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MYCIELSKI IDEALS GENERATED BY UNCOUNTABLE SYSTEMS

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1. Introduction. In the theory of infinite games one can treat sets for which the second player has a winning strategy as small sets. Usually we want small sets to be closed under some operations, e.g. to form an ideal. To obtain a good notion of smallness we have to consider either a lot of games or special kinds of games. Mycielski ideals introduced in [Myc] are based on the first idea.

Let $\mathcal{X}$ be a countable set with at least two elements.

For $A \subseteq \mathcal{X}^\omega$ and $X \in [\omega]^\omega$ let $\Gamma_{\mathcal{X}}(A, X)$ denote the infinite game between two players, I and II, in which both players choose the values of a sequence $c \in \mathcal{X}^\omega$. Player I chooses $c(n)$ for $n \notin X$, Player II chooses $c(n)$ if $n \in X$. Player I wins if and only if $c \in A$.

Denote by $\mathrm{STR}(\mathcal{X})$ the family of all functions $\sigma : \mathcal{X}^{<\omega} \to \mathcal{X}$. Elements of $\mathrm{STR}(\mathcal{X})$ are strategies in games $\Gamma_{\mathcal{X}}(A, X)$. Note that $\mathrm{STR}(\mathcal{X})$ can be equipped with the product topology and then, since $\mathcal{X}^{<\omega}$ is countable, it is homeomorphic to the space $\mathcal{X}^\omega$. For $\sigma, \tau \in \mathrm{STR}(\mathcal{X})$ and $X \in [\omega]^\omega$ let $\sigma *_X \tau \in \mathcal{X}^\omega$ be the result of the game $\Gamma_{\mathcal{X}}(A, X)$ when Player I follows the strategy σ and II follows τ, i.e.

$$\sigma *_X \tau(n) = \begin{cases} \sigma((\sigma *_X \tau)|n) & \text{if } n \notin X, \\ \tau((\sigma *_X \tau)|n) & \text{if } n \in X. \end{cases}$$

If we put $d(s) = d(\mathrm{lh}(s))$ for $d \in \mathcal{X}^\omega$ and $s \in \mathcal{X}^{<\omega}$ then the space $\mathcal{X}^\omega$ becomes a closed subset of $\mathrm{STR}(\mathcal{X})$. Hence the operation $*_X$ is also defined for elements of $\mathcal{X}^\omega$. Note that the function $*_X$ is continuous. By $\sigma *_X \mathcal{X}^\omega$ and $\mathcal{X}^\omega *_X \tau$ we will denote the images of $\mathcal{X}^\omega$ under the respective restrictions of the function $*_X$. These are the sets of all results of the game determined by X, in which the first (second) player uses the strategy σ (τ respectively).

A family $\mathcal{K} \subseteq [\omega]^\omega$ is said to be a *normal system* if for every $X \in \mathcal{K}$ there exist $X_1, X_2 \in \mathcal{K}$ such that $X_1, X_2 \subseteq X$ and $X_1 \cap X_2 = \emptyset$.

The *Mycielski ideal* $\mathfrak{M}_{\mathcal{X},\mathcal{K}}$ generated by a normal system $\mathcal{K}$ is the family of all sets $A \subseteq \mathcal{X}^\omega$ such that the second player has a winning strategy in

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every game $\Gamma_{\mathcal{X}}(A,X)$, $X\in\mathcal{K}$. In other words,

$$\mathfrak{M}_{\mathcal{X},\mathcal{K}}=\{A\subseteq\mathcal{X}^\omega:(\forall X\in\mathcal{K})(\exists\tau\in\mathrm{STR}(\mathcal{X}))((\mathcal{X}^\omega *_X \tau)\cap A=\emptyset)\}\,.$$

Theorem 1.1 [Mycielski, [Myc]]. *If $\mathcal{K}$ is a countable normal system then $\mathfrak{M}_{\mathcal{X},\mathcal{K}}$ is a σ-ideal such that*:

(a) *there exists a set $A\in\mathfrak{M}_{\mathcal{X},\mathcal{K}}$ such that $\mathcal{X}^\omega\setminus A$ is meager and has Lebesgue measure zero*,

(b) *if $A\in\mathfrak{M}_{\mathcal{X},\mathcal{K}}$ then there exists $B\in\mathfrak{M}_{\mathcal{X},\mathcal{K}}\cap\Pi^0_2(\mathcal{X}^\omega)$ such that $A\subseteq B$*,

(c) *there exist $\mathfrak{c}$ disjoint*, *closed subsets of $\mathcal{X}^\omega$ that do not belong to $\mathfrak{M}_{\mathcal{X},\mathcal{K}}$*,

(d) *if $\mathcal{X}$ is equipped with a group structure then $\mathfrak{M}_{\mathcal{X},\mathcal{K}}$ is invariant under translations in the product group $\mathcal{X}^\omega$.*

The proof of Theorem 1.1 suggested the following simplified versions of Mycielski ideals. Let

$$\mathfrak{M}^*_{\mathcal{X},\mathcal{K}}=\{A\subseteq\mathcal{X}^\omega:(\forall X\in\mathcal{K})(\exists d\in\mathcal{X}^\omega)((\mathcal{X}^\omega *_X d)\cap A=\emptyset)\}\,.$$

Obviously $\mathfrak{M}^*_{\mathcal{X},\mathcal{K}}\subseteq\mathfrak{M}_{\mathcal{X},\mathcal{K}}$ and the inclusion is proper. Also Theorem 1.1 holds for the ideal $\mathfrak{M}^*_{\mathcal{X},\mathcal{K}}$. For every normal system $\mathcal{K}$, $\mathfrak{M}_{\mathcal{X},\mathcal{K}}$ and $\mathfrak{M}^*_{\mathcal{X},\mathcal{K}}$ are σ-ideals on $\mathcal{X}^\omega$ satisfying condition (d) of 1.1.

So far the ideals $\mathfrak{M}_{\mathcal{X},\mathcal{K}}$ and $\mathfrak{M}^*_{\mathcal{X},\mathcal{K}}$ have been studied mainly either for countable $\mathcal{K}$ (cf. [Myc], [Men] and [BRo]) or for $\mathcal{K}=[\omega]^\omega$ (cf. [Ros] and [CRSW]). This paper concentrates on uncountable $\mathcal{K}$.

The ideals $\mathfrak{M}_{\mathcal{X},[\omega]^\omega}$ and $\mathfrak{M}^*_{\mathcal{X},[\omega]^\omega}$ are denoted by $\mathfrak{C}_{\mathcal{X}}$ and $\mathfrak{P}_{\mathcal{X}}$, respectively.

From now on, unless stated otherwise, $\mathcal{X}$ is assumed to be finite. $\mathcal{K}$ and $\mathcal{K}'$ stand for normal systems. $\mathbb{K}(\mathcal{X}^\omega)$ and $\mathbb{L}(\mathcal{X}^\omega)$ are the σ-ideals of meager and Lebesgue null subsets of $\mathcal{X}^\omega$ (the topology and the Lebesgue measure in $\mathcal{X}^\omega$ are the product topology and the product measure in $\mathcal{X}^\omega$ arising from the discrete topology and the measure weighting every point in $\mathcal{X}$ with $1/|\mathcal{X}|$). For the cardinal characteristics of the continuum used in this paper such as the unbounded number $\mathfrak{b}$, the dominating number $\mathfrak{d}$, the refinement number $\mathfrak{r}$ and others, see [Fre] and [Vau].

2. Relations between $\mathfrak{M}_{\mathcal{X},\mathcal{K}}$ and $\mathfrak{M}_{\mathcal{X},\mathcal{K}'}$. Here, and in the next section, we continue the study from [BRo] of the dependence of $\mathfrak{M}_{\mathcal{X},\mathcal{K}}$ on the generating system $\mathcal{K}$. We begin with the following easy observation.

Proposition 2.1. *If $\{\mathcal{K}_\alpha:\alpha<\kappa\}$ is a family of normal systems then $\mathcal{K}=\bigcup\{\mathcal{K}_\alpha:\alpha<\kappa\}$ is a normal system and $\mathfrak{M}_{\mathcal{X},\mathcal{K}}=\bigcap\{\mathfrak{M}_{\mathcal{X},\mathcal{K}_\alpha}:\alpha<\kappa\}$.* ∎

Two ideals of subsets of $\mathcal{X}^\omega$ are *orthogonal* if $\mathcal{X}^\omega$ can be covered by two sets, each from one ideal.

