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SYSTEMS OF CLAIRAUT TYPE

BY

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A characterization of systems of first order differential equations with (classical) complete solutions is given. Systems with (classical) complete solutions that consist of hyperplanes are also characterized.

0. Introduction. About 260 years ago Alex Claude Clairaut [2] studied the following equation which is now called the Clairaut equation:

$$y = x \cdot \frac{dy}{dx} + f\left(\frac{dy}{dx}\right).$$

It is usually taught in the first or second year university course of calculus and treated as an example of a non-linear equation which is easily solved. Moreover, it has a beautiful geometric structure: There exists a "general solution" that consists of lines $y = t \cdot x + f(t)$ where t is a parameter, and the singular solution is the envelope of that family.

In [6] we studied ordinary differential equations with geometric structure such as that of the Clairaut equation. In this note we shall be concerned with systems of first order partial differential equations (briefly, equations) with (classical) complete solutions, which are the natural generalization of the Clairaut equation. Since general solutions and the singular solution of the equation can be constructed from the complete solution, this class of equations plays a principal role in classical treatises (cf. Carathéodory [1], Courant–Hilbert [3], Forsyth [4], [5]). However, we have never seen characterizations for this class of equations. Our main result (Theorem 1.1) gives such a characterization. In §2 we shall give the proof of the main theorem. In §3, we shall study a class of equations with (classical) complete solutions that consist of hyperplanes, which is a direct generalization of the classical Clairaut equation.

All maps considered here are differentiable of class C^∞, unless stated otherwise.

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1. The main result. A first order differential equation is most naturally interpreted as being a closed subset of $J^1(\mathbb{R}^n,\mathbb{R})$. Unless the contrary is specifically stated, we use the following definition. A *system of partial differential equations of first order* (or briefly, an *equation*) is a submersion germ $F:(J^1(\mathbb{R}^n,\mathbb{R}),z_0)\to(\mathbb{R}^d,0)$ on the 1-jet space of functions of n variables, where $1\le d\le n$. Let θ be the canonical contact form on $J^1(\mathbb{R}^n,\mathbb{R})$ which is given by $\theta=dy-\sum_{i=1}^n p_i dx_i$, where (x,y,p) are the canonical coordinates of $J^1(\mathbb{R}^n,\mathbb{R})$. We define a *geometric solution* of $F=0$ to be an immersion $i:(L,q_0)\to(J^1(\mathbb{R}^n,\mathbb{R}),z_0)$ of an n-dimensional manifold such that $i^*\theta=0$ and $i(L)\subset F^{-1}(0)$ (i.e. a Legendrian submanifold which is contained in $F^{-1}(0)$). We say that z_0 is a π-*singular point* if $F(z_0)=0$ and $\operatorname{rank}\big(\frac{\partial F_i}{\partial p_j}(z_0)\big)<n$. We denote the set of π-singular points by $\Sigma_\pi(F)$ and write $\pi(\Sigma_\pi(F))=D_F$, where $\pi(x,y,p)=(x,y)$. We call D_F the *discriminant set* of the equation $F=0$.

An equation $F=0$ is said to be of *Clairaut type* if there exist smooth function germs $B_{ji},A^l_{ik}:(J^1(\mathbb{R}^n,\mathbb{R}),z_0)\to\mathbb{R}$ for $i,j=1,\dots,n$, $k=1,\dots,d$ and $l=1,\dots,d$ such that

$$(1)\qquad \frac{\partial F_l}{\partial x_i}+p_i\frac{\partial F_l}{\partial y}=\sum_{j=1}^n B_{ji}\frac{\partial F_l}{\partial p_j}+\sum_{k=1}^d A^l_{ik}F_k \qquad (i=1,\dots,n \text{ and } l=1,\dots,d)$$

and

$$(2)\qquad B_{ji}=B_{ij}\,,$$

$$(3)\qquad \frac{\partial B_{jk}}{\partial x_i}+p_i\frac{\partial B_{jk}}{\partial y}+\sum_{l=1}^n B_{li}\frac{\partial B_{jk}}{\partial p_l}=\frac{\partial B_{ji}}{\partial x_k}+p_k\frac{\partial B_{ji}}{\partial y}+\sum_{l=1}^n B_{lk}\frac{\partial B_{ji}}{\partial p_l}$$

at any $z\in(F^{-1}(0),z_0)$ for $i,j,k=1,\dots,n$.

