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ON EXTENSION OF THE GROUP OPERATION OVER THE ČECH–STONE COMPACTIFICATION

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The convolution of ultrafilters of closed subsets of a normal topological group $\mathbb{T}$ is considered as a substitute of the extension onto $(\beta\mathbb{T})^2$ of the group operation. We find a subclass of ultrafilters for which this extension is well-defined and give some examples of pathologies. Next, for a given locally compact group $\mathbb{L}$ and its dense subgroup $\mathbb{G}$, we construct subsets of $\beta\mathbb{G}$ algebraically isomorphic to $\mathbb{L}$. Finally, we check whether the natural mapping from $\beta\mathbb{G}$ onto $\beta\mathbb{L}$ is a homomorphism with respect to the extension of the group operation. All the results involve the existence of R-points.

0. Notation. We use the following notation: $\mathrm{card}(X)$ is the cardinality of a set X; ω is the first infinite cardinal as well as its ordinal type; $\mathbf{c}$ is 2^ω. For any topological space $\mathbb{T}$ we denote the operators of closure, interior and boundary in $\mathbb{T}$ by $\mathrm{cl}_\mathbb{T}$, $\mathrm{int}_\mathbb{T}$ and $\mathrm{bd}_\mathbb{T}$. Open-closed sets are briefly named clopen. Nwd means "nowhere dense". $F^{\rightarrow}X$ and $F^{\leftarrow}X$ are the image and preimage of a set X under a function F.

Let $\mathbf{Z}(\mathbb{T})$ be the ring of closed subsets of a given normal space $\mathbb{T}$. The *Stone space* $\mathbf{S}(\mathbb{T})$ over $\mathbb{T}$ is the space of ultrafilters on $\mathbf{Z}(\mathbb{T})$ with topology introduced by the closed base $\{\{u \in \mathbf{S}(\mathbb{T}) : X \in u\} : X \in \mathbf{Z}(\mathbb{T})\}$. It is well known that $\mathbf{S}(\mathbb{T})$ is homeomorphic to the Stone–Čech compactification $\beta\mathbb{T}$ of $\mathbb{T}$ [4, pp. 11–19], [5]. In the present paper they are identified. We treat $\mathbb{T}$ as a subset of $\beta\mathbb{T}$. Hence, any $t \in \mathbb{T}$ is identified with the principal ultrafilter $\{X \in \mathbf{Z}(\mathbb{T}) : t \in X\}$. The set $\beta\mathbb{T}\backslash\mathbb{T}$ is called the *growth*.

All spaces considered are assumed to have a countable base.

In the paper $\mathbb{L}$ is a locally compact topological group with group operation $+$ and neutral element e, while $\mathbb{G}$ is its dense subgroup. Basic results on topological groups [3] yield normality and translation-invariant metrizability of $\mathbb{L}$.

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Ψ is the continuous extension onto $\beta\mathbb{G}$ of the canonical injection $\mathbb{G} \to \beta\mathbb{L}$, $q \mapsto q$. We denote the set $\Psi^{\leftarrow}(\beta\mathbb{L}\setminus\mathbb{L})$ by $\mathbb{G}_\infty$ and $\Psi^{\leftarrow}(\mathbb{L})$ by $\beta\mathbb{G}_{\mathrm{fin}}$.

1. Imbedding $\beta\mathbb{G} \to \beta\mathbb{L}$. In this section the algebraic structures of $\mathbb{L}$ and $\mathbb{G}$ are not taken into account.

For each $U \in \beta\mathbb{G}$ let $\overline{U} = \{\mathrm{cl}_{\mathbb{L}}(X) : X \in U\}$.

PROPOSITION 1.1. (a) *For every $U \in \beta\mathbb{G}$, $\overline{U}$ has a unique extension to an ultrafilter from $\beta\mathbb{L}$.*

(b) *Let $U \in \beta\mathbb{G}$ and $u \in \beta\mathbb{L}$. Then $\Psi(U) = u$ if and only if $\overline{U} \subseteq u$.*

Proof. (a) Suppose that $u_1, u_2 \in \beta\mathbb{L}$, $\overline{U} \subseteq u_1$, $\overline{U} \subseteq u_2$ and $u_1 \neq u_2$. Fix disjoint $V_i \in u_i$ ($i = 1$ or 2) with nonempty interiors. Both sets $\mathbb{G} \cap V_i$ meet every element of U. This contradicts the fact that U is an ultrafilter.