Proposition 2.2. *The ideals $\mathfrak{M}_{\mathcal{X},\mathcal{K}}$ and $\mathfrak{M}_{\mathcal{X},\mathcal{K}'}$ are orthogonal if and only if $X\cap X'\neq\emptyset$ for every $X\in\mathcal{K}$, $X'\in\mathcal{K}'$.*

Proof. Assume that $\mathfrak{M}_{\mathcal{X},\mathcal{K}}$ and $\mathfrak{M}_{\mathcal{X},\mathcal{K}'}$ are orthogonal but $X \cap X' = \emptyset$ for some $X \in \mathcal{K}$, $X' \in \mathcal{K}'$. We find sets $A \in \mathfrak{M}_{\mathcal{X},\mathcal{K}}$ and $B \in \mathfrak{M}_{\mathcal{X},\mathcal{K}'}$ such that $A \cup B = \mathcal{X}^\omega$. Let $\tau, \tau' \in \mathrm{STR}(\mathcal{X})$ be winning strategies for Player II in the games $\Gamma_{\mathcal{X}}(A, X)$ and $\Gamma_{\mathcal{X}}(B, X')$ respectively. Let $c = \tau' *_X \tau$. Then $c \notin A \cup B$, because c is a result of the games $\Gamma_{\mathcal{X}}(A, X)$ and $\Gamma_{\mathcal{X}}(B, X')$ in which Player II uses strategies τ and τ' respectively. This contradicts our choice of A and B. The converse implication was actually shown in the proof of Lemma 1.1 of [BRo]. It was proved there that if $X \cap X' \neq \emptyset$ for every $X \in \mathcal{K}$, $X' \in \mathcal{K}'$ then for every $x \in \mathcal{X}$ the sets

$$A = \{c \in \mathcal{X}^\omega : (\forall X \in \mathcal{K})(\exists n \in X)(c(n) = x)\}, \qquad B = \mathcal{X}^\omega \setminus A$$

witness the orthogonality of $\mathfrak{M}_{\mathcal{X},\mathcal{K}}$ and $\mathfrak{M}_{\mathcal{X},\mathcal{K}'}$. ∎

For systems $\mathcal{K}$ and $\mathcal{K}'$ we write $\mathcal{K}' < \mathcal{K}$ whenever each element of $\mathcal{K}$ contains an element of $\mathcal{K}'$.

PROPOSITION 2.3. *$\mathfrak{M}_{\mathcal{X},\mathcal{K}'} \subseteq \mathfrak{M}_{\mathcal{X},\mathcal{K}}$ if and only if $\mathcal{K}' < \mathcal{K}$.*

Proof. First note that Player II has a winning strategy in $\Gamma_{\mathcal{X}}(A, X)$ provided $X' \subseteq X$ and he can win $\Gamma_{\mathcal{X}}(A, X')$. Hence $\mathfrak{M}_{\mathcal{X},\mathcal{K}'} \subseteq \mathfrak{M}_{\mathcal{X},\mathcal{K}}$ if $\mathcal{K}' < \mathcal{K}$. On the other hand, if there is an $X \in \mathcal{K}$ such that X contains no element of $\mathcal{K}'$ then the set

$$A = \{c \in \mathcal{X}^\omega : (\forall X' \in \mathcal{K}')(\exists n \in X')(c(n) = x)\}$$

from the previous proof belongs to $\mathfrak{M}_{\mathcal{X},\mathcal{K}'} \setminus \mathfrak{M}_{\mathcal{X},\mathcal{K}}$ (here x is a fixed element of $\mathcal{X}$). ∎

COROLLARY 2.4. (a) *For any cardinal $\kappa \leq \mathfrak{c}$, the intersection of κ Mycielski ideals generated by systems of size κ is a Mycielski ideal generated by a system of size κ.*

(b) *For each normal system $\mathcal{K}$ with $|\mathcal{K}| < \mathfrak{r}$ there exists a countable normal system $\mathcal{K}'$ such that the ideals $\mathfrak{M}_{\mathcal{X},\mathcal{K}}$ and $\mathfrak{M}_{\mathcal{X},\mathcal{K}'}$ are orthogonal.*

(c) *There exists a normal system $\mathcal{K}$ of size $\mathfrak{r}$ such that no Mycielski ideal is orthogonal to $\mathfrak{M}_{\mathcal{X},\mathcal{K}}$.*

Proof. (a) This is an immediate consequence of 2.1.

(b) This is a slight generalization of Lemma 1.1 from [BRo]. If $\mathcal{K}$ is of size less than $\mathfrak{r}$ and $L \subseteq \omega$ has the property that $(\forall X \in \mathcal{K})(L \cap X \neq \emptyset)$ then there exist $L_0, L_1 \in [L]^\omega$ such that $(\forall X \in \mathcal{K})(L_0 \cap X \neq \emptyset \neq L_1 \cap X)$ and $L_0 \cap L_1 = \emptyset$. Hence we can construct a countable normal system $\mathcal{K}'$ such that $X \cap X' \neq \emptyset$ for all $X' \in \mathcal{K}'$ and $X \in \mathcal{K}$. Applying Proposition 2.2 we see that $\mathfrak{M}_{\mathcal{X},\mathcal{K}}$ and $\mathfrak{M}_{\mathcal{X},\mathcal{K}'}$ are orthogonal.

(c) Let $\mathcal{R} \subseteq [\omega]^\omega$ be a family realizing the minimum in the definition of $\mathfrak{r}$. Let $\mathcal{K}$ be a normal system containing $\mathcal{R}$ such that $|\mathcal{K}| = \mathfrak{r}$. Let $\mathcal{K}'$ be another normal system on ω. Choose disjoint X_0, X_1 from $\mathcal{K}'$. The properties of $\mathcal{R}$ provide a set $X \in \mathcal{R} \subseteq \mathcal{K}$ such that either $X \subseteq^* X_0$ or $X \subseteq^* \omega \setminus X_0$. In the first case find $X' \subseteq X_1$, $X' \in \mathcal{K}'$ with $X \cap X' = \emptyset$. In the second case there is an $X' \subseteq X_0$, $X' \in \mathcal{K}'$ with the same property. Hence, by Proposition 2.2, $\mathfrak{M}_{\mathcal{X},\mathcal{K}}$ and $\mathfrak{M}_{\mathcal{X},\mathcal{K}'}$ cannot be orthogonal. ■

Remark. In the results above one can put $\mathfrak{M}^*_{\mathcal{X},\mathcal{K}}$ in place of $\mathfrak{M}_{\mathcal{X},\mathcal{K}}$.

It is interesting to know whether Mycielski ideals are similar to one another. Ideals generated by countable systems seem to be almost identical from the point of view of their structure.

Let BOREL$(\mathcal{X}^\omega)$ be the family of all Borel subsets of $\mathcal{X}^\omega$.

THEOREM 2.5. *For every countable system $\mathcal{K}$ the completion of the Boolean algebra* BOREL$(\mathcal{X}^\omega)/\mathfrak{M}_{\mathcal{X},\mathcal{K}}$ *is isomorphic to the collapsing algebra* Col$(\omega, \mathfrak{c})$.

Proof. Recall that if a notion of forcing $\mathbb{P}$ of cardinality $\mathfrak{c}$ satisfies $\mathbb{P} \Vdash$ "$\check{\mathfrak{c}}$ is countable" then RO$(\mathbb{P})$ = Col$(\omega, \mathfrak{c})$ (Theorem 25.11 of [Jec]). Since BOREL$(\mathcal{X}^\omega)/\mathfrak{M}_{\mathcal{X},\mathcal{K}}$ has size continuum it suffices to show that it collapses $\check{\mathfrak{c}}$ on ω. For $X \in \mathcal{K}$ and $\alpha < \mathfrak{c}$ choose $c_{\alpha,X} \in \mathcal{X}^\omega$ such that $c_{\alpha,X}|X \neq c_{\beta,X}|X$ provided $\alpha \neq \beta$. Put $\mathcal{D}_\alpha = \{[\mathcal{X}^\omega *_X c_{\alpha,X}] : X \in \mathcal{K}\}$. Note that the families $\mathcal{D}_\alpha$ are predense subsets of BOREL$(\mathcal{X}^\omega)/\mathfrak{M}_{\mathcal{X},\mathcal{K}}$. Indeed, assume that $B \notin \mathfrak{M}_{\mathcal{X},\mathcal{K}}$ is a Borel subset of $\mathcal{X}^\omega$. By Borel Determinacy we find $X \in \mathcal{K}$ and $\sigma \in$ STR$(\mathcal{X})$ such that $\sigma *_X \mathcal{X}^\omega \subseteq B$. Choose disjoint subsets of X, say $X_0, X_1 \in \mathcal{K}$. Let $\sigma' \in$ STR$(\mathcal{X})$ be defined by

$$\sigma'(s) = \begin{cases} \sigma(s), & \mathrm{lh}(s) \notin X_0, \\ c_{\alpha,X_0}(\mathrm{lh}(s)), & \mathrm{lh}(s) \in X_0. \end{cases}$$

Then $\sigma' *_{X_1} \mathcal{X}^\omega \subseteq B \cap (\mathcal{X}^\omega *_{X_0} c_{\alpha,X_0})$ and $B \cap (\mathcal{X}^\omega *_{X_0} c_{\alpha,X_0}) \notin \mathfrak{M}_{\mathcal{X},\mathcal{K}}$. It is easy to find a BOREL$(\mathcal{X}^\omega)/\mathfrak{M}_{\mathcal{X},\mathcal{K}}$-name $\dot{\tau}$ such that $\Vdash$ "$\dot{\tau} : \check{\mathfrak{c}} \to \mathcal{K}$ and $\dot{\tau}(\alpha) = X$ implies $[\mathcal{X}^\omega *_X c_{\alpha,X}] \in \dot{\Gamma}$", where $\dot{\Gamma}$ is a name for the generic set. Since the sets $\mathcal{X}^\omega *_X c_{\alpha,X}$ (with fixed X) are disjoint, $\Vdash$ "$\dot{\tau}$ is one-to-one". The proof is complete. ■

PROBLEM 2.6. Can there exist countable normal systems $\mathcal{K}$ and $\mathcal{K}'$ such that the Boolean algebras BOREL$(\mathcal{X}^\omega)/\mathfrak{M}_{\mathcal{X},\mathcal{K}}$ and BOREL$(\mathcal{X}^\omega)/\mathfrak{M}_{\mathcal{X},\mathcal{K}'}$ are not isomorphic, or the algebras $\mathcal{P}(\mathcal{X}^\omega)/\mathfrak{M}_{\mathcal{X},\mathcal{K}}$ and $\mathcal{P}(\mathcal{X}^\omega)/\mathfrak{M}_{\mathcal{X},\mathcal{K}'}$ are not isomorphic?