We also say that an $(n-d+1)$-parameter family of function germs

$$f:(\mathbb{R}^{n-d+1}\times\mathbb{R}^n,(t_0,x_0))\to(\mathbb{R},y_0)$$

is a (*classical*) *complete solution* of $F=0$ if $F_k\big(x,f(t,x),\frac{\partial f}{\partial x}(t,x)\big)=0$ for $k=1,\dots,d$ and $\operatorname{rank}\big(\frac{\partial f}{\partial t_i},\frac{\partial^2 f}{\partial t_i\partial x_j}\big)=n-d+1$. Our main result is the following.

THEOREM 1.1. *For an equation germ $F=0$, the following are equivalent.*

(i) $F=0$ *is a Clairaut type equation.*

(ii) $F=0$ *has a* (*classical*) *complete solution.*

In this case, if $\Sigma_\pi(F)\ne\emptyset$, then $\Sigma_\pi(F)$ is a geometric solution (*i.e. the* s i n g u l a r s o l u t i o n) *of $F=0$ and the discriminant set D_F is the envelope of the family of graphs of the complete solution.*

By the classical existence theorem (see [7]), if $F = 0$ is a π-regular equation, there exists a (classical) complete solution. Thus we can assert that a π-regular equation is of Clairaut type by the above theorem.

We now give two examples which describe the above assertion.

EXAMPLES 1.2. 1) The following equation is a generalization of the classical Clairaut equation:

$$F_i(p_1, \dots, p_n) = 0 \qquad (i = 1, \dots, d-1)\,,$$
$$F_d(x, y, p) = y - \sum_{i=1}^{n} p_i x_i - f(p_1, \dots, p_n) = 0\,,$$

where $F_i,\ f$ are function germs. Since $F = (F_1, \dots, F_d)$ is a submersion, we have $\operatorname{rank}(\partial F_i/\partial x_j) = d-1$. Thus the set $F^{-1}(0)$ is locally parametrized by an immersion $a(t) = (a_1(t), \dots, a_n(t))$, where $t = (t_1, \dots, t_{n-d+1})$. It follows that we get a complete solution

$$y = \sum_{i=1}^{n} a_i(t) x_i + f(a_1(t), \dots, a_n(t))\,.$$

We can easily check that

$$\frac{\partial F_l}{\partial x_i} + p_i \frac{\partial F_l}{\partial y} = 0$$

on $F^{-1}(0)$. This means that we can choose $B_{ij} = 0$.

2) Consider the equation $F_1 = p_1^2 - y = 0,\ F_2 = p_2 = 0$ $(n = 2)$. Then we have

$$\frac{\partial F_1}{\partial x_1} + p_1 \frac{\partial F_1}{\partial y} = -p_1, \qquad \frac{\partial F_1}{\partial x_2} + p_2 \frac{\partial F_1}{\partial y} = -p_2\,,$$
$$\frac{\partial F_2}{\partial x_1} + p_1 \frac{\partial F_2}{\partial y} = 0, \qquad \frac{\partial F_2}{\partial x_2} + p_2 \frac{\partial F_2}{\partial y} = 0$$

and

$$\frac{\partial F_1}{\partial p_1} = 2p_1, \quad \frac{\partial F_1}{\partial p_2} = 0, \quad \frac{\partial F_2}{\partial p_1} = 0, \quad \frac{\partial F_2}{\partial p_2} = 1\,.$$

