COLLOQUIUM MATHEMATICUM

VOL. LXVI 1993 FASC. 2

COERCIVE INEQUALITIES ON WEIGHTED SOBOLEV SPACES

BY

AGNIESZKA KAŁAMAJSKA (WARSZAWA)

1. Introduction. Coercive inequalities play an important role in many disciplines of P.D.E. They are applied to derive existence and regularity results in various boundary value problems. The most well known is the Korn inequality

$$\|\nabla u\|_{L^p(\Omega)} \le C\bigg\{\|u\|_{L^p(\Omega)} + \sum_{i,j=1}^{n}\bigg\|\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\bigg\|_{L^p(\Omega)}\bigg\},$$

which has been used to obtain results of existence, uniqueness and regularity for the principal boundary value problems in linearized elastostatics (see e.g. [V]). Koshelev noticed that the classical Korn inequality is insufficient for treating certain boundary value problems in the elasticity theory. To solve some of the problems he proved the Korn inequality in a weighted version with power-type, radial weights $\varrho(x) = |x - x_0|^\alpha$ (see [Kos]). Kondrat'ev and Oleĭnik [KO] extended this result to a wider class of weights of type $|x|^\alpha$. As is known if $-n < \alpha < n(p-1)$ then such a weight belongs to the Muckenhoupt class A_p (see [T], Sec. IX, Corollary 4.4). Thus it is natural to ask whether coercive inequalities hold in weighted L^p spaces with Muckenhoupt weights. A good example of such nonradial weights are functions of the form $(\mathrm{dist}(x,\partial\Omega))^\alpha$ where $-1 < \alpha < p-1$ and Ω is for example a bounded Lipschitz-boundary domain. Weighted Sobolev spaces with such weights have been investigated by Kufner in [Ku] for needs of equations with perturbed ellipticity.

We prove that the classical coercive inequalities (see e.g. [BIN], [S]) extend to inequalities in a weighted version with Muckenhoupt weights (Theorem 6).

Weighted coercive inequalities relate to equivalent norms in weighted Sobolev spaces. In recent time much attention has been paid to the study of weighted Sobolev spaces (see e.g. [GK], [Ku], [LO]). Their understanding leads to the generalization of regularity results in many problems of P.D.E.

1991 *Mathematics Subject Classification*: 46E35, 35B45.

I would like to express my thanks to Professor Bogdan Bojarski for his attention.

2. Preliminaries. Let Ω be an open subset of $\mathbb{R}^n$, and $\varrho \geq 0$ a locally integrable function. By $L^p_\varrho(\Omega)$ we denote the weighted L^p-space on Ω, i.e. the space of functions f for which $\int_\Omega |f|^p \varrho\, dx$ is finite. If $\varrho \equiv 1$ then ϱ will be omitted in notation.

We use the following definition of weighted Sobolev spaces:

$$W^{m,p}_\varrho(\Omega) := \{f \in D'(\Omega) : D^\alpha f \in L^p_\varrho(\Omega),\ |\alpha| \leq m\}$$

with the norm

$$\|f\|_{W^{m,p}_\varrho(\Omega)} := \sum_{|\alpha|\leq m} \|D^\alpha f\|_{L^p_\varrho(\Omega)}\,.$$

THEOREM 1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain, starshaped with respect to a ball B. Choose $\omega \in C_0^\infty(B)$ such that $\int_B \omega\, dx = 1$. Then for any $f \in W^{m,1}(\Omega)$,*

$$f(x) = P^{m-1}_\omega f(x) + \sum_{|\alpha|=m} \int_\Omega K_\alpha(x,y) D^\alpha f(y)\, dy \qquad a.e.\ in\ \Omega\,,$$

where

$$P^{m-1}_\omega f(x) = \int_\Omega \bigg\{ \sum_{|\beta|<m} D^\beta_y \bigg(\frac{(y-x)^\beta}{\beta!}\omega(y)\bigg)\bigg\} f(y)\, dy$$

($P^{m-1}_\omega f(x)$ is a polynomial of degree less than m) and

$$K_\alpha(x,y) = \frac{(-1)^m m}{\alpha!}\frac{(y-x)^\alpha}{|y-x|^n} \int_{|y-x|}^\infty \omega\bigg(x + t\frac{y-x}{|y-x|}\bigg) t^{n-1} dt\,.$$

(See [Ma], Th. 1.1.10/1 for the proof.)