In the presence of the continuum hypothesis we have the following theorem.

THEOREM 2.7 [Mendez [Men], Balcerzak [Bal]]. *Assume CH. Suppose that $\mathcal{K}$ and $\mathcal{K}'$ are countable. Then*

(a) *there exists a bijection $f : \mathcal{X}^\omega \to \mathcal{X}^\omega$ such that $f = f^{-1}$ and for every set $A \subseteq \mathcal{X}^\omega$, $f[A] \in \mathfrak{M}_{\mathcal{X},\mathcal{K}}$ if and only if $A \in \mathbb{K}$,*

(b) *there exists a bijection* $g:\mathcal{X}^\omega\to\mathcal{X}^\omega$ *such that for every set* $A\subseteq\mathcal{X}^\omega$, $g[A]\in\mathfrak{M}_{\mathcal{X},\mathcal{K}'}$ *if and only if* $A\in\mathfrak{M}_{\mathcal{X},\mathcal{K}}$. ■

3. Relation of $\mathfrak{M}_{\mathcal{X},\mathcal{K}}$ to $\mathbb{K}(\mathcal{X}^\omega)$ and $\mathbb{L}(\mathcal{X}^\omega)$. In Theorem 1.1 we mentioned the result of Mycielski that for countable $\mathcal{K}$ the ideal $\mathfrak{M}_{\mathcal{X},\mathcal{K}}$ is orthogonal to the ideal $\mathbb{K}(\mathcal{X}^\omega)\cap\mathbb{L}(\mathcal{X}^\omega)$. Actually Mycielski's argument shows that every set in $\mathfrak{M}_{\mathcal{X},\mathcal{K}}$ can be covered by a comeager set from $\mathfrak{M}_{\mathcal{X},\mathcal{K}}$ if $|\mathcal{K}|<\mathrm{add}(\mathbb{K})$ and by a conull set from $\mathfrak{M}_{\mathcal{X},\mathcal{K}}$ if $|\mathcal{K}|<\mathrm{add}(\mathbb{L})$, and that the same is true for the ideal $\mathfrak{M}^*_{\mathcal{X},\mathcal{K}}$. Hence, for small uncountable generating systems, the ideals $\mathfrak{M}^*_{\mathcal{X},\mathcal{K}}$ are orthogonal to the ideal $\mathbb{K}(\mathcal{X}^\omega)$ (respectively $\mathbb{L}(\mathcal{X}^\omega)$). Below we describe the systems $\mathcal{K}$ for which the ideals $\mathfrak{M}_{\mathcal{X},\mathcal{K}}$ and $\mathbb{K}(\mathcal{X}^\omega)$ are orthogonal and we give some information on the orthogonality of $\mathfrak{M}_{\mathcal{X},\mathcal{K}}$ and $\mathbb{L}(\mathcal{X}^\omega)$. Recall first that if $\mathcal{X}$ is infinite then each ideal $\mathfrak{M}^*_{\mathcal{X},\mathcal{K}}$ is orthogonal to $\mathbb{K}(\mathcal{X}^\omega)\cap\mathbb{L}(\mathcal{X}^\omega)$ (cf. [Ros]). For finite $\mathcal{X}$ the situation is more complicated.

For $X\in[\omega]^\omega$ let $\mu_X\in\omega^\omega$ be an increasing enumeration of X.

We will say that a family $\mathcal{F}\subseteq[\omega]^\omega$ is *unbounded* if

$$(\forall Y\in[\omega]^\omega)(\exists X\in\mathcal{F})(\exists^\infty n)([\mu_Y(n),\mu_Y(n+1))\cap X=\emptyset)\,.$$

A family $\mathcal{F}\subseteq[\omega]^\omega$ will be called *dominating* whenever

$$(\forall Y\in[\omega]^\omega)(\exists X\in\mathcal{F})(\forall^\infty n)(|[\mu_Y(n),\mu_Y(n+1))\cap X|\le 1)\,.$$

Note that $\mathcal{F}$ is unbounded if and only if $\{\mu_X:X\in\mathcal{F}\}$ is an unbounded family in $(\omega^\omega,\le^*)$. The notion of a dominating family in $[\omega]^\omega$ is close to that of a dominating family in $(\omega^\omega,\le^*)$. Namely, $\{\mu_X:X\in\mathcal{F}\}$ is a dominating family in ω^ω provided $\mathcal{F}$ is dominating. Moreover, every dominating family in ω^ω naturally produces a dominating family in $[\omega]^\omega$ (of the same cardinality).

THEOREM 3.1. *Suppose that $\mathcal{X}$ is a finite set. Then the ideal $\mathfrak{M}_{\mathcal{X},\mathcal{K}}$ ($\mathfrak{M}^*_{\mathcal{X},\mathcal{K}}$) is not orthogonal to $\mathbb{K}(\mathcal{X}^\omega)$ if and only if the system $\mathcal{K}$ is unbounded.*

Proof. ($\Rightarrow$) Suppose $\mathcal{K}$ is not an unbounded family and $Y\in[\omega]^\omega$ is a witness for it. Fix $x_0\in\mathcal{X}$. Define

$$G=\{c\in\mathcal{X}^\omega:(\exists^\infty n)(c|[\mu_Y(n),\mu_Y(n+1))\equiv x_0)\}\in\Pi^0_2(\mathcal{X}^\omega)\,.$$

Clearly, G is dense in $\mathcal{X}^\omega$ and hence it is comeager in $\mathcal{X}^\omega$. We show that G belongs to $\mathfrak{M}^*_{\mathcal{X},\mathcal{K}}$. Let $X\in\mathcal{K}$ and let $d\in\mathcal{X}^\omega$ be such that $d(n)\neq x_0$ for $n\in X$. Suppose $c\in\mathcal{X}^\omega *_X d$. Then $X\cap[\mu_Y(n),\mu_Y(n+1))\neq\emptyset$ implies $c|[\mu_Y(n),\mu_Y(n+1))\not\equiv x_0$. Hence $c\notin G$ and $(\mathcal{X}^\omega *_X d)\cap G=\emptyset$.

($\Leftarrow$) Suppose $\mathcal{K}$ is unbounded and $G\in\Pi^0_2(\mathcal{X}^\omega)$ is dense in $\mathcal{X}^\omega$. We prove that $G\notin\mathfrak{M}_{\mathcal{X},\mathcal{K}}$. Due to finiteness of $\mathcal{X}$ we find a set $Y\in[\omega]^\omega$ and sequences $s_n:[\mu_Y(n),\mu_Y(n+1))\to\mathcal{X}$, $n\in\omega$, such that $\{c\in\mathcal{X}^\omega:(\exists^\infty n)(s_n\subseteq c)\}\subseteq G$. We find $X\in\mathcal{K}$ for which infinitely often $[\mu_Y(n),\mu_Y(n+1))\cap X=\emptyset$. For this X the first player can win the game $\Gamma_{\mathcal{X}}(G,X)$: the winning strategy for him may be described by "play according to s_n whenever $[\mu_Y(n), \mu_Y(n+1)) \cap X = \emptyset$". ∎

Let $\mathrm{BAIRE}(\mathcal{X}^\omega)$ be the family of all subsets of $\mathcal{X}^\omega$ with the property of Baire.

COROLLARY 3.2. *Suppose that $\mathcal{X}$ is a finite set.*

(a) *If $|\mathcal{K}| < \mathfrak{b}$ then $\mathfrak{M}^*_{\mathcal{X},\mathcal{K}}$ is orthogonal to $\mathbb{K}(\mathcal{X}^\omega)$.*
(b) *If $\mathcal{K}$ is unbounded then $\mathfrak{M}_{\mathcal{X},\mathcal{K}} \cap \mathrm{BAIRE}(\mathcal{X}^\omega) \subseteq \mathbb{K}(\mathcal{X}^\omega)$.*

Proof. (a) This is an immediate consequence of 3.1.

(b) Suppose that $A \in \mathfrak{M}_{\mathcal{X},\mathcal{K}} \cap \mathrm{BAIRE}(\mathcal{X}^\omega)$ is nonmeager in $\mathcal{X}^\omega$. Equip $\mathcal{X}$ with a group structure (with a neutral element x_0) and put $\mathbb{Q} = \{c \in \mathcal{X}^\omega : (\forall^\infty n)(c(n) = x_0)\}$. Then $A+\mathbb{Q} \in \mathfrak{M}_{\mathcal{X},\mathcal{K}}$ and $A+\mathbb{Q}$ is comeager in $\mathcal{X}^\omega$ (due to the 0-1 law for category). Applying 3.1 we conclude that $\mathcal{K}$ cannot be unbounded. ∎

In Proposition 1.4 of [BRo] another observation illustrating the dependence of $\mathfrak{M}_{\mathcal{X},\mathcal{K}}$ on $\mathcal{K}$ was formulated. Here is a slight modification of it.