(b) ($\Rightarrow$) Assume that $\Psi(U) = u$. Let $X \in \mathbf{Z}(\mathbb{G})$ be an element of U. Then $U \in \mathrm{cl}_{\beta\mathbb{G}}(X)$ and hence $u \in \Psi^{\rightarrow}(\mathrm{cl}_{\beta\mathbb{G}}(X))$. Since $\Psi^{\rightarrow}(\mathrm{cl}_{\beta\mathbb{G}}(X))$ is compact we get $\mathrm{cl}_{\mathbb{L}}(X) = \Psi^{\rightarrow}(\mathrm{cl}_{\beta\mathbb{G}}(X)) \in u$.

($\Leftarrow$) Suppose that $\overline{U} \subseteq u$. If $\Psi(U) = v$, then $\overline{U} \subseteq v$ and $u = v$ by (a). ∎

PROPOSITION 1.2. *Let $u \in \beta\mathbb{L}$. Then* $\mathrm{card}(\Psi^{\leftarrow}\{u\})$ *is either* 1 *or* $2^{\mathfrak{c}}$.

Proof. If $u \in \Psi^{\rightarrow}(\mathbb{G})$, its preimage is obviously one-element. Otherwise $\Psi^{\leftarrow}\{u\} \subseteq \beta\mathbb{G}\setminus\mathbb{G}$. If it is infinite, one can find its countable discrete subset E. Since $\Psi^{\leftarrow}\{u\}$ is a closed subset of $\beta\mathbb{G}\setminus\mathbb{G}$, $E \in \mathbf{Z}(\mathbb{G}\cup E)$. $\mathbb{G}\cup E$ is normal as it is a regular Lindelöf space. Hence $\mathrm{cl}_{\beta\mathbb{G}}(E) \cong \beta E \cong \beta\omega$. In particular, $\mathrm{card}(\beta E) = 2^{\mathfrak{c}}$.

Suppose now that $\Psi^{\leftarrow}\{u\} \subseteq \beta\mathbb{G}\setminus\mathbb{G}$ is finite and contains n ($n > 1$) elements $U_1, \ldots, U_n$. There are pairwise disjoint closed sets $V_i \in U_i$, $1 \leq i \leq n$. By Proposition 1.1(b), $V = \mathrm{cl}_{\mathbb{L}}(V_1) \cap \ldots \cap \mathrm{cl}_{\mathbb{L}}(V_n) \in u$. Note that $V = \mathrm{bd}_{\mathbb{L}}(V) \subseteq \mathbb{L}\setminus\mathbb{G}$. As stated above, $\mathbb{L}$ is a metric space and hence V is G_δ. Fix any sequence $\{G_i\}$ of open subsets of $\mathbb{L}$ such that $\mathrm{cl}_{\mathbb{L}}(G_{i+1}) \subseteq \mathrm{int}_{\mathbb{L}}(G_i)$ and $\bigcap\{G_i : i < \omega\} = V$. It can be assumed that $G_1 \cap \mathbb{G} \subseteq V_1$ (one could take $\mathrm{int}_{\mathbb{G}}(V_1) \neq \emptyset$). We define

$$V' = \bigcup\{\mathrm{cl}_{\mathbb{G}}(G_{4n}\setminus G_{4n+1}) : i < \omega\},$$
$$V'' = \bigcup\{\mathrm{cl}_{\mathbb{G}}(G_{4n+2}\setminus G_{4n+3}) : i < \omega\}.$$

Note that $V' \cup V'' \subseteq V_1$, $V' \cap V'' = \emptyset$ and $V \subseteq \mathrm{cl}_{\mathbb{L}}(V') \cap \mathrm{cl}_{\mathbb{L}}(V'')$. Hence at least one of V', V'' is not a member of U_1 and we have got an $(n+1)$th element of $\Psi^{\leftarrow}\{u\}$. This proves that the only possible finite cardinality of $\Psi^{\leftarrow}\{u\}$ is 1. ∎

DEFINITION 1.3. (a) $U \in \beta\mathbb{G}_{\mathrm{fin}}\setminus\mathbb{G}$ is *regular* if there is $\mathfrak{u} \in \beta(\mathbb{L}\setminus\{\Psi(U)\})$ such that $U = \{\mathbb{G} \cap A : A \in \mathfrak{u}\}$.

