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SPACE-LIKE SURFACES IN AN ANTI-DE SITTER SPACE

BY

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1. Introduction. It is well-known that complete minimal submanifolds of a unit sphere $S^{n+p}(1)$ with $S = n/(2-1/p)$ are the Clifford torus and the Veronese surface, where S is the squared norm of the second fundamental form (cf. Chern, do Carmo and Kobayashi [4] and Cheng [1]). The related problem of complete maximal space-like submanifolds in an anti-de Sitter space was studied by Ishihara [5]. He proved that if M is an n-dimensional complete maximal space-like submanifold in an anti-de Sitter space $H_p^{n+p}(c)$ of constant curvature $-c$ ($c > 0$) and with index p, then $S \le npc$, and $S = npc$ if and only if $M = H^{n_1}(n_1c/n) \times \ldots \times H^{n_p}(n_pc/n)$, where $H^{n_i}(c_i)$ is an n_i-dimensional hyperbolic space of constant curvature $-c_i$.

On the other hand, we know that the hyperbolic Veronese surface $H^2(c/3)$ is a maximal space-like submanifold in $H_2^4(c)$ defined by

$$u_1 = yz/\sqrt{3c}\,, \quad u_2 = xz/\sqrt{3c}\,, \quad u_3 = xy/\sqrt{3c}\,,$$
$$u_4 = (x^2-y^2)/(2\sqrt{3c}) \quad \text{and} \quad u_5 = (x^2+y^2+2z^2)/(6\sqrt{c})\,,$$

where (x,y,z) and (u_1,u_2,u_3,u_4,u_5) are the natural coordinate systems in $\mathbb{R}_1^3$ and $\mathbb{R}_3^5$ respectively.

In this paper, we consider the space-like surfaces in an anti-de Sitter space. In Section 2, we prepare some formulas and notations which are used in the paper. In Section 3, we give a sharper estimate of S on complete maximal space-like surfaces than the one due to Ishihara [5] and give a characterization of the hyperbolic Veronese surface and of $H^1(c/2) \times H^1(c/2)$. The complete space-like surfaces with parallel mean curvature vector in an anti-de Sitter space are studied in Section 4. In the final section, we present a complete maximal space-like surface in $H_2^6(c)$.

2. Formulas and notations. Throughout this paper all manifolds are assumed to be smooth and connected. Let $H_p^{n+p}(c)$ be an *anti-de Sitter space*, that is, $H_p^{n+p}(c)$ is an indefinite space form with index p and of con-

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stant curvature $-c$ ($c > 0$). An n-dimensional submanifold M of $H_p^{n+p}(c)$ is said to be *space-like* if the metric induced on M from the ambient space $H_p^{n+p}(c)$ is positive definite. We choose a local field of orthonormal frames $e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+p}$ adapted to the indefinite Riemannian metric in $H_p^{n+p}(c)$ such that $e_1, \ldots, e_n$ are tangent to M. Let $\omega_1, \ldots, \omega_n$ be a field of dual frames on M. The *second fundamental form* of M is given by

$$\mathfrak{a} = -\sum h_{ij}^a \omega_i \omega_j e_a \,, \tag{2.1}$$

where $h_{ij}^a = h_{ji}^a$ for any $a = n+1, \ldots, n+p$. The *mean curvature vector* $\vec{h}$ and the *mean curvature* H of M are defined by

$$\vec{h} = -\sum \Big(\sum_i h_{ii}^a\Big) e_a \Big/ n \tag{2.2}$$

and

$$H = \sqrt{\sum \Big(\sum_i h_{ii}^a\Big)^2} \Big/ n \,. \tag{2.3}$$

If $H = 0$, we call M *maximal*. The *Gaussian equations* of M are

$$R_{ijkl} = -c(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) - \sum (h_{il}^a h_{jk}^a - h_{ik}^a h_{jl}^a) \,, \tag{2.4}$$

$$R_{abij} = -\sum (h_{ik}^a h_{kj}^b - h_{ik}^b h_{jl}^a) \,. \tag{2.5}$$

The covariant derivative $\nabla \mathfrak{a}$ of the second fundamental form $\mathfrak{a}$ of M has components h_{ijk}^a given by

$$\sum h_{ijk}^a \omega_k = dh_{ij}^a + \sum h_{kj}^a \omega_{ik} + \sum h_{ik}^a \omega_{jk} + \sum h_{ij}^b \omega_{ba} \,.$$

Thus we can get the *Codazzi equation*

$$h_{ijk}^a = h_{ikj}^a \,.$$

3. Maximal space-like surfaces

THEOREM 1. *Let M be a complete maximal space-like surface in an anti-de Sitter space $H_p^{2+p}(c)$. Then $S \le 2c$, and $S = 2c$ if and only if $M = H^1(c/2) \times H^1(c/2)$ and $p = 1$.*

Remark 1. The estimate $S \le 2c$ in Theorem 1 is sharper than $S \le 2pc$ which has been obtained by Ishihara [5]. Our result does not depend on p.