PROPOSITION 3.3. *For each $A \in \mathbb{K}(\mathcal{X}^\omega)$, there exists an unbounded normal system $\mathcal{K}$ on ω such that $A \in \mathfrak{M}^*_{\mathcal{X},\mathcal{K}}$.* ∎

Since the ideals $\mathbb{K}(\mathcal{X}^\omega)$ and $\mathbb{L}(\mathcal{X}^\omega)$ are orthogonal it follows from Proposition 3.3 that

COROLLARY 3.4. *There exists an unbounded normal system $\mathcal{K}$ on ω (of power $\mathfrak{c}$) such that $\mathfrak{M}^*_{\mathcal{X},\mathcal{K}}$ is orthogonal to $\mathbb{L}(\mathcal{X}^\omega)$.* ∎

For our next result we need Bartoszyński's description of sets of measure zero.

A set $H \subseteq \mathcal{X}^\omega$ is called *small* if there exist a partition $\{I_n : n \in \omega\}$ of ω and a sequence $\langle J_n : n \in \omega\rangle$ such that

(i) I_n's are intervals, $J_n \subseteq \mathcal{X}^{I_n}$,
(ii) $\sum_{n\in\omega} |J_n| \cdot |\mathcal{X}|^{-|I_n|} < \infty$ and
(iii) $H \subseteq \{c \in \mathcal{X}^\omega : (\exists^\infty n)(c|I_n \in J_n)\} \stackrel{\mathrm{def}}{=} (I_n, J_n)_{n=0}^\infty$.

Note that small sets are of measure zero.

Bartoszyński's theorem says that every set from $\mathbb{L}(\mathcal{X}^\omega)$ can be covered by the union of two small sets (cf. [Bar]).

PROPOSITION 3.5. *Suppose $\mathcal{K}$ is a dominating normal system on ω. Then $\mathfrak{M}_{\mathcal{X},\mathcal{K}}$ is not orthogonal to $\mathbb{L}(\mathcal{X}^\omega)$.*

Proof. We have to show that $\mathfrak{M}_{\mathcal{X},\mathcal{K}} \cap \mathbb{L}^c(\mathcal{X}^\omega) = \emptyset$. Suppose $H \in \mathbb{L}(\mathcal{X}^\omega)$ and $(I_n, J_n)_{n=0}^\infty$, $(I_n^*, J_n^*)_{n=0}^\infty$ are two small sets which cover H. Let $Y \in [\omega]^\omega$

be such that each segment $[\mu_Y(n), \mu_Y(n+1))$ contains some interval I_k as well as some interval I^*_l. Next find $X \in \mathcal{K}$ such that

$$(\forall^\infty n)(|[\mu_Y(n), \mu_Y(n+1)) \cap X| \le 1).$$

Note that then $|I_n \cap X| \le 2$ and $|I^*_n \cap X| \le 2$ for all but finitely many n. Let $\underset{\sim}{J}_n$ (respectively $\underset{\sim}{J}^*_n$) be a family of all functions from I_n (I^*_n) into $\mathcal{X}$ which agree with some element of J_n (J^*_n) on the set $I_n \setminus X$ ($I^*_n \setminus X$). The sets $(I_n, \underset{\sim}{J}_n)^\infty_{n=0}$ and $(I^*_n, \underset{\sim}{J}^*_n)^\infty_{n=0}$ are small because $|\underset{\sim}{J}_n| \le |J_n| \cdot |\mathcal{X}|^{|X \cap I_n|}$ and $|\underset{\sim}{J}^*_n| \le |J^*_n| \cdot |\mathcal{X}|^{|X \cap I^*_n|}$. Take $c \in \mathcal{X}^\omega \setminus ((I_n, \underset{\sim}{J}_n)^\infty_{n=0} \cup (I^*_n, \underset{\sim}{J}^*_n)^\infty_{n=0})$. Clearly, $c *_X \mathcal{X}^\omega$ is disjoint from $(I_n, \underset{\sim}{J}_n)^\infty_{n=0} \cup (I^*_n, \underset{\sim}{J}^*_n)^\infty_{n=0}$, and consequently from H. Hence $\mathcal{X}^\omega \setminus H \notin \mathfrak{M}_{\mathcal{X},\mathcal{K}}$. ■

Corollary 3.6. *If $\mathcal{K}$ is a dominating normal system on ω then*

$$\mathfrak{M}_{\mathcal{X},\mathcal{K}} \cap \mathrm{MEASURE}(\mathcal{X}^\omega) \subseteq \mathbb{L}(\mathcal{X}^\omega). \ ■$$

Problem 3.7. (a) Is $\mathfrak{M}_{\mathcal{X},\mathcal{K}}$ orthogonal to $\mathbb{L}(\mathcal{X}^\omega)$, provided $\mathcal{K}$ is not dominating? What if $|\mathcal{K}| < \mathfrak{d}$?

(b) Suppose $A \in \mathbb{L}(\mathcal{X}^\omega)$. Does there exist a countable normal system $\mathcal{K}$ such that $A \in \mathfrak{M}_{\mathcal{X},\mathcal{K}}$? Note that the full measure analogue of Proposition 3.3 is impossible because of Corollary 3.2.

4. Notions of forcing connected with $\mathfrak{C}_\mathcal{X}$ and $\mathfrak{P}_\mathcal{X}$. In 2.5 we showed that for countable $\mathcal{K}$ the Boolean algebra $\mathrm{BOREL}(\mathcal{X}^\omega)/\mathfrak{M}_{\mathcal{X},\mathcal{K}}$ as a notion of forcing is equivalent to the collapsing algebra $\mathrm{Col}(\omega, \mathfrak{c})$. Easy arguments prove that the forcing $\mathrm{BOREL}(\omega^\omega)/\mathfrak{C}_\omega$ also collapses $\check{\mathfrak{c}}$ onto ω. If $\mathcal{X}$ is finite, however, $\mathrm{BOREL}(\mathcal{X}^\omega)/\mathfrak{C}_\mathcal{X}$ becomes a nontrivial notion of forcing. Due to the Borel Determinacy we can describe this order more precisely. Every Borel set that does not belong to $\mathfrak{C}_\mathcal{X}$ contains a set of the form $\sigma *_X \mathcal{X}^\omega$ for some $\sigma \in \mathrm{STR}(\mathcal{X})$, $X \in [\omega]^\omega$. Such a set is actually the body of a perfect tree T on $\mathcal{X}$ with the property that, for some $X \in [\omega]^\omega$, $(\forall s \in T, \mathrm{lh}(s) \in X)(\mathrm{succ}_T(s) = \mathcal{X})$. Let $\mathbb{Q}_\mathcal{X} = \{T \subseteq \mathcal{X}^{<\omega} :$ T is a perfect tree & $(\exists X \in [\omega]^\omega)(\forall s \in T, \mathrm{lh}(s) \in X)(\mathrm{succ}_T(s) = \mathcal{X})\}$ be ordered by inclusion. By the above remarks we see that $\mathbb{Q}_\mathcal{X}$ can be densely embedded in $\mathrm{BOREL}(\mathcal{X}^\omega)/\mathfrak{C}_\mathcal{X}$. Note that $\mathbb{Q}_\mathcal{X}$ as an ordered set contains the Silver forcing $\mathbb{S}_\mathcal{X} = \{p : p$ is a function & $\mathrm{dom}(p) \subseteq \omega$ & $\mathrm{rng}(p) \subseteq \mathcal{X}$ & $\omega \setminus \mathrm{dom}(p)$ is infinite$\}$ and is contained in the Sacks perfect set forcing for $\mathcal{X}^\omega$. As in those forcings, we can define orders $\le_n$ in $\mathbb{Q}_\mathcal{X}$ by $T_1 \le_n T_2$ if and only if $T_1 \le T_2$ and the first n elements of the sets $\{m \in \omega : (\forall s \in \mathcal{X}^m \cap T_2)(\mathrm{succ}_{T_2}(s) = \mathcal{X})\}$ and $\{m \in \omega : (\forall s \in \mathcal{X}^m \cap T_1)(\mathrm{succ}_{T_1}(s) = \mathcal{X})\}$ are the same. Standard arguments show the following:

Proposition 4.1. (a) *If $T_{n+1} \le_{n+1} T_n$ and $T_n \in \mathbb{Q}_\mathcal{X}$ then there exists T from $\mathbb{Q}_\mathcal{X}$ such that $T \le_n T_n$ for all n.*

(b) *If* $T \Vdash$ "$\dot{\tau} \in V$" *and* $n \in \omega$ *then there are* $T' \leq_n T$ *and* $A \in [V]^{|\mathcal{X}|^n}$ *such that* $T' \Vdash$ "$\dot{\tau} \in A$". ∎

COROLLARY 4.2. (a) $\mathbb{Q}_{\mathcal{X}}$ *satisfies Axiom* A *of Baumgartner* [Bau].