(b) $U \in \mathbb{G}_\infty$ is *regular* if there is $\mathfrak{u} \in \beta\mathbb{L}$ such that $\{U\} = \Psi^{\leftarrow}\{\mathfrak{u}\}$. ∎

2. Remote points in $\beta\mathbb{T}$. Let $\mathbb{T}$ be any normal space.

DEFINITION 2.1. An ultrafilter $Z \in \beta\mathbb{T}$ is a *remote point* (*R-point*) if it does not contain any nwd set. Z is a *far point* if it does not contain any discrete set.

The following R-point existence theorem is due to van Douwen [1]:

THEOREM 2.2. *If $\mathbb{T}$ has a countable base then every nonvoid G_δ subset of $\beta\mathbb{T}$ included in $\beta\mathbb{T}\setminus\mathbb{T}$ contains $2^{\mathfrak{c}}$ R-points.* ∎

Since $\mathbb{L}$ has a countable base we have

COROLLARY 2.3. *There are R-points in the growth of any noncompact dense-in-itself subset of $\mathbb{L}$. In particular, for any $u \in \mathbb{L}\setminus\mathbb{G}$, $\Psi^{\leftarrow}\{u\}$ contains R-points.* ∎

For given $X \subseteq \mathbb{L}$ and $a \in \mathbb{L}$, $X + a$ is defined by $\{x + a : x \in X\}$.

DEFINITION 2.4. Let $T, U \in \beta\mathbb{G}$. T is *similar* to U if there is $r \in \mathbb{L}$ such that $T_r(A) = \mathbb{G} \cap \mathrm{cl}_{\mathbb{L}}(A - r) \in T$ for any $A \in U$.

PROPOSITION 2.5. (a) *Any ultrafilter similar to a regular R-point is a regular R-point.*
(b) *Similarity of regular R-points is an equivalence relation.*

Proof. (a) Let T be an ultrafilter similar to a regular R-point U. There is $r \in \mathbb{L}$ such that $T_r(A) \in T$ for $A \in U$. Let N be a nwd subset of $\mathbb{L}$. By regularity of U there exists $M \in U$ whose closure in $\mathbb{L}\setminus\{\Psi(U)\}$ is disjoint from $N + r$. Then $T_r(M) \in T$ and $T_r(M) \cap N = \emptyset$. Hence T is a regular R-point.

(b) Let $S, T, U \in \beta\mathbb{G}$ be regular R-points. It is clear that similarity is reflexive. To prove symmetry suppose that S is similar to U. There is $r \in \mathbb{L}$ such that $T_r(A) \in T$ for all $A \in U$. Suppose that $B \in T$ and $T_{-r}(B) \notin U$. There is $C \in U$ which does not meet $T_{-r}(B)$. Although the set $T_r(T_{-r}(B)) \cap T_r(C)$ is not necessarily void, it is nwd and avoids a certain element of the regular R-point U. Since B and $T_r(T_{-r}(B))$ are equal up to a nowhere dense set, there is an element of the regular R-point T which is disjoint from $T_r(C)$, which contradicts the assumption of similarity.

Assume now that S is similar to T and T is similar to U. There are $q, r \in \mathbb{L}$ such that $T_r(A) \in T$ for any $A \in U$ and $T_q(B) \in S$ for any $B \in T$. Let $A \in U$. By regularity of U there is $C \in U$ which avoids $\mathrm{bd}_{\mathbb{L}}(\mathrm{cl}_{\mathbb{L}}(A))$. One can assume that $C \subseteq A$ (in the other case take $C \cap A$). Note that $T_r(T_q(C))$ is a subset of $T_{q+r}(A)$ and belongs to S. Hence $T_{q+r}(A) \in S$ as well and transitivity is proved. ∎

Example 3.4 will show that similarity is not transitive on $\beta\mathbb{G}$ or even on $\beta\mathbb{G}\setminus\mathbb{G}$.