It follows that

$$\frac{\partial F_1}{\partial x_1} + p_1 \frac{\partial F_1}{\partial y} = -\frac{1}{2} \cdot \frac{\partial F_1}{\partial p_1} + 0 \cdot \frac{\partial F_1}{\partial p_2} + 0 \cdot F_1 + 0 \cdot F_2,$$
$$\frac{\partial F_1}{\partial x_2} + p_2 \frac{\partial F_1}{\partial y} = 0 \cdot \frac{\partial F_1}{\partial p_1} + 0 \cdot \frac{\partial F_1}{\partial p_2} + 0 \cdot F_1 - 1 \cdot F_2,$$
$$\frac{\partial F_2}{\partial x_1} + p_1 \frac{\partial F_2}{\partial y} = -\frac{1}{2} \cdot \frac{\partial F_2}{\partial p_1} + 0 \cdot \frac{\partial F_2}{\partial p_1} + 0 \cdot F_1 + 0 \cdot F_2,$$
$$\frac{\partial F_2}{\partial x_2} + p_2 \frac{\partial F_2}{\partial y} = 0 \cdot \frac{\partial F_2}{\partial p_1} + 0 \cdot \frac{\partial F_2}{\partial p_2} + 0 \cdot F_1 + 0 \cdot F_2.$$

The complete solution is given by $y = \frac{1}{4}(x_1 + t)^2$.

In classical textbooks (see [1], [3], [4], [5]), the notion of singular solution appears together with the notion of complete solutions. Namely, the singular solution is defined to be the envelope of the family of graphs of the complete solution. Theorem 1.1 gives a characterization of the class of equations having a complete solution as the class of Clairaut type equations.

2. Proof of Theorem 1.1. We need some elementary properties of Legendrian singularities. For a Legendrian immersion germ $i:(L,q_0)\to J^1(\mathbb{R}^n,\mathbb{R})$, $q_0\in L$ is said to be a *Legendrian singular point* if $\pi\circ i$ is not an immersion at q_0. Then we have the following lemma.

LEMMA 2.1. *For an equation* $F:(J^1(\mathbb{R}^n,\mathbb{R}),z_0)\to(\mathbb{R}^d,0)$, *the following are equivalent.*

(i) $F=0$ *has a* (*classical*) *complete solution.*

(ii) *There exists a foliation on* $F^{-1}(0)$ *by geometric solutions of* $F=0$ *whose leaves are Legendrian nonsingular.*

Proof. Suppose that $f:(\mathbb{R}^{n-d+1}\times\mathbb{R}^n,(t_0,x_0))\to(\mathbb{R},y_0)$ is a (classical) complete solution of $F=0$. Then we define a map germ $j^1_*f:(\mathbb{R}^{n-d+1}\times\mathbb{R}^n,(t_0,x_0))\to(J^1(\mathbb{R}^n,\mathbb{R}),z_0)$ by

$$j^1_*f(t,x)=\left(x,f(t,x),\frac{\partial f}{\partial x}(t,x)\right).$$

We can easily show that j^1_*f is an immersion if and only if $\operatorname{rank}\left(\frac{\partial f}{\partial t_i},\frac{\partial^2 f}{\partial t_i\partial x_j}\right)=n-d+1$. It follows that j^1_*f gives a local parametrization of $F^{-1}(0)$ and the family $\{\mathrm{Image}\,j^1_*f_t\}_{t\in(\mathbb{R}^{n-d+1},t_0)}$ gives the desired foliation, where $f_t(x)=f(t,x)$.

For the converse, we remark that q_0 is a Legendrian nonsingular point of a Legendrian immersion $i:(L,q_0)\to J^1(\mathbb{R}^n,\mathbb{R})$ if and only if $\widetilde{\pi}\circ i$ is a local diffeomorphism at q_0, where $\widetilde{\pi}(x,y,p)=x$.