(b) $\mathbb{Q}_{\mathcal{X}} \Vdash$ "$(\forall A \in \mathbb{L})(\exists B \in \mathbb{L} \cap V)(A \subseteq B)$". ∎

Remark. With every set from $\mathfrak{C}_{\mathcal{X}}$ we can associate a dense subset of $\mathbb{Q}_{\mathcal{X}}$. Namely, for $A \subseteq \mathcal{X}^\omega$ we put $D_A = \{T \in \mathbb{Q}_{\mathcal{X}} : [T] \cap A = \emptyset\}$. It is obvious that D_A is open dense in $\mathbb{Q}_{\mathcal{X}}$ provided $A \in \mathfrak{C}_{\mathcal{X}}$. Moreover, one can consider the following ideal on $\mathcal{X}^\omega$ connected with $\mathbb{Q}_{\mathcal{X}}$:

$$\mathbb{IQ}_{\mathcal{X}} = \{A \subseteq \mathcal{X}^\omega : (\forall T \in \mathbb{Q}_{\mathcal{X}})(\exists T' \in \mathbb{Q}_{\mathcal{X}}, T' \leq T)([T'] \cap A = \emptyset)\}\,.$$

An easy application of the fusion property proves that $\mathbb{IQ}_{\mathcal{X}}$ is a σ-ideal of subsets of $\mathcal{X}^\omega$. Clearly $\mathfrak{C}_{\mathcal{X}} \subseteq \mathbb{IQ}_{\mathcal{X}}$.

We do not have any reasonable description of the algebra $\mathrm{BOREL}(\mathcal{X}^\omega)/\mathfrak{P}_{\mathcal{X}}$. Since $\mathrm{BOREL}(\omega^\omega)/\mathfrak{P}_\omega$ collapses $\check{\mathfrak{c}}$ onto ω, the only nontrivial case here is $\mathcal{X}$ finite. It was noted in [CRSW] that the Silver forcing $\mathbb{S}_{\mathcal{X}}$ is connected with $\mathfrak{P}_{\mathcal{X}}$ in the following way. Consider the σ-ideal determined by $\mathbb{S}_{\mathcal{X}}$: $\mathbb{IS}_{\mathcal{X}} = \{A \subseteq \mathcal{X}^\omega : (\forall p \in \mathbb{S}_{\mathcal{X}})(\exists q \in \mathbb{S}_{\mathcal{X}}, q \leq p)([q] \cap A = \emptyset)\}$ (here $[q] = \{c \in \mathcal{X}^\omega : q \subseteq c\}$ for $q \in \mathbb{S}_{\mathcal{X}}$. Then $\mathfrak{P}_{\mathcal{X}} \subseteq \mathbb{IS}_{\mathcal{X}}$. Unfortunately, we do not know whether $\mathbb{S}_{\mathcal{X}}$ can be densely embedded in $\mathrm{BOREL}(\mathcal{X}^\omega)/\mathfrak{P}_{\mathcal{X}}$.

5. Cardinal coefficients. In this section we study the cardinal coefficients of the ideals $\mathfrak{M}_{\mathcal{X},\mathcal{K}}$ and $\mathfrak{M}^*_{\mathcal{X},\mathcal{K}}$, especially their covering numbers. Recall first that the cardinal coefficients of $\mathfrak{M}_{\mathcal{X},\mathcal{K}}$ if $\mathcal{K}$ is countable or if $\mathcal{X}$ is infinite are as follows (cf. [Ros]).

THEOREM 5.1. (a) *Suppose* $\mathcal{K}$ *is countable. Then*

$$\mathrm{non}(\mathfrak{M}_{\mathcal{X},\mathcal{K}}) = \mathrm{non}(\mathfrak{M}^*_{\mathcal{X},\mathcal{K}}) = \mathrm{cof}(\mathfrak{M}_{\mathcal{X},\mathcal{K}}) = \mathrm{cof}(\mathfrak{M}^*_{\mathcal{X},\mathcal{K}}) = \mathfrak{c}$$

and

$$\mathrm{cov}(\mathfrak{M}_{\mathcal{X},\mathcal{K}}) = \mathrm{cov}(\mathfrak{M}^*_{\mathcal{X},\mathcal{K}}) = \mathrm{add}(\mathfrak{M}_{\mathcal{X},\mathcal{K}}) = \mathrm{add}(\mathfrak{M}^*_{\mathcal{X},\mathcal{K}}) = \omega_1\,.$$

(b) $\mathrm{add}(\mathfrak{P}_\omega) = \mathrm{cov}(\mathfrak{P}_\omega) = \mathrm{add}(\mathfrak{C}_\omega) = \mathrm{cov}(\mathfrak{C}_\omega) = \omega_1$, $\mathrm{non}(\mathfrak{C}_\omega) = \mathrm{non}(\mathfrak{P}_\omega) = \mathfrak{c}$, $\mathrm{cof}(\mathfrak{P}_\omega) > \mathfrak{c}$, *and if* $\mathrm{cov}(\mathbb{K}) = \mathfrak{c}$ *then* $\mathrm{cof}(\mathfrak{C}_\omega) > \mathfrak{c}$. ∎

If we drop the countability assumption we have the following.

PROPOSITION 5.2. $\mathrm{add}(\mathfrak{M}^*_{\mathcal{X},\mathcal{K}}) = \mathrm{cov}(\mathfrak{M}^*_{\mathcal{X},\mathcal{K}})$ *provided for every* $X \in \mathcal{K}$, $\mathcal{K} \cap \mathcal{P}(X)$ *is isomorphic to* $\mathcal{K}$. *In any case*, $\mathrm{cov}(\mathfrak{M}^*_{\mathcal{X},\mathcal{K}}) \geq \mathrm{cov}(\mathfrak{M}_{\mathcal{X},\mathcal{K}})$. *In particular*, $\mathrm{add}(\mathfrak{P}_{\mathcal{X}}) = \mathrm{cov}(\mathfrak{P}_{\mathcal{X}}) \geq \mathrm{cov}(\mathfrak{C}_{\mathcal{X}})$.

Remark. The extra assumption above is essential. There may exist a system $\mathcal{K}$ such that $\mathrm{add}(\mathfrak{M}^*_{2,\mathcal{K}}) < \mathrm{cov}(\mathfrak{M}^*_{2,\mathcal{K}})$. E.g. take a normal system $\mathcal{K}$ such that for some $X_1, X_2 \in \mathcal{K}$, $|\mathcal{K} \cap \mathcal{P}(X_1)| = \omega$ but $\mathcal{K} \cap \mathcal{P}(X_2) = \mathcal{P}(X_2)$. Then $\mathrm{add}(\mathfrak{M}^*_{2,\mathcal{K}}) = \omega_1$ (cf. 5.1(a)) while it is possible that $\mathrm{cov}(\mathfrak{M}^*_{2,\mathcal{K}}) > \omega_1$ (cf. 5.11, 5.12).

Applying 3.1 and Rothberger's result saying that if $\mathbb{I}$, $\mathbb{J}$ are orthogonal, translation invariant ideals on a group $\mathbf{X}$ then $\mathrm{cov}(\mathbb{I}) \le \mathrm{non}(\mathbb{J})$ (cf. [Fre]) we obtain

PROPOSITION 5.3. *If $\mathcal{K}$ is not unbounded then* $\mathrm{cov}(\mathfrak{M}^*_{\mathcal{X},\mathcal{K}}) \le \mathrm{non}(\mathbb{K})$. ■

Remark. By Proposition 5.3 we know that $\mathrm{cov}(\mathfrak{M}^*_{\mathcal{X},\mathcal{K}}) \le \mathrm{non}(\mathbb{K})$ provided $|\mathcal{K}| < \mathfrak{b}$. In Proposition 5.7 we improve this to $\mathrm{cov}(\mathfrak{M}^*_{\mathcal{X},\mathcal{K}}) \le \mathfrak{b}$.

A double indexed sequence $\{X_{\xi,\nu} : \xi < \eta,\ \nu < \kappa\} \subseteq [\omega]^\omega$ is called a *κ-support* for $\mathcal{K}$ if

(1) $(\forall X \in \mathcal{K})(\forall \nu < \kappa)(\exists \xi < \eta)(X_{\xi,\nu} \subseteq X)$,

and a *special κ-support* for $\mathcal{K}$ if additionally

(2) $X_{\xi,\nu} \ne X_{\xi',\nu'}$ provided $(\xi,\nu) \ne (\xi',\nu')$.

Note that if $\kappa \le \mathfrak{c}$ then there exists a special κ-support for $[\omega]^\omega$ which is also a special κ-support for all $\mathcal{K}$.

A *κ-covering system* for $\mathcal{K}$ and $\mathcal{X}$ is a sequence of partial functions $\{f_{\xi,\nu} : \xi < \eta,\ \nu < \kappa\}$ such that:

(3) $\mathrm{dom}(f_{\xi,\nu}) \in [\omega]^\omega$, $\mathrm{rng}(f_{\xi,\nu}) \subseteq \mathcal{X}$,
(4) $\{\mathrm{dom}(f_{\xi,\nu}) : \xi < \eta,\ \nu < \kappa\}$ is a κ-support for $\mathcal{K}$,
(5) no function $c \in \mathcal{X}^\omega$ is such that for each $\nu < \kappa$ there is a $\xi < \eta$ with $f_{\xi,\nu} \subseteq c$.