3. Extension of the group operation: $\beta\mathbb{G}_{\mathrm{fin}}$ **and** $\mathbb{L}$**.** For each $p \in \mathbb{G}$ let $t_p : \mathbb{G} \to \mathbb{G}$ be the right translation $t_p(x) = x + p$, and let βt_p be its continuous extension onto $\beta\mathbb{G}$. For any $U \in \beta\mathbb{G}$ we have

$$\beta t_p(U) = \{X + p : X \in U\}.$$

PROPOSITION 3.1. *Let $\mathbb{G}$ be a dense-in-itself noncompact topological group. There is no continuous extension of the group operation from $\mathbb{G}^2$ onto $(\beta\mathbb{G})^2$.*

Proof. We show that for fixed $U \in \beta\mathbb{G} \setminus \mathbb{G}$ the mapping $p \mapsto \beta t_p(U)$ is not continuous. Let $\{p_n : n < \omega\} \subseteq \mathbb{G}$ be a discrete-in-itself sequence with a limit point $p \in \mathbb{G}$. Let $U \in \beta\mathbb{G}\setminus\mathbb{G}$. Take any open $X \subseteq \mathbb{G}$ such that $\beta t_p(U) \in \mathrm{cl}_{\beta\mathbb{G}}(X)$. The sequence $\{\beta t_{p_n}(U)\}$ can be disjoint from X, since U is fixed independently of p_n's. Thus the range of p over the mapping need not be the limit point of the range of $\{p_n : n < \omega\}$. ∎

DEFINITION 3.2. Let $\mathbb{T}$ be a topological semigroup. For each $T, U \in \beta\mathbb{T}$ we put $T + U = \{X \in \mathbf{Z}(\mathbb{T}) : (\exists A \in U)(\forall a \in A)\, X \in \beta t_a(T)\}$.

Note that the family $T + U$ is a filter on $\mathbf{Z}(\mathbb{T})$. In fact, for any $X, Y \in T + U$ we have $\{a \in \mathbb{T} : X \in \beta t_a(T)\} \supseteq A \in U$ and $\{a \in \mathbb{T} : Y \in \beta t_a(T)\} \supseteq B \in U$. Since T and U are filters, $\{a : X \cap Y \in \beta t_a(T)\} \supseteq \{a : X \in \beta t_a(T)\} \cap \{a : Y \in \beta t_a(T)\} \supseteq A \cap B \in U$. Every superset of a given $X \in T + U$ is obviously a member of $T + U$. Finally, $T + U \neq \emptyset$ (i.e. $\mathbb{G} \in T + U$). ∎

By an *extended group operation* on $\beta\mathbb{T}$ we mean every function that assigns to the pair (T, U) any extension of $T + U$ to an ultrafilter. 3.1 states that in the general case there can be no continuous extended group operation on $\beta\mathbb{T}$. Examples of its uniqueness and nonuniqueness will be given below. For the case $\mathbb{T} = \omega$, 3.2 defines the sum of ultrafilters, which has been investigated by many authors (e.g. Frolík [2]). If $\mathbb{T}$ is a group, we can rewrite 3.2 in the form

$$T + U = \{X \in \mathbf{Z}(\mathbb{T}) : (\exists A \in U)\ A \subseteq \{a \in \mathbb{G} : X - a \in T\}\}. \ ∎$$

PROPOSITION 3.3. *Let $t, u \in \mathbb{L}\setminus\mathbb{G}$.*

(a) *If $T \in \Psi^{\leftarrow}\{t\}$ and $U \in \Psi^{\leftarrow}\{u\}$ then every extension of $T + U$ to an ultrafilter is an element of $\Psi^{\leftarrow}\{t + u\}$.*

(b) *If $T \in \mathbb{G}_\infty$ or $U \in \mathbb{G}_\infty$ then every extension of $T+U$ to an ultrafilter is an element of $\mathbb{L}_\infty$.*

Proof. (a) By 1.1(a) it is sufficient to show that $\overline{T + U}$ is a subfamily of the principal ultrafilter $t + u$. Fix any $X \in T + U$. Let $A \in U$ be such that $X - a \in T$ for all $a \in A$. Then $(\forall a \in A)\ t \in \mathrm{cl}_{\mathbb{L}}(X - a)$ and $u \in \mathrm{cl}_{\mathbb{L}}(A)$. Hence $t + u \in \mathrm{cl}_{\mathbb{L}}(X)$ for any $X \in T + U$.

(b) Case $T \in \mathbb{G}_\infty$. $X \in T + U$ yields $X - a \in T$ for some $a \in \mathbb{G}$. Then for any conditionally compact (in $\mathbb{L}$) $C \subseteq X$ we have $(X \backslash C) - a \in T$ as well. Hence $T + U$ has no extension in $\mathbb{L}$.

