J f\|^2_{\tilde{\eta}'} \le C\widetilde{\eta'}(a)^{-1} \sum_{j=0}^{N} \|\mathcal{A}^j f\|^2_{\tilde{\eta}}\,. \ ∎$$

Proof of the first part of Theorem (1.5). Let $\varphi \in C_c^\infty(B(0,2))$. Theorem (3.1) and (2.1) imply that $\varphi * \mu_t \in S^\infty(G)$. From Lemma (5.1) and from the fact that our semigroup is holomorphic on weighted Hilbert spaces we deduce that for every multi-index I and every $\eta > \eta'$ there are constants N and C such that for $t \in (0,1)$,

$$(5.4)\qquad |X^I(\varphi * \mu_t)(a)| \le C\widetilde{\eta'}(a)^{-1/2}t^{-N}\|\varphi\|_{\tilde\eta} \le C'\widetilde{\eta'}(a)^{-1/2}t^{-N}\|\varphi\|_{L^2}\,.$$

Hence, the linear functional $\Lambda\varphi = X^I(\varphi * \mu_t)(a)$ on $L^2(B(0,2))$ is bounded and its norm is estimated by $C\widetilde{\eta'}(a)^{-1/2}t^{-N}$. From the Riesz theorem, we get

$$(5.5)\qquad \|X^I\mu_t\|_{L^2(B(0,2)a)} \le C\widetilde{\eta'}(a)^{-1/2}t^{-N}, \quad t \in (0,1)\,.$$

By the Sobolev inequality, we obtain $d\mu_t(x) = p_t(x)\,dx$ with $p_t \in S^\infty(G)$ and for every η and every multi-index I there exist constants C and N such that

$$(5.6)\qquad |X^I p_t(x)| + |Y^I p_t(x)| \le Ct^{-N}\widetilde{\eta}(x)^{-1}, \quad t \in (0,1)\,.$$

It follows from (5.6) that the p_t belong to the domain of the operator $\mathcal{A}^k$ for every natural k and

$$(5.7)\qquad |\mathcal{A}^k p_t(x)| \le Ct^{-N(k)}\widetilde{\eta}(x)^{-1} \quad \text{for } t \in (0,1)\,.$$

Hence, by (5.6) and (5.7), the function $Y^I p_{2t}(x) = Y^I p_t * p_t(x)$ is differentiable with respect to t and

$$(5.8)\qquad \partial_t^k Y^I p_{2t}(x) = (Y^I p_t) * (\mathcal{A}^k p_t)(x)\,.$$

The equality (5.8) combined with (5.6) and (5.7) implies

$$|\partial_t^k Y^I p_t(x)| \le C\widetilde{\eta}(x)^{-1}t^{-N(k,I)}\,,$$

which by (2.1) gives (1.6). ∎

In order to prove the second part of Theorem (1.5) we need the following lemma in the spirit of Duflo [Du, Proposition 14].

LEMMA (5.9). *For every η and every natural number k there exist a relatively compact neighborhood U of the origin and a constant C such that*

$$\int\limits_{x\notin U} p_t(x)\widetilde{\eta}(x)\,dx \le Ct^k, \quad t \in (0,1)\,.$$

Proof. Fix k. Let r be such that $\operatorname{supp}\nu \subset U_0 = B(0,r)$ and let $U = U_0^{k+2}$. For $\psi \in C_c^\infty(U)$ with $0 \le \psi \le 1$ and $\psi = 1$ on U_0^{k+1} we define a family of functions $\widetilde{\eta}_n \in C_c^\infty(G)$ by

$$(5.10)\qquad \widetilde{\eta}_n(x) = \widetilde{\eta}(x)(1-\psi(x))\psi(\delta_{1/n}x)\,.$$

Obviously there exists a constant C independent of n such that

$$|X^I\widetilde{\eta}_n(x)| \le C\widetilde{\eta}(x), \qquad |\mathcal{A}^k\widetilde{\eta}_n(x)| \le C\widetilde{\eta}(x)\,, \tag{5.11}$$

$$\lim_{n\to\infty}\widetilde{\eta}_n(x) = \widetilde{\eta}(x) \qquad \text{for } x \notin U\,. \tag{5.12}$$

Now, define a function h_n by

$$h_n(t) = \int_G \widetilde{\eta}_n(x)p_t(x)\,dx = T_t\widetilde{\eta}_n(0)\,.$$

Obviously $h_n \in C^\infty[0,\infty)$. Moreover,

$$\partial_t^j h_n(t) = \int_G (\mathcal{A}^j\widetilde{\eta}_n)(x)p_t(x)\,dx, \qquad j = 0,1,\ldots,k\,. \tag{5.13}$$

By (3.10), (5.11) and (5.13), we obtain

$$|\partial_t^j h_n(t)| < C, \qquad j = 0,1,\ldots,k,\ t\in(0,1), \text{ with } C \text{ independent of } n\,.$$