The existence of κ-covering systems is connected with the covering number of $\mathfrak{M}^*_{\mathcal{X},\mathcal{K}}$ in the following way:

LEMMA 5.4. *There exists a κ-covering system for $\mathcal{K}$ and $\mathcal{X}$ if and only if* $\mathrm{cov}(\mathfrak{M}^*_{\mathcal{X},\mathcal{K}}) \le \kappa$.

Proof. Assume that $\{f_{\xi,\nu} : \xi < \eta,\ \nu < \kappa\}$ is a κ-covering system for $\mathcal{K}$ and $\mathcal{X}$, and put $A_\nu = \{c \in \mathcal{X}^\omega : (\forall \xi < \eta)(\neg f_{\xi,\nu} \subseteq c)\}$. Then obviously $A_\nu \in \mathfrak{M}^*_{\mathcal{X},\mathcal{K}}$ and $\bigcup\{A_\nu : \nu < \kappa\} = \mathcal{X}^\omega$ (the last is a consequence of (5)). On the other hand, suppose $\{A_\nu : \nu < \kappa\} \subseteq \mathfrak{M}^*_{\mathcal{X},\mathcal{K}}$ is such that $\bigcup\{A_\nu : \nu < \kappa\} = \mathcal{X}^\omega$. We choose functions $c_{X,\nu} \in \mathcal{X}^\omega$ such that for every $X \in \mathcal{K}$ and $\nu < \kappa$, $(\mathcal{X}^\omega *_X c_{X,\nu}) \cap A_\nu = \emptyset$. Then $\{c_{X,\nu}|X : X \in \mathcal{K},\ \nu < \kappa\}$ is a κ-covering system for $\mathcal{K}$ and $\mathcal{X}$. ■

The easy lemma below has interesting consequences.

LEMMA 5.5. *Suppose $\mathcal{K}' < \mathcal{K}$ and $\mathcal{X}' \subseteq \mathcal{X}$. Every κ-covering system for $\mathcal{K}'$ and $\mathcal{X}'$ is a covering system for $\mathcal{K}$ and $\mathcal{X}$.* ■

PROPOSITION 5.6. *Assume that $\mathcal{K}' < \mathcal{K}$ and $\mathcal{X}' \subseteq \mathcal{X}$. Then*

$$\mathrm{cov}(\mathfrak{M}^*_{\mathcal{X},\mathcal{K}}) \le \mathrm{cov}(\mathfrak{M}^*_{\mathcal{X}',\mathcal{K}'})\,. \ ■$$

The basic estimate of $\mathrm{cov}(\mathfrak{M}^*_{\mathcal{X},\mathcal{K}})$ is given by the following

PROPOSITION 5.7. *There exists a* $|\mathcal{K}|^+$*-covering system for* $\mathcal{K}$ *and* $\mathcal{X}$. *Consequently*, $\mathrm{cov}(\mathfrak{M}_{\mathcal{X},\mathcal{K}}) \le \mathrm{cov}(\mathfrak{M}^*_{\mathcal{X},\mathcal{K}}) \le |\mathcal{K}|^+$.

Proof. For $|\mathcal{K}| = \mathfrak{c}$ this is obvious by $\mathrm{cov}(\mathfrak{M}^*_{\mathcal{X},\mathcal{K}}) \le \mathfrak{c}$ and 5.4. Assume that $|\mathcal{K}| < \mathfrak{c}$. Choose $f_{\alpha,X} : X \to \mathcal{X}$ for $\alpha < |\mathcal{K}|^+$, $X \in \mathcal{K}$ such that $f_{\alpha,X} \ne f_{\beta,X}$ provided $\alpha < \beta < |\mathcal{K}|^+$. Then clearly $\{f_{\alpha,X} : \alpha < |\mathcal{K}|^+, X \in \mathcal{K}\}$ is a $|\mathcal{K}|^+$-covering system for $\mathcal{K}$ and $\mathcal{X}$. ■

Remark. Note that the above estimate cannot be improved. If $|\mathcal{K}| = \omega$ then $\mathrm{cov}(\mathfrak{M}^*_{\mathcal{X},\mathcal{K}}) = \omega_1$. But even if $\mathcal{K}$ is uncountable we may have $\mathrm{cov}(\mathfrak{M}^*_{\mathcal{X},\mathcal{K}}) = |\mathcal{K}|^+$ (compare 5.11).

Let $\mathcal{B} = \{S : S : \omega \to [\omega]^{<\omega} \ \&\ (\forall n \in \omega)(|S(n)| = 2^n)\}$ and let $\pi : [\omega]^{<\omega} \to \omega$ be a bijection. For $X \in [\omega]^\omega$ we define $\varphi_X : \omega \to [\omega]^{<\omega}$ by $\varphi_X(n) =$ "the set of the first 2^{n+2} elements of X". If $X \in [\omega]^\omega$ and $S \in \mathcal{B}$ are such that $(\forall n)(\pi(\varphi_X(n)) \in S(n))$ then we write $X \mathrel{\widehat{\in}} S$.

The following useful lemma was proved in [CRSW].

LEMMA 5.8. *There exists a* (*Borel*) *function* $F : \mathcal{B} \times [\omega]^\omega \to 2^\omega$ *such that if* $X_1 \mathrel{\widehat{\in}} S$, $X_2 \mathrel{\widehat{\in}} S$ *and the partial functions* $F(S, X_1)|X_1$, $F(S, X_2)|X_2$ *are compatible then* $X_1 = X_2$. ■

THEOREM 5.9. *Suppose* $|\mathcal{K}| < \mathrm{add}(\mathbb{L})$. *Then there exists an* ω_1*-covering system for* $\mathcal{K}$ *and* 2. *Consequently*, *for each* $\mathcal{X}$, $\mathrm{cov}(\mathfrak{M}^*_{\mathcal{X},\mathcal{K}}) = \omega_1$.

Proof. By 5.1(a) and 5.4 we may assume that $\mathrm{add}(\mathbb{L}) > \omega_1$. Let F be the function given by 5.8. Let $\{X_{\xi,\nu} : \xi < |\mathcal{K}|,\ \nu < \omega_1\}$ be a special ω_1-support for $\mathcal{K}$. Due to Bartoszyński's well known characterization of $\mathrm{add}(\mathbb{L})$ (cf. [Fre]) we find $\mathcal{L} \in [\mathcal{B}]^\omega$ such that $(\forall \xi < |\mathcal{K}|)(\forall \nu < \omega_1)(\exists S_{\xi,\nu} \in \mathcal{L})(X_{\xi,\nu} \mathrel{\widehat{\in}} S_{\xi,\nu})$. For each ξ and ν put $f_{\xi,\nu} = F(S_{\xi,\nu}, X_{\xi,\nu})|X_{\xi,\nu}$. To show that $\{f_{\xi,\nu} : \xi < |\mathcal{K}|, \nu < \omega_1\}$ is an ω_1-covering system for $\mathcal{K}$ and 2 we should verify the condition (5) only. But assuming that $c \in 2^\omega$ is a couterexample for (5), we have $(\forall \nu < \omega_1)(\exists \xi < |\mathcal{K}|)(f_{\xi,\nu} \subseteq c)$. Since $\mathcal{L}$ is countable, we find different $\nu, \mu < \omega_1$ and suitable $\xi, \vartheta < |\mathcal{K}|$ such that $S_{\xi,\nu} = S_{\vartheta,\mu} = S$. Then $F(S, X_{\xi,\nu})|X_{\xi,\nu}$ and $F(S, X_{\vartheta,\mu})|X_{\vartheta,\mu}$ are included in c. The properties of F give that $X_{\xi,\nu} = X_{\vartheta,\mu}$, contrary to condition (2) of a special ω_1-support. The last part of the theorem follows from 5.4 and 5.5. ■

Recall that Lemma 5.8 was applied in [CRSW] to show (after a slight reformulation) the following

THEOREM 5.10. *There exists a* $\mathrm{cof}(\mathbb{L})^+$*-covering system for* $[\omega]^\omega$ *and* 2. *Consequently*, *for each* $\mathcal{X}$ *and* $\mathcal{K}$, $\mathrm{cov}(\mathfrak{M}^*_{\mathcal{X},\mathcal{K}}) \le \mathrm{cof}(\mathbb{L})^+$. ■

We have no reasonable lower bound for $\mathrm{cov}(\mathfrak{M}^*_{\mathcal{X},\mathcal{K}})$ but it can be large.

An almost disjoint family $\{A_\alpha : \alpha < \kappa\} \subseteq [\omega]^\omega$ has the *Uniformization Property* (UP) if for every system of functions $f_\alpha : A_\alpha \to 2$ there is a function $f : \bigcup\{A_\alpha : \alpha < \kappa\} \to 2$ such that for every $\alpha < \kappa$ we have $f_\alpha \subseteq^* f$.

Shelah showed that the existence of uncountable almost disjoint families with UP is consistent with ZFC (cf. [She]).