Case $U \in \mathbb{G}_\infty$. Let $X \in T + U$. If X were conditionally compact, then since $X - a \in T$ for some $a \in \mathbb{G}$, there would be $\Psi(T) \in \mathbb{L}$; but then $\{a : X - a \in T\}$ could not contain an "unbounded" member of U. ∎

Example 3.4. Let $X = \mathrm{cl}_{\mathbb{L}}(X) \subseteq \mathbb{G}$ be an infinite discrete set. Let $R \in \mathrm{cl}_{\beta\mathbb{G}}(X) \backslash X$ and $r = \Psi(R)$. Let $s \in \mathbb{L} \backslash \mathbb{G}$, $s + r \in \mathbb{L} \backslash \mathbb{G}$, $S \in \Psi^{\leftarrow}\{s\}$ and suppose S is a regular R-point. Then:

(a) $S + R$ is a remote point, but not a regular one.

(b) R is similar to $S + R$, but $S + R$ is not similar to R. ∎

4. Regularity of the extended group operation. Now we shall try to answer the question if (and under what conditions) $T + U$ has a unique extension to an ultrafilter.

Proposition 4.1. *Let $\mathbb{G}$ be totally disconnected. Let $T, U \in \beta\mathbb{G}$ and $\Psi(T) = t \in \mathbb{L} \backslash \mathbb{G}$. The filter $T + U$ has no less than two different extensions iff there exists a clopen $X \subseteq \mathbb{G}$ and sets $B_X, B_Y \subseteq \mathbb{G}$ such that*:

(1) *For each $V \in U$, $\{a \in \mathbb{G} : X - a \in T\} \cap V \neq \emptyset$ and $\{a \in \mathbb{G} : (\mathbb{G} \backslash X) - a \in T\} \cap V \neq \emptyset$.*

(2) *$B_X \subseteq \{a \in \mathbb{G} : X - a \in T\}$ and $B_Y \subseteq \{a \in \mathbb{G} : (\mathbb{G} \backslash X) - a \in T\}$.*

(3) *$B_X \cup B_Y = \mathrm{bd}_{\mathbb{G}}(\mathbb{G} \cap \mathrm{cl}_{\mathbb{L}}(X - t)) = \mathrm{bd}_{\mathbb{G}}(\mathbb{G} \cap \mathrm{cl}_{\mathbb{L}}((\mathbb{G} \backslash X) - t))$.*

Proof. Assume that $T + U$ has more than one extension. Then there exist clopen subsets X and $Y = \mathbb{G} \backslash X$ of $\mathbb{G}$ such that:

(i) For every closed C contained in $\{a \in \mathbb{G} : X - a \in T\}$ or in $\{a \in \mathbb{G} : Y - a \in T\} = \mathbb{G} \backslash \{a \in \mathbb{G} : X - a \in T\}$ there exists $B \in U$ disjoint from C.

(ii) Both X and Y meet every element of $T + U$.

Hence for $V \in U$ we get $\{a \in \mathbb{G} : X - a \in T\} \cap V \neq \emptyset$ and $\{a \in \mathbb{G} : Y - a \in T\} \cap V \neq \emptyset$.

On the other hand, since $\Psi(T) \in \mathbb{L}$, we obtain

$$\{a \in \mathbb{G} : X - a \in T\} = \mathrm{int}_{\mathbb{G}}(\mathbb{G} \cap \mathrm{cl}_{\mathbb{L}}(X - t)) \cup B_X$$

and

$$\{a \in \mathbb{G} : Y - a \in T\} = \mathrm{int}_{\mathbb{G}}(\mathbb{G} \cap \mathrm{cl}_{\mathbb{L}}(Y - t)) \cup B_Y\,,$$

where B_X and B_Y are certain subsets of $\mathrm{bd}_{\mathbb{G}}(\mathbb{G} \cap \mathrm{cl}_{\mathbb{L}}(X - t)) = \mathrm{bd}_{\mathbb{G}}(\mathbb{G} \cap \mathrm{cl}_{\mathbb{L}}(Y - t))$. ∎

Theorem 4.2. *Let $\mathbb{G}$ be a totally disconnected first Baire category group. There are ultrafilters $T, U \in \beta\mathbb{G}$ such that $T + U$ has more than one extension* *to an ultrafilter. The set of such ultrafilters is dense in $\beta\mathbb{G}_{\mathrm{fin}} \times \beta\mathbb{G}$ and in $\Psi^{\leftarrow}\{t\} \times \Psi^{\leftarrow}\{u\}$ for any $t, u \in \mathbb{L}$ whenever $t + u \in \mathbb{L}\backslash\mathbb{G}$.*

Proof. Fix $t \in \mathbb{L}$ and $T \in \Psi^{\leftarrow}\{t\}$. Fix a sequence $\{G_n : n < \omega\}$ of pairwise disjoint nwd closed subsets of $\mathbb{G}$ such that $\mathbb{G} = \bigcup\{G_n : n < \omega\}$ and an open base $\{B_n : n < \omega\}$ of $\mathbb{L}$.