Since $\partial_t^j h_n(0) = 0$ for $j = 0,1,\ldots,k$, we get $h_n(t) \le Ct^k$ for $t \in (0,1)$, which by (5.12) and the Lebesgue convergence theorem ends the proof of the lemma. ∎

Proof of the second part of Theorem (1.5)

LEMMA (5.14). *For every weight $\widetilde{\eta'}$, every natural number k and every multi-index I there exist constants C and r such that*

$$|X^Ip_t(x)| \le Ct^k\widetilde{\eta'}(x)^{-1} \qquad \textit{for } |x|>r,\ t<1\,. \tag{5.15}$$

Proof. Let $\eta > \eta'$. For a multi-index I let N and C be constants such that

$$\sum_{\|J\|\le\|I\|}(|X^Jp_t(x)| + |Y^Jp_t(x)|) \le Ct^{-N}\widetilde{\eta}(x)^{-1} \qquad \text{for } t\in(0,1)\,. \tag{5.16}$$

By Lemma (5.9) for a fixed natural number k there are constants l and C such that

$$\int_{|x|>l} p_t(x)\widetilde{\eta}(x)\,dx \le Ct^{N+k} \qquad \text{for } t\in(0,1)\,. \tag{5.17}$$

Let $\varphi \in C_c^\infty(G)$ with $\varphi(x) = 1$ for $|x| < 2l$ and $0 \le \varphi \le 1$. Then by (2.1), we get

$$\begin{aligned}
&|\widetilde{\eta'}(x)X^Ip_t(x)| \\
&\quad\le C\int_G |((1-\varphi)p_{t/2})(xy^{-1})X^Ip_{t/2}(y)|\widetilde{\eta'}(xy^{-1})\widetilde{\eta'}(y)\,dy \\
&\qquad + C\sum_{\|J\|\le\|I\|}\int_G |w_J(x)|\,|Y^J(\varphi p_{t/2})(xy^{-1})p_t(y)|\widetilde{\eta'}(xy^{-1})\widetilde{\eta'}(y)\,dy\,,
\end{aligned}$$

which combined with (5.16), (5.17) gives (5.15). ∎

COROLLARY (5.18). *For every multi-index I, every weight $\widetilde{\eta}'$ and any nonnegative integers s, k there exist C and r such that*

$$|\partial_t^s X^I p_t(x)| \le C t^k \widetilde{\eta}'(x)^{-1} \quad \textit{for } t < 1,\ |x| > r\,. \tag{5.19}$$

Proof. Let us remark that $\partial_t p_t = \mathcal{A} p_t$ and ∂_t and X^I commute. Using the fact that the distribution $\mathcal{A}$ has compact support, Lemma (5.4) and Sobolev inequalities we obtain (5.19). ∎

REFERENCES

[BG] T. Byczkowski and P. Graczyk, *Malliavin calculus for stable processes on Heisenberg group*, Probab. Math. Statist. 13 (1992), 277–292.

[Da] E. B. Davies, *One Parameter Semigroups*, Academic Press, 1980.

[Du] M. Duflo, *Représentations de semi-groupes de mesures sur un groupe localement compact*, Ann. Inst. Fourier (Grenoble) 28 (3) (1978), 225–249.

[Dz] J. Dziubański, *On semigroups generated by subelliptic operators on homogeneous groups*, Colloq. Math. 64 (1993), 215–231.

[FS] G. B. Folland and E. M. Stein, *Hardy Spaces on Homogeneous Groups*, Princeton University Press, 1982.

[G] P. Głowacki, *Stable semi-groups of measures as commutative approximate identities on non-graded homogeneous groups*, Invent. Math. 83 (1986), 557–582.

[G1] —, *Lipschitz continuity of densities of stable semigroups of measures*, Colloq. Math. 66 (1993), 29–47.

[GH] P. Głowacki and A. Hulanicki, *A semi-group of probability measures with non-smooth differentiable densities on a Lie group*, ibid. 51 (1987), 131–139.

[HS] W. Hebisch and A. Sikora, *A smooth subadditive homogeneous norm on a homogeneous group*, Studia Math. 96 (1990), 231–236.

[HN] B. Helffer et J. Nourrigat, *Caractérisation des opérateurs hypoelliptiques homogènes invariants à gauche sur un groupe nilpotent gradué*, Comm. Partial Differential Equations 4 (1979), 899–958.

[H] A. Hulanicki, *A class of convolution semi-groups of measures on a Lie group*, in: Lecture Notes in Math. 828, Springer, 1980, 82–101.

[Hu] G. A. Hunt, *Semi-groups of measures on Lie groups*, Trans. Amer. Math. Soc. 81 (1956), 264–293.

[S] E. M. Stein, *Topics in Harmonic Analysis Related to the Littlewood–Paley Theory*, Princeton Univ. Press, 1970.

INSTITUTE OF MATHEMATICS
WROCŁAW UNIVERSITY
PL. GRUNWALDZKI 2/4
50-384 WROCŁAW, POLAND

Reçu par la Rédaction le 5.2.1993