PROPOSITION 5.11. *Assume that there exists an almost disjoint family of cardinality κ with UP. Then for every cardinal $\lambda \le \kappa$ there exists a normal system $\mathcal{K}$ such that $|\mathcal{K}| = \lambda$ and $\operatorname{cov}(\mathfrak{M}^*_{\mathcal{X},\mathcal{K}}) = \lambda^+$. In particular*, $\operatorname{cov}(\mathfrak{P}_2) > \kappa$. ∎

As we saw in Section 4, $\mathfrak{C}_{\mathcal{X}} \subseteq \mathbb{IQ}_{\mathcal{X}}$. Hence $\operatorname{cov}(\mathbb{IQ}_{\mathcal{X}}) \le \operatorname{cov}(\mathfrak{C}_{\mathcal{X}})$. Since $\mathbb{Q}_{\mathcal{X}}$ satisfies Baumgartner's Axiom A we obtain

PROPOSITION 5.12. *PFA implies* $\operatorname{cov}(\mathfrak{C}_{\mathcal{X}}) > \omega_1$. ∎

Remark. The above result was formulated by Recław for $\mathfrak{P}_{\mathcal{X}}$. Proposition 5.12 strengthens his observation. Let us also recall that MA does not imply $\operatorname{cov}(\mathfrak{P}_{\mathcal{X}}) > \omega_1$. This is a result of Steprāns (cf. [CRSW]).

Finally, we show that the covering numbers of the ideals $\mathfrak{C}_{\mathcal{X}}$ can be different for different finite $\mathcal{X}$.

THEOREM 5.13. *Suppose $k \ge 2$. Then*

$$\mathrm{CON}(\mathrm{ZFC} + \operatorname{cov}(\mathfrak{P}_k) = \operatorname{cov}(\mathfrak{C}_k) = \omega_2 = \mathfrak{c} + (\forall j > k)(\operatorname{cov}(\mathfrak{C}_j) = \omega_1)).$$

Proof. Suppose that $V \models \mathrm{CH}$. Let $\langle \mathbb{P}_\alpha : \alpha < \omega_2\rangle$ be a countable support iteration of forcings $\mathbb{Q}_k$. Then $\mathbb{P}_{\omega_2}$ preserves cardinal numbers and $\Vdash_{\omega_2}$ "$\mathfrak{c} = \omega_2$". Suppose that for $\alpha < \omega_1$ we have a $\mathbb{P}_{\omega_2}$-name $\dot{A}_\alpha$ such that $\Vdash_{\omega_2}$ "$\dot{A}_\alpha \in \mathfrak{C}_k$". Note that each set from $\mathfrak{C}_k$ is determined by a function from $[\omega]^\omega$ into $\mathrm{STR}(k)$. Thus we have $\mathbb{P}_{\omega_2}$-names $\dot{\tau}_\alpha$ such that for each $\alpha < \omega_1$,

$$\Vdash_{\omega_2} \text{“}\dot{\tau}_\alpha : [\omega]^\omega \to \mathrm{STR}(k) \ \&\ (\forall X \in [\omega]^\omega)(k^\omega *_X \dot{\tau}_\alpha(X) \cap \dot{A}_\alpha = \emptyset)\text{”}.$$

By standard arguments we find $\beta < \omega_2$ such that the sequence $\langle \dot{\tau}_\alpha | ([\omega]^\omega \cap V^{\mathbb{P}_\beta}) : \alpha < \omega_1\rangle$ belongs to $V^{\mathbb{P}_\beta}$. Let $\dot{c}_\beta$ be a $\mathbb{P}_\beta$-name such that $\Vdash_\beta$ "$\dot{c}_\beta$ is a name for the $\mathbb{Q}_k$-generic real". Then obviously

$$\Vdash_\beta \text{“}\mathbb{Q}_k \Vdash (\forall \alpha < \omega_1)(\exists X \in [\omega]^\omega \cap V^{\mathbb{P}_\beta})(\dot{c}_\beta \in k^\omega *_X \dot{\tau}_\alpha(X))\text{”}$$

and consequently $\Vdash_{\omega_2}$ "$\dot{c}_\beta \notin \bigcup_{\alpha<\omega_1} \dot{A}_\alpha$". We have thus proved $\Vdash_{\omega_2}$ "$\operatorname{cov}(\mathfrak{C}_k) = \omega_2$". To show that $\Vdash_{\omega_2}$ "$(\forall i > k)(\operatorname{cov}(\mathfrak{C}_i) = \omega_1)$" we have to strengthen 4.1(b).

A tree $T \subseteq \omega^{<\omega}$ is a *k-tree* if $(\forall s \in T)(|\mathrm{succ}_T(s)| \le k)$. A notion of forcing $\mathbb{P}$ has the *k-localization property* if

$$\mathbb{P} \Vdash (\forall f \in \omega^\omega)(\exists T \in V)(\text{“}T \text{ is a } k\text{-tree on } \omega\text{” } \&\ f \in [T]).$$

A slight modification of Theorem 2.3 of [NRo] shows that every countable support iteration of forcings $\mathbb{Q}_k$ has the k-localization property. Hence, in $V^{\mathbb{P}_{\omega_2}}$, if $i > k$ then i^ω can be covered by ω_1 k-trees. Note that if $T \subseteq i^{<\omega}$ is a k-tree then $[T] \in \mathfrak{C}_i$. Consequently, $\Vdash_{\omega_2}$ "$\operatorname{cov}(\mathfrak{C}_i) = \omega_1$" for every $i > k$. ∎

Remark. Similarly to the above theorem one can build a model for $(\forall i \le k)(\mathrm{cov}(\mathfrak{C}_i) = \omega_2)$ & $(\forall i > k)(\mathrm{cov}(\mathfrak{C}_i) = \omega_1)$. But we do not know whether in these models $\mathrm{cov}(\mathfrak{P}_{k+1}) = \omega_1$ holds true. The problem "Can the covering numbers of the ideals $\mathfrak{P}_{\mathcal{X}}$ be different for distinct $\mathcal{X}$" remains open.

6. Compact sets from ideals. Let $\mathcal{K}(\mathcal{X}^\omega)$ denote the space of all compact subsets of $\mathcal{X}^\omega$ equipped with the Vietoris topology. The subbase of this topology consists of all sets $U(G) = \{F \in \mathcal{K}(\mathcal{X}^\omega) : F \subseteq G\}$, $V(G) = \{F \in \mathcal{K}(\mathcal{X}^\omega) : F \cap G \ne \emptyset\}$ for open $G \subseteq \mathcal{X}^\omega$ (cf. [Kur]).

A recent result of Kechris, Louveau and Woodin (cf. [KLW]) shows that if $\mathbb{I}$ is a σ-ideal on a Polish space $\mathbf{X}$ then its trace on compact sets is either very simple (Π^0_2) or very complicated (at least Π^1_1). The compact sets of uniqueness form a Π^1_1-complete set (cf. [KLW]). The strongly porous compact sets (cf. [Lar]), the nowhere dense compact sets and Lebesgue null sets (cf. [KLW]) are Π^0_2 in $\mathcal{K}(\mathbb{R})$. For Mycielski ideals generated by countable systems a similar result was proved by Balcerzak.

THEOREM 6.1 [Balcerzak, [BRo]]. *Suppose $\mathcal{K}$ is countable. Then $\mathfrak{M}_{\mathcal{X},\mathcal{K}} \cap \mathcal{K}(\mathcal{X}^\omega)$ and $\mathfrak{M}^*_{\mathcal{X},\mathcal{K}} \cap \mathcal{K}(\mathcal{X}^\omega)$ are Π^0_2, hence comeager subsets of $\mathcal{K}(\mathcal{X}^\omega)$.* ■

Since each system $\mathcal{K}$ is the union of $|\mathcal{K}|$ countable systems, putting 2.1 and 6.1 together we get

COROLLARY 6.2. (a) *If $|\mathcal{K}| < \mathrm{add}(\mathbb{K})$ then $\mathfrak{M}^*_{\mathcal{X},\mathcal{K}} \cap \mathcal{K}(\mathcal{X}^\omega)$ (and hence $\mathfrak{M}_{\mathcal{X},\mathcal{K}} \cap \mathcal{K}(\mathcal{X}^\omega)$) is comeager in $\mathcal{K}(\mathcal{X}^\omega)$.*

(b) *If $|\mathcal{K}| < \mathrm{cov}(\mathbb{K})$ then $\mathfrak{M}^*_{\mathcal{X},\mathcal{K}} \cap \mathcal{K}(\mathcal{X}^\omega)$ (and hence $\mathfrak{M}_{\mathcal{X},\mathcal{K}} \cap \mathcal{K}(\mathcal{X}^\omega)$) is nonmeager in $\mathcal{K}(\mathcal{X}^\omega)$.* ■

We now describe the traces of $\mathfrak{C}_{\mathcal{X}}$ and of $\mathfrak{P}_{\mathcal{X}}$ on compact sets. The following easy technical lemma was mentioned in [BRo].