First we shall construct a nwd $D \subseteq \mathbb{L}$ and next an ultrafilter $U \in \beta D$. We consider two cases:

Case 1: Fix $u \in \mathbb{L}$ such that $t + u \in \mathbb{L}\backslash\mathbb{G}$.

Let D_0 be a fixed compact subset of $\mathbb{L}$ such that $D_0 \cap (\mathbb{G} + t)$ is clopen in $\mathbb{G} + t$ and $t + u \in D_0$. By induction we define sets W_n, V_n and D_n as follows.

Let $k = \min\{i < \omega : B_i \subseteq D_n\}$. Fix $a_{n+1}, b_{n+1} \in D_n \cap B_k \cap (\mathbb{G} + t)$. Fix an open subset A of D_n with $A \subseteq D_n\backslash(G_{2n} \cup G_{2n+1})$ and $t + u \in A$. Then W_{n+1} is an open subset of $\mathbb{L}$ such that

$$G_{2n} \subseteq W_{n+1} \subseteq \mathrm{int}_{\mathbb{L}}(D_n\backslash A), \quad W_{n+1} \text{ is clopen in } \mathbb{G},$$
$$a_{n+1} \in \mathrm{bd}_{\mathbb{L}}(W_{n+1}), \quad b_{n+1} \notin \mathrm{bd}_{\mathbb{L}}(W_{n+1}) \quad \text{and} \quad W_{n+1} - a_{n+1} \in T;$$

next, V_{n+1} is an open subset of $\mathbb{L}$ such that

$$G_{2n+1} \subseteq V_{n+1} \subseteq \mathrm{int}_{\mathbb{L}}(D_n\backslash(A \cup W_{n+1})), \quad V_{n+1} \text{ is clopen in } \mathbb{G},$$
$$b_{n+1} \in \mathrm{bd}_{\mathbb{L}}(V_{n+1}) \quad \text{and} \quad V_{n+1} - b_{n+1} \in T,$$

and

$$D_{n+1} = D_n\backslash(W_{n+1} \cup V_{n+1}).$$

Finally, we put

$$D = \mathbb{G} \cap \Big(\bigcap\{D_n : n < \omega\} - t\Big).$$

Case 2: Fix $u \in \mathbb{L}_\infty$. The construction goes similarly.

Let $U \in \Psi^{\leftarrow}\{u\}$ be any R-point with respect to D (see Corollary 2.3). The clopen set $X = \mathbb{G} \cap (\bigcup\{W_i : i < \omega\})$ and boundary sets $B_X = \{a \in D : X - a \in T\}$, $B_Y = \{a \in D : Y - a \in T\}$ satisfy conditions (1), (2) and (3) of 4.1, so $T + U$ has at least two different extensions. ■

THEOREM 4.3. *Let $t \in \mathbb{L}$. If $U \in \beta\mathbb{G}$ is a regular R-point and $T \in \Psi^{\leftarrow}\{t\}$, then $T+U$ has exactly one extension to an ultrafilter. It is a principal ultrafilter if $\Psi(U) \in \mathbb{L}$ and $\Psi(U) + t \in \mathbb{G}$, and a regular R-point similar to U otherwise.*

Proof. Let $S \in \beta\mathbb{G}$ be the extension of $T+U$. If $\Psi(U) \in \mathbb{L}$ and $\Psi(U)+t \in \mathbb{G}$ then by Proposition 3.3, S is a principal ultrafilter.

Suppose now that $u = \Psi(U) \in \mathbb{L}\backslash\mathbb{G}$ and $t + u \in \mathbb{L}\backslash\mathbb{G}$. Let A be any element of U. Every closed set containing $T_t(A)$ in its interior belongs to $T + U$. By regularity of U the set $\mathrm{bd}_{\mathbb{L}}(A)$ avoids some member of U. Hence

$\mathrm{bd}_{\mathbb{L}}(T_t(A))$ avoids an element of $T+U$ and, consequently, $T_t(A)$ meets every member of $T+U$. Thus $A \in S$ and hence S is similar to U. By Proposition 2.5(a), S is a regular R-point. Hence the extension is unique.