Let $P_j = (P_{j1}, \ldots, P_{jk})$ $(j = 1, \ldots, N)$ be scalar differential operators of order m, acting on vector-valued functions $f = (f_1, \ldots, f_k)$:

$$P_j f = \sum_{i=1}^k P_{ji} f_i, \qquad P_{ji} g(x) = \sum_{|\alpha|\leq m} a_{\alpha,j,i}(x) D^\alpha g(x)\,.$$

Denote by P_j^m the principal part of P_j, involving differentiations of highest order, and by P_j^0 the part involving differentiations of order less than m. We say that P_j is homogeneous if $P_j^0 = 0$. Let $P_{ji}(x,\xi)$ be the corresponding characteristic polynomials. If P_j has constant coefficients then we write simply $P_{ji}(\xi)$.

We will be interested in Sobolev type spaces of vector-valued functions

$$L^{\{P_j\},p}_\varrho(\Omega) = \{f = (f_1, \ldots, f_k) : f_i \in D',\ P_j f \in L^p_\varrho(\Omega)\}\,.$$

If $\varrho \equiv 1$ then we omit ϱ in our notation. For example one of such spaces is

$$L^{m,p}_{\varrho}(\Omega) = \{f = (f_1, \ldots, f_k) : f_i \in D',\ \nabla^m f \in L^p_{\varrho}(\Omega)\}$$

where $\nabla^m f$ stands for the vector $\{D^\alpha f\}_{|\alpha|=m}$.

If $f \in L^1_{\mathrm{loc}}$ then Mf denotes the Hardy–Littlewood maximal function of f. We will require that $\varrho \in A_p$ $(1 < p < \infty)$, that is, ϱ satisfies the Muckenhoupt condition

$$\sup_Q \frac{1}{|Q|} \int_Q \varrho\, dx \biggl\{ \frac{1}{|Q|} \int_Q \varrho^{-1/p-1}\, dx \biggr\}^{p-1} < \infty$$

where Q are cubes in $\mathbb{R}^n$. Muckenhoupt's theorem (see e.g. [T]) states that for $1 < p < \infty$ the operator $f \mapsto Mf$ is bounded in L^p_ϱ if $\varrho \in A_p$.

Theorem 2. *Let Ω be a bounded, starshaped domain, and $\{P_j\}_{j=1,\ldots,N}$ a family of differential operators acting on vector-valued functions $f = (f_1, \ldots, f_k)$, with the following properties*:

- *P_j are homogeneous of order m and have constant coefficients*,
- *the matrix $\{P_{ji}(\xi)\}^{j=1,\ldots,N}_{i=1,\ldots,k}$ has rank k for any $\xi \neq (0, \ldots, 0)$ with complex ξ_i $(i = 1, \ldots, n)$.*

Then there exist vector-valued functions $K_j(x, y)$ $(j = 1, \ldots, N)$, $K_j(x, y) = (K_{j1}, \ldots, K_{jk})$, satisfying the following conditions:

(i) $K_{ji} \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{x = y\})$,

(ii) $K_{ji}(x, \cdot) \equiv 0$ *near the boundary of Ω for $x \in \Omega$*,

(iii) $|D^\alpha_x D^\beta_y K_{ji}(x, y)| \le C/|x - y|^{n-m+|\alpha|+|\beta|}$ *for any $x, y \in \Omega$*,