Proof of Theorem 1. From the Gaussian equations (2.4) and (2.5), we can get, making use of the same computations as in Ishihara [5],

$$(1/2)\Delta S = \sum (h_{ijk}^a)^2 - ncS + \sum N(H_a H_b - H_b H_a) + \sum (S_a)^2 \,, \tag{3.1}$$

where $S_a = \sum (h_{ij}^a)^2$, $H_a = (h_{ij}^a)$ and $N(A) = \mathrm{tr}(A^t A)$.

We consider the linear map

(3.2) $$B : T^{\perp}M \to TM \otimes TM\,, \qquad B(e_a) = \sum h^a_{ij} e_i \otimes e_j\,,$$

where $T^{\perp}M$ and TM are the normal bundle and the tangent bundle to M respectively, and e_1 and e_2 are tangent to M. For any x in M, since $e_1 \otimes e_2$, $e_1 \otimes e_1$ and $e_2 \otimes e_2$ are a basis of $T_xM \otimes T_xM$ and $\sum h^a_{ii} = 0$, we have

(3.3) $$B(e_a) = 2h^a_{12} e_1 \otimes e_2 + h^a_{11}(e_1 \otimes e_1 - e_2 \otimes e_2) \quad \text{for } a \geq 3\,.$$

Hence the rank of B is not greater than 2. Thus we can choose $e_5, \ldots, e_{2+p}$ such that $B(e_a) = 0$ for $a \geq 5$. From (3.3) we have $h^a_{ij} = 0$ for $a \geq 5$. Let $S_{ab} = \sum h^a_{ij} h^b_{ij}$. We can take e_3 and e_4 such that (S_{ab}) is diagonal. Thus

$$\sum h^3_{ij} h^4_{ij} = 0\,.$$

On the other hand, we choose e_1 and e_2 such that $h^3_{ij} = \lambda_i \delta_{ij}$. Hence we can take $e_1, \ldots, e_{2+p}$ such that

(3.4) $$H_3 = (h^3_{ij}) = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \qquad H_4 = (h^4_{ij}) = \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix},$$
$$H_a = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{for } a \geq 5\,.$$

From (3.4), we have

(3.5) $$\sum N(H_a H_b - H_b H_a) = 16\lambda^2\mu^2\,,$$

(3.6) $$\sum (S_a)^2 = 4(\lambda^4 + \mu^4)\,.$$

Now (3.1), (3.5) and (3.6) yield

(3.7) $$\begin{aligned} (1/2)\Delta S &= \sum (h^a_{ijk})^2 - 2cS + 4(\lambda^4 + \mu^4) + 16\lambda^2\mu^2 \\ &= \sum (h^a_{ijk})^2 + (S - 2c)S + 8\lambda^2\mu^2 \\ &= \sum (h^a_{ijk})^2 + S(3S/2 - 2c) - 2(\lambda^2 - \mu^2)^2\,. \end{aligned}$$

From the result due to Ishihara [5], we know that $S \leq 2pc$. According to the Gaussian equation (2.4), we see that the Ricci curvature is bounded from below. Hence from (3.7) and the generalized maximum principle given below, due to Omori and Yau, we obtain

$$0 \geq \sup S(\sup S - 2c)\,.$$

Hence $S \leq 2c$.

Generalized maximum principle (cf. Omori [6] and Yau [7]). *Let M be a complete Riemannian manifold whose Ricci curvature is bounded from* *below. Let F be a C^2-function bounded from above on M. Then there exists a sequence $\{p^m\}$ of points in M such that*

$$\lim F(p^m) = \sup F\,, \quad \lim \|\nabla F(p^m)\| = 0 \quad \textit{and} \quad \limsup \Delta F(p^m) \le 0\,.$$

If $S = 2c$, from Theorem 3 in [5], we have $M = H^1(c/2) \times H^1(c/2)$.