The case $U \in \mathbb{G}_\infty$ can be proved in a similar way. ■

From Propositions 2.5 and 4.3 we obtain

PROPOSITION 4.4. *There exists a family* $\mathbf{L} = \{R_\xi : \xi \in \mathbb{L}\} \subseteq \beta\mathbb{G}$ *such that* $R_\xi \in \Psi^{\leftarrow}\{\xi\}$ *and* $\mathbf{L}$ *with the extended group operation on* $\beta\mathbb{G}$ *is isomorphic to the group* $\mathbb{L}$.

Proof. For any $\xi \in \mathbb{G}$ let R_ξ be the principal ultrafilter generated by $\{\xi\}$. Let $\{R_\xi : \xi \in \mathbb{L}\backslash\mathbb{G}\}$ be any similarity class of regular R-points. By Propositions 3.3 and 4.3 the filter $R_\xi + R_\zeta$ extends uniquely to $R_{\xi+\zeta}$.

In the light of 3.1, $\mathbb{L}$ and $\mathbf{L}$ are algebraically (but not topologically) isomorphic. ■

EXAMPLE 4.5. Let $\mathbb{R}$ be the field of real numbers and $\mathbb{Q}$ the field of rationals. By 4.4, there are subsets $\mathbf{R}_+$ and $\mathbf{R}_*$ of $\beta\mathbb{Q}$ isomorphic to the additive and multiplicative groups of $\mathbb{R}$, but there is no family isomorphic to the field. In fact, if we assume that for fixed $r \in \mathbb{R}\backslash\mathbb{Q}$, $\mathbf{R}_+ \cap \Psi^{\leftarrow}\{r\} = \mathbf{R}_* \cap \Psi^{\leftarrow}\{r\}$, we get $\mathbf{R}_+ \cap \mathbf{R}_* \cap \Psi^{\leftarrow}\{-r\} = \emptyset$. ■

PROPOSITION 4.6. *Assume that* $\mathrm{card}(\mathbb{G}) = \omega$. *Let* $T, U \in \beta\mathbb{G}$.

(a) *If* $T \in \mathbb{G}_\infty$ *is a regular R-point, then every extension of* $T+U$ *is a regular R-point from* $\mathbb{L}_\infty$.

(b) *If* T *and* U *are regular R-points then* $T+U$ *extends either to a principal ultrafilter or to a regular R-point.*

Proof. (a) By Proposition 3.3 the filter $T+U$ has an extension in $\mathbb{L}_\infty$. Let N be a nwd subset of $\mathbb{L}$ and let $A \in U$. Fix an enumeration $(a_i)_{i<\omega}$ of elements of A. Let $\{G_n : n < \omega\}$ be an increasing sequence of conditionally compact (in $\mathbb{L}$) neighborhoods of the neutral element in $\mathbb{G}$ such that $\mathbb{G} = \bigcup\{G_n\}$.

For any $i < \omega$ let $X_i \in T$ be such that $X_i \subseteq \mathbb{G}\backslash G_i$ and $X_i \cap (N - a_i) = \emptyset$. Then the set $X = \bigcup\{X_i + a_i : i < \omega\}$ is closed, belongs to $T+U$ and avoids N. Hence the extension of $T+U$ is a regular R-point.

(b) follows directly from Propositions 3.3(a), 4.3 and (a). ■

5. Extension of the group operation: $\beta\mathbb{G}$ and $\beta\mathbb{L}$.

This part of the paper is devoted to verification whether Ψ is a homomorphism with respect to the extended group operations on $\beta\mathbb{G}$ and on $\beta\mathbb{L}$.

From now on we assume that $t, u \in \beta\mathbb{L}$, $T, U \in \beta\mathbb{G}$, $\Psi(U) = u$ and $\Psi(T) = t$.