LEMMA 6.3. *If $A \in \mathcal{K}(\mathcal{X}^\omega)$, $X \in [\omega]^\omega$ and τ is a winning strategy for the second player in the game $\Gamma_{\mathcal{X}}(A, X)$, then there is an integer $N > 0$ such that for each $c \in \mathcal{X}^\omega$ with $(\forall n < N, n \in X)(c(n) = \tau(c|n))$ we have $c \notin A$.* ■

THEOREM 6.4. *$\mathfrak{C}_{\mathcal{X}} \cap \mathcal{K}(\mathcal{X}^\omega)$, $\mathfrak{P}_{\mathcal{X}} \cap \mathcal{K}(\mathcal{X}^\omega) \in \Pi^1_1 \setminus \Sigma^1_1$ and both are meager subsets of $\mathcal{K}(\mathcal{X}^\omega)$.*

Proof. First we show that $\mathfrak{C}_{\mathcal{X}} \cap \mathcal{K}(\mathcal{X}^\omega)$ and $\mathfrak{P}_{\mathcal{X}} \cap \mathcal{K}(\mathcal{X}^\omega)$ are coanalytic. For $A \in \mathcal{K}(\mathcal{X}^\omega)$, applying 6.3, we have

$$\begin{aligned} A \in \mathfrak{C}_{\mathcal{X}} & \\ &\equiv (\forall X \in [\omega]^\omega)(\exists \sigma \in \mathrm{STR})(\forall \tau \in \mathrm{STR})(\tau *_X \sigma \notin A) \\ &\equiv (\forall X \in [\omega]^\omega)(\exists N \in \omega)(\exists \sigma : \mathcal{X}^{<N} \to \mathcal{X})(\forall \tau \in \mathrm{STR})(\tau *_{X\cap N} \sigma \notin A)\,, \end{aligned}$$

and similarly for $\mathfrak{P}_{\mathcal{X}}$:

$$A \in \mathfrak{P}_{\mathcal{X}} \equiv (\forall X \in [\omega]^\omega)(\exists N \in \omega)(\exists d \in \mathcal{X}^N)(\forall c \in \mathcal{X}^\omega)(c *_{X\cap N} d \notin A)\,.$$

The last formulas represent Π^1_1 subsets of $\mathcal{K}(\mathcal{X}^\omega)$.

To prove $\mathfrak{C}_\mathcal{X}\cap\mathcal{K}(\mathcal{X}^\omega)\in\mathbb{K}(\mathcal{K}(\mathcal{X}^\omega))$, note that $\mathfrak{C}_\mathcal{X}\cap\mathcal{K}(\mathcal{X}^\omega)$ has the Baire property (since Π^1_1 implies the Baire property). So, it is enough to show

Claim. *If $G\in\Pi^0_2(\mathcal{K}(\mathcal{X}^\omega))$ is nonmeager then $G\setminus\mathfrak{C}_\mathcal{X}\neq\emptyset$.*

Suppose that $G=\bigcap_{n\in\omega}G_n$ is dense in $W=V([s_0])\cap\ldots\cap V([s_{k-1}])\cap U(\bigcup_{i<k}[s_i])$, $s_0,\ldots,s_{k-1}\in\mathcal{X}^{n_0}$, and G_n are open. Construct inductively a perfect tree $T\subseteq\mathcal{X}^{<\omega}$ and a set $X=\{n_0,n_1,\ldots\}$ as follows: $T\cap\mathcal{X}^{n_0}=\{s_0,\ldots,s_{k-1}\}$. Having defined $n_i\in\omega$ and $T\cap\mathcal{X}^{n_i}$ consider $U(\bigcup\{[s]:s\in T\cap\mathcal{X}^{n_i}\})\cap\bigcap\{V([s^\wedge x]):s\in T\cap\mathcal{X}^{n_i},x\in\mathcal{X}\}$. It is an open subset of W, G_i is dense in W, hence, for $s\in T\cap\mathcal{X}^{n_i}$ and $x\in\mathcal{X}$ there are nonempty $t(s,x)\subseteq\mathcal{X}^{<\omega}$ such that $s^\wedge x\subseteq\bigcap t(s,x)$ and $U(\bigcup\{[t]:t\in t(s,x),s\in T\cap\mathcal{X}^{n_i},x\in\mathcal{X}\})\cap\bigcap\{V([t]):t\in t(s,x),s\in T\cap\mathcal{X}^{n_i},x\in\mathcal{X}\}$ is contained in G_i. Clearly, we may assume that $\mathrm{lh}(t)=n_{i+1}$ for all $t\in t(s,x)$, $s\in T\cap\mathcal{X}^{n_i}$, $x\in\mathcal{X}$. Put $T\cap\mathcal{X}^{n_{i+1}}=\{t:t\in t(s,x),s\in T\cap\mathcal{X}^{n_i},x\in\mathcal{X}\}$. Our construction provides $[T]\in G$. Moreover, for each $n\in X$ and $s\in T\cap\mathcal{X}^n$ we have $\mathrm{succ}_T(s)=\mathcal{X}$. Hence $[T]\notin\mathfrak{C}_\mathcal{X}$.

It follows from the above that also $\mathfrak{P}_\mathcal{X}\cap\mathcal{K}(\mathcal{X}^\omega)\in\mathbb{K}(\mathcal{K}(\mathcal{X}^\omega))$.

Now, if $\mathfrak{C}_\mathcal{X}\cap\mathcal{K}(\mathcal{X}^\omega)$ or $\mathfrak{P}_\mathcal{X}\cap\mathcal{K}(\mathcal{X}^\omega)$ were analytic then it would be of type Π^0_2 (due to the result of Kechris, Louveau and Woodin mentioned earlier). But $\mathfrak{C}_\mathcal{X}\cap\mathcal{K}(\mathcal{X}^\omega)$ and $\mathfrak{P}_\mathcal{X}\cap\mathcal{K}(\mathcal{X}^\omega)$ are dense in $\mathcal{K}(\mathcal{X}^\omega)$ (they contain all finite sets) and therefore they would be comeager in $\mathcal{K}(\mathcal{X}^\omega)$, contrary to what we have proved. ■

As an application of 6.4 consider a mapping $\Phi:\mathcal{P}(\mathcal{X}^\omega\times\mathcal{X}^\omega)\to\mathcal{P}(\mathcal{X}^\omega)$ given by the formula $\Phi(A)=\{c\in\mathcal{X}^\omega:A_c\notin\mathfrak{C}_\mathcal{X}\}$, where A_c is the vertical section of A at c.

Proposition 6.5. (a) $\Phi[\Sigma^0_1(\mathcal{X}^\omega\times\mathcal{X}^\omega)]=\Sigma^0_1(\mathcal{X}^\omega)$.
(b) $\Phi[\Pi^0_1(\mathcal{X}^\omega\times\mathcal{X}^\omega)]=\Phi[\Sigma^0_3(\mathcal{X}^\omega\times\mathcal{X}^\omega)]=\Sigma^1_1(\mathcal{X}^\omega)$.
(c) $\Phi[\mathrm{BOREL}(\mathcal{X}^\omega\times\mathcal{X}^\omega)]\subseteq\Sigma^1_2(\mathcal{X}^\omega)$.

Proof. (a) and (c) are obvious.
(b) Suppose $A\in\Pi^0_2(\mathcal{X}^\omega\times\mathcal{X}^\omega)$. Then

$$A_c\notin\mathfrak{C}_\mathcal{X}\equiv(\exists K\in\mathcal{K}(\mathcal{X}^\omega))(K\notin\mathfrak{C}_\mathcal{X}\ \&\ K\subseteq A_c)\,.$$

The formula $K\subseteq A_c$ represents Π^0_2-subsets of $\mathcal{K}(\mathcal{X}^\omega)\times\mathcal{X}^\omega$. Apply 6.4 to obtain $\Phi(A)\in\Sigma^1_1(\mathcal{X}^\omega)$. Since $\Phi(\bigcup_{n\in\omega}A_n)=\bigcup_{n\in\omega}\Phi(A_n)$ we have shown $\Phi[\Sigma^0_3(\mathcal{X}^\omega\times\mathcal{X}^\omega)]\subseteq\Sigma^1_1(\mathcal{X}^\omega)$. Suppose now that $B\in\Sigma^1_1(\mathcal{X}^\omega)$. 6.4 implies that $\mathfrak{C}_\mathcal{X}\cap\mathcal{K}(\mathcal{X}^\omega)$ is Π^1_1-complete (cf. [KLW]) and therefore we can find a continuous function $f:\mathcal{X}^\omega\to\mathcal{K}(\mathcal{X}^\omega)$ such that $f^{-1}[\mathfrak{C}_\mathcal{X}]=\mathcal{X}^\omega\setminus B$. Put $A=\{(c,d)\in\mathcal{X}^\omega\times\mathcal{X}^\omega:d\in f(c)\}\in\Pi^0_1(\mathcal{X}^\omega\times\mathcal{X}^\omega)$. Clearly $\Phi(A)=B$.

Problem 6.6. Describe $\Phi[\mathrm{BOREL}(\mathcal{X}^\omega\times\mathcal{X}^\omega)]$.

Note that an analogous mapping may be defined for every σ-ideal. The ideals $\mathbb{L}$ and $\mathbb{K}$ are regular from that standpoint since for them $\Phi[\Sigma^0_\alpha(\mathcal{X}^\omega \times \mathcal{X}^\omega)] = \Sigma^0_\alpha(\mathcal{X}^\omega)$ for $\alpha < \omega_1$.

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