Thus we complete the proof of Theorem 1.

COROLLARY 1. *The Gaussian curvature of the complete maximal space-like surface in an anti-de Sitter space $H_p^{2+p}(c)$ is nonpositive.*

Proof. From the Gaussian equation (2.4), we have

$$K = -c + S/2 \tag{3.8}$$

where K is the Gaussian curvature. Theorem 1 implies $K \le 0$.

In particular, when $p = 1$, we have

THEOREM 2. *Let M be a complete maximal space-like surface in an anti-de Sitter space $H_1^3(c)$ with $\inf K > -c$. Then $K = 0$ and $M = H^1(c/2) \times H^1(c/2)$.*

Proof. Since the codimension of M is one, from (3.1) we have

$$(1/2)\Delta S = \sum (h_{ijk})^2 + S(S - 2c)\,. \tag{3.9}$$

We choose e_1 and e_2 such that $h_{ij} = \lambda_i \delta_{ij}$. Because M is maximal, we get $\sum h_{ii} = 0$. Hence $\sum (h_{iik}) = 0$ for any k. Now,

$$\begin{aligned} |\nabla S|^2 &= 4 \sum \Big(\sum h_{ij} h_{ijk} \Big)^2 = 4 \sum \Big(\sum \lambda_i h_{iik} \Big)^2 \\ &= 4 \sum (\lambda_1 h_{11k} + \lambda_2 h_{22k})^2 = 4(\lambda_1 - \lambda_2)^2 \sum (h_{11k})^2 \\ &= 2(\lambda_1 - \lambda_2)^2 \sum (h_{iik})^2 = 4S \sum (h_{iik})^2\,. \end{aligned}$$

Moreover,

$$\sum (h_{ijk})^2 = 3 \sum_{i \ne k} (h_{iik})^2 + \sum (h_{kkk})^2 = 2 \sum (h_{iik})^2\,.$$

Hence

$$2S \sum (h_{ijk})^2 = |\nabla S|^2\,. \tag{3.10}$$

From (3.9) and (3.10), we have

$$S \Delta S = |\nabla S|^2 + 2S^2(S - 2c)\,. \tag{3.11}$$

Thus $\inf S = 0$ or $\inf S \ge 2c$ from the generalized maximum principle. According to (3.8), we get $\inf K = -c$ or $\inf K \ge 0$. From the assumption and Theorem 1, we obtain $S = 2c$ and $M = H^1(c/2) \times H^1(c/2)$.

THEOREM 3. *Let M be a complete maximal space-like surface in an anti-de Sitter space $H_p^{2+p}(c)$ $(p > 1)$ with parallel second fundamental form. If*

$S \leq 4c/3$, then $M = H^2(c)$ is totally geodesic or $M = H^2(c/3)$ is the hyperbolic Veronese surface.

Proof. Since the second fundamental form is parallel, we have $\sum(h^a_{ijk})^2 = 0$ and S is constant. From (3.7) we obtain

$$S(3S/2 - 2c) - 2(\lambda^2 - \mu^2)^2 = 0\,.$$

Hence $\lambda^2 = \mu^2$ and $S = 0$ or $S = 4c/3$. If $S = 0$, $M = H^2(c)$ is totally geodesic. If $S = 4c/3$, $M = H^2(c/3)$ is the hyperbolic Veronese surface.

THEOREM 4. *Let M be an n-dimensional complete maximal space-like hypersurface in an anti-de Sitter space $H_1^{n+1}(c)$ with parallel second fundamental form. Then M is $H^n(c)$ or $H^{n_1}(n_1c/n) \times H^{n-n_1}[(n-n_1)c/n]$.*

Proof. Since M is a hypersurface, from (3.1) we have

$$(1/2)\Delta S = \sum(h_{ijk})^2 + S(S - nc)\,.$$

By the same proof as for Theorem 3, we get $S = 0$ or $S = nc$. From the result due to Ishihara [5], we know that Theorem 4 is true.

COROLLARY 2. *Let M be a complete isoparametric maximal space-like hypersurface in an anti-de Sitter space $H_1^{n+1}(c)$. Then $M = H^n(c)$ or $H^{n_1}(n_1c/n) \times H^{n-n_1}[(n-n_1)c/n]$ $(n > n_1 \geq 1)$.*

Proof. Since M is isoparametric, we know that the second fundamental form of M is parallel. From Theorem 4, we conclude that Corollary 2 is true.