Suppose that there exists a foliation which satisfies (ii). Then we have an $(n-d+1)$-parameter family of smooth sections $s:(\mathbb{R}^{n-d+1}\times\mathbb{R}^n,(t_0,x_0))\to(J^1(\mathbb{R}^n,\mathbb{R}),z_0)$ of $\widetilde{\pi}$ (i.e. $\widetilde{\pi}\circ s(t,x)=x$) such that s is an immersion, $s(\mathbb{R}^{n-d+1}\times\mathbb{R}^n)=F^{-1}(0)$ and $s_t^*\theta=0$ for any $t\in(\mathbb{R}^{n-d+1},t_0)$, where $s_t(x)=s(t,x)$. It follows that there exists a family of function germs $f:(\mathbb{R}^{n-d+1}\times\mathbb{R}^n,(t_0,x_0))\to(\mathbb{R}^n,y_0)$ such that $j^1_*f(t,x)=s(t,x)$. Since s is an immersion, f is a (classical) complete solution of $F=0$.

Now we can give the proof that (i) implies (ii) in Theorem 1.1.

Proof of Theorem 1.1, (i)⇒(ii). By the assumption, there exist function germs $B_{ij},A^l_{ik}:(J^1(\mathbb{R}^n,\mathbb{R}),z_0)\to\mathbb{R}$ such that formulas (1), (2) and (3) hold.

We consider linearly independent vector fields

$$V_i = \frac{\partial}{\partial x_i} + p_i \frac{\partial}{\partial y} - \sum_{j=1}^{n} B_{ji} \frac{\partial}{\partial p_j} \qquad (i = 1, \ldots, n)$$

on $(J^1(\mathbb{R}^n, \mathbb{R}), z_0)$. Let $c(t)$ be an integral curve of V_i such that $c(0) \in F^{-1}(0)$. Then we can show that

$$\frac{dF_l(c(t))}{dt}\bigg|_{t=0} = \frac{\partial F_l}{\partial x_i} + p_i \frac{\partial F_l}{\partial y} - \sum_{j=1}^{n} B_{ji} \frac{\partial F_l}{\partial p_j} = 0\,.$$

It follows that $V_i(z) \in T_z F^{-1}(0)$ for any $z \in F^{-1}(0)$. Since the V_i are linearly independent, we can define an n-dimensional distribution E on $F^{-1}(0)$ which is generated by the vectors $V_i(z)$ at each $z \in F^{-1}(0)$. By direct calculation, we have

$$[V_i, V_k] = \sum_{j=1}^{n} \bigg(\frac{\partial B_{ji}}{\partial x_k} - \frac{\partial B_{jk}}{\partial x_i} + p_k \frac{\partial B_{ji}}{\partial y} - p_i \frac{\partial B_{jk}}{\partial y} + \sum_{l=1}^{n} B_{lk} \frac{\partial B_{ji}}{\partial p_l} - \sum_{l=1}^{n} B_{li} \frac{\partial B_{jk}}{\partial p_l} \bigg) \frac{\partial}{\partial p_j}$$

for any $i, k = 1, \ldots, n$. By the assumption, $[V_i, V_k](z) \in E_z$ for any $z \in F^{-1}(0)$. Thus the distribution E is integrable and there exists an n-dimensional foliation on $F^{-1}(0)$ by the Frobenius theorem. Since $\theta(V_i) = 0$, the leaves of this foliation are Legendrian submanifolds. By the definition of V_i, we have $d\widetilde{\pi}(V_i) = \partial/\partial x_i$. It follows that the leaves are Legendrian nonsingular. Hence this foliation gives a (classical) complete solution by Lemma 2.1.

The converse direction is fairly direct.

Proof of Theorem 1.1, (ii)⇒(i). Let $y = f(t, x)$ be a complete solution of $F = 0$. Calculating the x_i-derivative of $F_l\big(x, f(t,x), \frac{\partial f}{\partial x}(t,x)\big) = 0$, we have

$$\frac{\partial F_l}{\partial x_i} + \frac{\partial f}{\partial x_i} \frac{\partial F_l}{\partial y} + \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i} \frac{\partial F_l}{\partial p_j} = 0$$

at $\big(x, f(t,x), \frac{\partial f}{\partial x}(t,x)\big) \in F^{-1}(0)$.