The following statement results from Propositions 1.1(b) and 3.3:

Remark 5.1. If $t, u \in \mathbb{L}$ and S is any extension of $T+U$ to an ultrafilter, then $\Psi(S) = t + u$. ■

PROPOSITION 5.2. *Let $t \in \mathbb{L}$ and $u \in \beta\mathbb{L}\backslash\mathbb{L}$. The filter $\overline{T+U}$ is a subfamily of $t+u$.*

Proof. If $X \in \mathbf{Z}(\mathbb{G})$ is an element of $T+U$ then there exists $A \in U$ such that $A \subseteq \{a \in \mathbb{G} : X - a \in T\} \subseteq \{a \in \mathbb{L} : \mathrm{cl}_{\mathbb{L}}(X-a) \in t\} = \mathrm{cl}_{\mathbb{L}}(X) - t$. The set $\mathrm{cl}_{\mathbb{L}}(X) - t$ is closed, so $\mathrm{cl}_{\mathbb{L}}(A) \subseteq \mathrm{cl}_{\mathbb{L}}(X) - t$ and hence $\mathrm{cl}_{\mathbb{L}}(X) \in t+u$. ■

PROPOSITION 5.3. *If* $\mathrm{card}(\Psi^{\leftarrow}\{u\}) = 1$ *then* $\overline{T+U}$ *and* $t+u$ *have some common extension to an ultrafilter.*

Proof. Let a closed $X \subseteq \mathbb{G}$ be a member of $T+U$. There exists $A \in U$ such that $A \subseteq \{a \in \mathbb{G} : X - a \in T\}$. Note that

$$\{a \in \mathbb{G} : X - a \in T\} \subseteq \{a \in \mathbb{G} : \mathrm{cl}_{\mathbb{L}}(X) - a \in t\} \subseteq \{a \in \mathbb{L} : \mathrm{cl}_{\mathbb{L}}(X) - a \in t\}.$$

For any $Y \in t+u$ the set $\{a \in \mathbb{L} : Y - a \in t\}$ contains an element B' of u, but the set $B = \mathbb{G} \cap B'$ is, by Proposition 1.1(b), a member of U. Hence $A \cap B \subseteq \{a \in \mathbb{G} : (Y \cap \mathrm{cl}_{\mathbb{L}}(X)) - a \in t\}$. Therefore $Y \cap \mathrm{cl}_{\mathbb{L}}(X) \neq \emptyset$, so the family $(\overline{T+U}) \cup (t+u)$ has the finite intersection property. ■

EXAMPLE 5.4. Let $u \in \mathbb{R}\backslash\mathbb{Q}$. There exists $t \in \beta\mathbb{R}\backslash\mathbb{R}$ such that $\overline{T+U}$ and $t+u$ have no common extension. One can require $\mathrm{card}(\Psi^{\leftarrow}\{t\})$ to be 1.

Proof. Let k be any natural number greater than u. Let $X_n = \mathbb{G} \cap [kn, kn + u - 1/n]$ and let $X = \bigcup\{X_n : n < \omega\}$. Let $t \in \mathrm{cl}_{\beta\mathbb{R}}(k\omega)\backslash\mathbb{R}$. Note that $X \in T+U$, because

$$\{a \in \mathbb{G} : X - a \in T\} \supseteq \{a \in \mathbb{G} : (\exists m \in \omega)\, X - a \supseteq k(\omega - m)\} \supseteq \mathbb{Q} \cap [0, u).$$

Moreover, $\{a \in \mathbb{L} : \mathrm{cl}_{\mathbb{R}}(X) - a \in t\} \supseteq [0, u)$, and $(\mathrm{cl}_{\mathbb{R}}(X) - u) \cap k\omega = \emptyset$. Hence $\mathrm{cl}_{\mathbb{R}}(X) \cap (k\omega + u) = \emptyset$. Consequently, $\mathrm{cl}_{\mathbb{R}}(X) \notin t+u$. ■

COROLLARY 5.5. (a) *Ψ is a homomorphism $\beta\mathbb{G}_{\mathrm{fin}} \to \mathbb{L}$ with respect to the extended group operation.*

(b) *Ψ need not be a homomorphism $\beta\mathbb{G} \to \beta\mathbb{L}$.*

(c) *There exists a version of the extended group operation such that Ψ is a homomorphism of the set of regular R-points in $\beta\mathbb{G}\backslash\beta\mathbb{G}_{\mathrm{fin}}$ onto the set of R-points in $\beta\mathbb{L}$.*

(d) *Ψ need not be a homomorphism of the set of regular R-points in $\beta\mathbb{G}$ onto its image.*

Proof. (a) follows from Proposition 3.3. Parts (b) and (d) follow from 5.4. Part (c) follows from Proposition 5.3. ■

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