4. Space-like surfaces with parallel mean curvature vector

THEOREM 5. *Let M be a complete space-like surface with parallel mean curvature vector in an anti-de Sitter space $H_2^{2+p}(c)$. Then $S \leq 2c + 4H^2$ if $p = 1$, $S \leq (8/3)c + (14/3)H^2$ if $p = 2$ and $S \leq 2(p-1)c + 2pH^2$ if $p > 2$.*

Proof. If the mean curvature H is zero, then from Theorem 1, we have $S \leq 2c$. Hence, next we suppose $H \neq 0$. We choose e_3 such that $\vec{h} = He_3$. Then we have

$$H_aH_3 = H_3H_a \quad \text{for any } a \geq 3 \text{ (cf. Cheng [2])}\,, \tag{4.1}$$

$$\operatorname{tr} H_3 = 2H\,, \quad \operatorname{tr} H_a = 0 \quad \text{for } a > 3\,. \tag{4.2}$$

By setting $\mu_{ij} = h^3_{ij} - H\delta_{ij}$ and $\tau^a_{ij} = h^a_{ij}$ for $a > 3$, we have

$$|\mu|^2 = \sum(\mu_{ij})^2 = \sum(h^3_{ij})^2 - 2H^2\,,$$

$$|\tau|^2 = \sum(\tau^a_{ij})^2 = \sum(h^a_{ij})^2\,,$$

$$S = |\mu|^2 + |\tau|^2 + 2H^2\,. \tag{4.3}$$

It can be seen that $|\mu|^2$ and $|\tau|^2$ are independent of the choice of the frame fields and are functions globally defined on M. Making use of the similar proof as in Cheng [2], we get

$$(4.4)\quad (1/2)\Delta|\mu|^2 = \sum(h^3_{ijk})^2 - 2c\sum(h^3_{ij})^2 + 4cH^2 - 2H\,\mathrm{tr}(H_3)^3 + \sum \mathrm{tr}(H_3H_a)^2 + [\mathrm{tr}(H_3)^2]^2\,.$$

For a fixed index a, since $H_aH_3 = H_3H_a$, we can choose e_1 and e_2 such that $h^a_{ij} = \lambda^a_i\delta_{ij}$ and $h^3_{ij} = \lambda_i\delta_{ij}$. Hence $\mathrm{tr}(H_3H_a)^2 = (1/2)(|\mu|^2 + 2H^2)\,\mathrm{tr}(H_a)^2$, which does not depend on the choice of frame fields. Thus

$$\sum \mathrm{tr}(H_3H_a)^2 = (1/2)(|\mu|^2 + 2H^2)|\tau|^2\,.$$

We choose e_1 and e_2 such that $h^3_{ij} = \lambda_i\delta_{ij}$. We know

$$(4.5)\quad \lambda_1 + \lambda_2 = 2H\,,$$

$$(4.6)\quad \begin{aligned} 2H\,\mathrm{tr}(H_3)^3 &= 2H((\lambda_1)^3 + (\lambda_2)^3)\\ &= 2H(\lambda_1+\lambda_2)((\lambda_1)^2 + (\lambda_2)^2 - \lambda_1\lambda_2)\\ &= 6H^2(|\mu|^2 + 2H^2) - 8H^4\,. \end{aligned}$$

Hence from (4.4) and (4.6), we obtain

$$(4.7)\quad (1/2)\Delta|\mu|^2 \ge (-2c - 2H^2 + |\mu|^2)|\mu|^2 + (1/2)(|\mu|^2 + 2H^2)|\tau|^2\,.$$

If $p = 1$, making use of the same proof as in Cheng [2], we have $|\mu|^2 \le 2c + 2H^2$. Hence $S \le 2c + 4H^2$.

If $p > 1$, making use of similar calculations to [2], we have

$$(4.8)\quad \begin{aligned} (1/2)\Delta|\tau|^2 \ge\ & -2c|\tau|^2 + |\tau|^4/(p-1) + \sum h^a_{km}h^3_{mk}h^3_{ij}h^a_{ij}\\ & - 2\sum h^3_{ik}h^a_{km}h^3_{mj}h^a_{ij} + \sum h^a_{im}h^3_{mk}h^3_{kj}h^a_{ij}\\ & - 2H\sum h^a_{im}h^3_{mj}h^a_{ij} + \sum h^3_{ik}h^3_{km}h^a_{mj}h^a_{ij}\,. \end{aligned}$$