Since $j^1_* f$ is an immersion germ, there exist function germs B_{ji}: $(J^1(\mathbb{R}^n, \mathbb{R}), z_0) \to \mathbb{R}$ such that

$$B_{ji} \circ j^1_* f = \frac{\partial^2 f}{\partial x_j \partial x_i} \qquad \text{for } i, j = 1, \ldots, n\,.$$

For any $z \in F^{-1}(0)$, there exists $(t, x) \in (\mathbb{R}^{n-d+1} \times \mathbb{R}^n, (t_0, x_0))$ such that

$\big(x, f(t,x), \frac{\partial f}{\partial x}(t,x)\big) = z$. Then we have

$$\frac{\partial F_l}{\partial x_i} + p_i \frac{\partial F_l}{\partial y} = \sum_{j=1}^{n} B_{ji} \frac{\partial F_l}{\partial p_j} \quad \text{on } F^{-1}(0)\,.$$

This means that there exists a function germ $A^l_{ik} : (J^1(\mathbb{R}^n, \mathbb{R}), z_0) \to \mathbb{R}$ such that

$$\frac{\partial F_l}{\partial x_i} + p_i \frac{\partial F_l}{\partial y} = \sum_{j=1}^{n} B_{ji} F_{p_j} + \sum_{k=1}^{d} A^l_{ik} F_k$$

for $i, j = 1, \ldots, n$ and $l = 1, \ldots, d$.

On the other hand, calculating the x_k-derivative of

$$\frac{\partial^2 f}{\partial x_j \partial x_i}(t,x) = B_{ji}\left(x, f(t,x), \frac{\partial f}{\partial x}(t,x)\right),$$

we have

$$\frac{\partial^3 f}{\partial x_j \partial x_i \partial x_k} = \frac{\partial B_{ji}}{\partial x_k} + \frac{\partial B_{ji}}{\partial y}\frac{\partial f}{\partial x_k} + \sum_{l=1}^{n} \frac{\partial B_{ji}}{\partial p_l} \frac{\partial f}{\partial x_l \partial x_k}\,.$$

Since $\frac{\partial f}{\partial x_k}(t,x) = p_k$, $\frac{\partial^2 f}{\partial x_l \partial x_k} = B_{lk}$ and f is smooth, $F = 0$ is Clairaut type. This completes the proof that (ii) implies (i).

Proof of the second part of Theorem 1.1. By the first part of the theorem, we may assume that there exists a (classical) complete solution $y = f(t,x)$ of $F = 0$ and $\Sigma_\pi(F) \neq \emptyset$. By the definition, $j^1_* f(t,x) \in \Sigma_\pi(F)$ if and only if

$$\operatorname{rank}\begin{pmatrix} E & \frac{\partial f}{\partial x} \\ 0 & \frac{\partial f}{\partial t} \end{pmatrix} = n \quad \text{at } (t,x)\,.$$

This is equivalent to the fact that $\frac{\partial f}{\partial t_i}(t,x) = 0$. The Jacobian matrix of this equation is given by $J\big(\frac{\partial f}{\partial t_1}, \ldots, \frac{\partial f}{\partial t_{n-d+1}}\big) = \big(\frac{\partial f}{\partial t_i \partial x_j}, \frac{\partial f}{\partial t_i \partial t_k}\big)$. Since

$$\operatorname{rank}\left(\frac{\partial f}{\partial t_1}, \frac{\partial f}{\partial t_i \partial x_j}\right) = \operatorname{rank}\left(0, \frac{\partial f}{\partial t_i \partial x_j}\right) = n - d + 1$$

at the point (t,x) with $j^1_* f(t,x) \in \Sigma_\pi(F)$, we have $\operatorname{rank} J\big(\frac{\partial f}{\partial t_1}, \ldots, \frac{\partial f}{\partial t_{n-d+1}}\big) = n - d + 1$. It follows that $\Sigma_\pi(F) = j^1_* f\big(\big\{\frac{\partial f}{\partial t_i} = 0 \mid i = 1, \ldots, n-d+1\big\}\big)$ is an n-dimensional submanifold.