For a fixed index a, since $H_aH_3 = H_3H_a$, we choose e_1 and e_2 such that $h^a_{ij} = \lambda^a_i\delta_{ij}$ and $h^3_{ij} = \lambda_i\delta_{ij}$. Then we get, for fixed a,

$$\begin{aligned} &\sum h^a_{km}h^3_{mk}h^3_{ij}h^a_{ij} - 2\sum h^3_{ik}h^a_{km}h^3_{mj}h^a_{ij} + \sum h^a_{im}h^3_{mk}h^3_{kj}h^a_{ij}\\ &\qquad - 2H\sum h^a_{im}h^3_{mj}h^a_{ij} + \sum h^3_{ik}h^3_{km}h^a_{mj}h^a_{ij}\\ &= \Big(\sum \lambda_i\lambda^a_i\Big)^2 - 2H\sum \lambda_i(\lambda^a_i)^2\\ &= (\lambda_1 - \lambda_2)^2(\lambda^a_1)^2 - 4H^2(\lambda^a_1)^2 \qquad \text{(by (4.2))}\\ &= (|\mu|^2 - 2H^2)\,\mathrm{tr}(H_a)^2\,. \end{aligned}$$

Both sides of the equality above do not depend on the choice of frame fields. Therefore we have

$$(4.9)\quad \sum h^a_{km}h^3_{mk}h^3_{ij}h^a_{ij} - 2\sum h^3_{ik}h^a_{km}h^3_{mj}h^a_{ij} + \sum h^a_{im}h^3_{mk}h^3_{kj}h^a_{ij}$$

$$- 2H \sum h^a_{im} h^3_{mj} h^a_{ij} + \sum h^3_{ik} h^3_{km} h^a_{mj} h^a_{ij}$$
$$= (|\mu|^2 - 2H^2)|\tau|^2 .$$

Hence, from (4.8) and (4.9), we have

$$(1/2)\Delta|\tau|^2 \geq -2c|\tau|^2 + |\tau|^4/(p-1) + (|\mu|^2 - 2H^2)|\tau|^2 . \tag{4.10}$$

Now (4.7)+(4.10) implies, from (4.3),

$$\begin{aligned}(1/2)\Delta(S - 2H^2) \geq & - (2c + 2H^2)(S - 2H^2) + |\mu|^4 \\ & + [1/(p-1)]|\tau|^4 + (3/2)|\tau|^2|\mu|^2 + H^2|\tau|^2 \\ \geq & \begin{cases} -(2c + 2H^2)(S - 2H^2) + (3/4)(S - 2H^2)^2 & \text{if } p = 2, \\ -(2c + 2H^2)(S - 2H^2) + (S - 2H^2)^2/(p-1) & \text{if } p > 2. \end{cases}\end{aligned}$$

Making use of a similar proof to [2], we have

$$S \leq \begin{cases} (8/3)c + (14/3)H^2 & \text{if } p = 2, \\ 2(p-1)c + 2pH^2 & \text{if } p > 2. \end{cases}$$

Remark 2. When $p = 1$, the hyperbolic cylinder satisfies $S = 2c + 4H^2$. Hence the estimate in Theorem 5 is optimal, which has been obtained by the author and Nakagawa [3] if $p = 1$.

5. An example of a complete maximal space-like surface in $H^6_2(c)$. We consider the space-like immersion of $H^2(c/2)$ into $H^6_2(c)$ defined by

$$\begin{aligned} u_1 &= [1/(24\sqrt{c})]x(x^2 + y^2 + 4z^2) , & u_5 &= (\sqrt{10/c}/12)xyz , \\ u_2 &= [1/(24\sqrt{c})]y(x^2 + y^2 + 4z^2) , & u_6 &= (\sqrt{6/c}/72)z(3x^2 + 3y^2 + 2z^2) , \\ u_3 &= (\sqrt{15/c}/72)x(x^2 - 3y^2) , & u_7 &= (\sqrt{10/c}/24)z(x^2 - y^2)/24 , \\ u_4 &= (\sqrt{15/c}/72)y(3x^2 - y^2) , & & \end{aligned}$$

where (x, y, z) and $(u_1, \ldots, u_7)$ are the natural coordinate systems in $\mathbb{R}^3_1$ and $\mathbb{R}^7_3$ respectively. It is obvious that $H^2(c/6)$ is a complete space-like surface in $H^6_2(c)$. We can also easily prove that $H^2(c/6)$ is maximal in $H^6_2(c)$.

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