On the other hand, $(j^1_* f)^*\theta = 0$ if and only if $\frac{\partial f}{\partial t_i}(t,x) = 0$. This means that $\Sigma_\pi(F)$ is a Legendrian submanifold. Furthermore, we consider the family of graphs of the complete solution which is given by the equation $f(t,x) - y = 0$. Then we can show that the set

$$\left\{ (x, f(t,x)) \,\middle|\, \text{there exists } t \in (\mathbb{R}^n, t_0) \text{ such that } \frac{\partial f}{\partial t_i}(t,x) = 0 \ (i = 1, \dots, n-d+1) \right\}$$

is the envelope of this family by the usual method of elementary calculus. This set is equal to the discriminant set D_F by the previous arguments. This completes the proof of Theorem 1.1.

3. The Clairaut system. In this section we shall study equations with (classical) complete solutions that consist of hyperplanes.

THEOREM 3.1. *For an equation* $F : (J^1(\mathbb{R}^n, \mathbb{R}), z_0) \to (\mathbb{R}^d, 0)$ *with* $\Sigma_\pi(F) \neq \emptyset$, *the following are equivalent.*

(i) *There exist smooth function germs* $A^l_{ik} : (J^1(\mathbb{R}^n, \mathbb{R}), z_0) \to \mathbb{R}$ *such that*

$$\frac{\partial F_l}{\partial x_i} + p_i \frac{\partial F_l}{\partial y} = \sum_{k=1}^{d} A^l_{ik} F_k \quad \text{for } i = 1, \dots, n,\ l = 1, \dots, d\,.$$

(ii) *There exists a* (*classical*) *complete solution of* $F = 0$ *such that all members are hyperplanes.*

(iii) *There exists a submersion germ* $G : (\mathbb{R}^n, p_0) \to (\mathbb{R}^d, 0)$ *and a function germ* $f : (\mathbb{R}^n, p_0) \to \mathbb{R}$ *such that*

$$F^{-1}(0) = \left\{ (x, y, p) \,\middle|\, G(p_1, \dots, p_n) = 0 \text{ and } y = \sum_{i=1}^{n} x_i p_i - f(p_1, \dots, p_n) \right\}.$$

Proof. Suppose that the equation $F = 0$ satisfies (i). By the proof of Theorem 1.1, the vector fields $V_i = \frac{\partial}{\partial x_i} + p_i \frac{\partial}{\partial y}$ generate a completely integrable distribution E. By the definition of V_i, maximal integral submanifolds of E are affine Legendrian subspaces in $J^1(\mathbb{R}^n, \mathbb{R})$, so that (ii) follows.

Suppose that a family of hyperplanes $y = \sum_{i=1}^n a_i(t) x_i + b(t)$ is a complete solution of $F = 0$, where $t \in (\mathbb{R}^{n-d+1}, t_0)$. Since $\Sigma_\pi(F) \neq \emptyset$, we can calculate that rank $\left(\frac{\partial a_i(t)}{\partial t_j}(t_0)\right) = n - d + 1$, so that the germ $a : (\mathbb{R}^{n-d+1}, t_0) \to (\mathbb{R}^n, p_0)$ defined by $a(t) = (a_1(t), \dots, a_n(t))$ is an immersion germ. It follows that there exists a submersion germ $G : (\mathbb{R}^n, p_0) \to (\mathbb{R}^{d-1}, 0)$ such that $(G^{-1}(0), p_0) = (\mathrm{Image}\, a, p_0)$. We can also find a function germ $f : (\mathbb{R}^n, p_0) \to \mathbb{R}$ such that $f \circ a(t) = b(t)$. Then we have the following inclusion:

$$F^{-1}(0) \supset \left\{ (x, y, p) \,\middle|\, G(p_1, \dots, p_n) = 0 \text{ and } y = \sum_{i=1}^{n} x_i p_i - f(p_1, \dots, p_n) \right\}.$$

However, both manifolds are of codimension d, so their germs are equal. This completes the proof that (ii) implies (iii). The remaining assertion can be proved by direct calculation just as in the proof of Theorem 1.1.

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