(iv) *there exists a positive integer $l \ge m$ and scalar differential operators $P_{j,i,\alpha}$ $(j = 1, \ldots, N,\ i = 1, \ldots, k,\ |\alpha| = l)$ of order $l - m$, homogeneous, with constant coefficients*, *satisfying*

$$K_{ji}(x, y) = \sum_{|\alpha|=l} (P_{j,i,\alpha})_y K_\alpha(x, y)\,,$$

(v) *for any $f \in L^{\{P_j\},1}_{\mathrm{loc}}(\Omega)$ and almost every $x \in \Omega$*

$$f_i(x) = P^{l-1}_\omega f_i(x) + \sum_{j=1}^N \int_\Omega K_{ji}(x, y) P_j f(y)\, dy$$

where $P^{l-1}_\omega f_i$ is as in Theorem 1.

The proof can be found in [Ka], Ths. 4 and 6. Note that in the scalar case the third assumption on P_j means that the $P_j(\xi)$ with complex ξ_i $(i = 1, \ldots, n)$ have no common nontrivial zeros.

Observe that if $1 < p < \infty$ then the representation from Theorem 2 is valid for every $f \in L^{\{P_j\},p}_\varrho(\Omega)$, since then $L^{\{P_j\},p}_\varrho(\Omega) \subseteq L^{\{P_j\},1}(\Omega)$.

The following result is a consequence of Theorem 9 of [Ka] and the inclusion $L^p_\varrho(\Omega) \subseteq L^1(\Omega)$ $(1 < p < \infty,\ \varrho \in A_p)$.

THEOREM 3. *Let Ω be a bounded domain with the cone property*, $1 < p < \infty$, *and $\varrho \in A_p$. Then there exists a constant C such that for every $f \in W^{m,p}_\varrho(\Omega)$, and every multiindex β with $|\beta| = k,\ 0 < k < m$,*

$$\|D^\beta f\|_{L^p_\varrho(\Omega)} \le C\Big\{\|f\|_{L^p_\varrho(\Omega)} + (\|f\|_{L^p_\varrho(\Omega)})^{1-k/m}\Big(\sum_{|\alpha|=m}\|D^\alpha f\|_{L^p_\varrho(\Omega)}\Big)^{k/m}\Big\}.$$

By C we denote the general constant. It may stand for different constants even in the same proof.

3. A norm equivalence condition for homogeneous operators with constant coefficients. We will need the following facts:

LEMMA 1 (see [H]). *Let $\psi \in L^1$ be a radial-decreasing function, and $f \in L^1_{\mathrm{loc}}$. Then the convolution $\psi * f$ satisfies*

$$|\psi * f(x)| \le C\|\psi\|_{L^1} Mf(x)$$

almost everywhere, with a constant independent of f.

Applying Muckenhoupt's theorem and the above lemma we easily derive

COROLLARY 1. *If Ω is any bounded domain, and $\varrho \in A_p$, $1 < p < \infty$, then weakly singular operators on Ω are bounded in $L^p_\varrho(\Omega)$.*

LEMMA 2 (see [Mi], Sec. II/8). *Let $K \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{x = y\})$ and $h \in L^1(\Omega)$.*

(i) *If $|\frac{\partial}{\partial x_i} K(x,y)| \le C/|x-y|^{n-1}$ and $|K(x,y)| \le C/|x-y|^{n-2}$ then*

$$\frac{\partial}{\partial x_i}\int_\Omega K(x,y)h(y)\,dy = \int_\Omega \frac{\partial}{\partial x_i}K(x,y)h(y)\,dy\,.$$

(ii) *If*

$$K(x,y) = \frac{c\Big(x, \dfrac{x-y}{|x-y|}\Big)}{|x-y|^{n-1}}$$

and $c \in C^\infty(\mathbb{R}^n \times S^{n-1})$ then

$$\frac{\partial}{\partial x_i}\int_\Omega K(x,y)h(y)\,dy = \int_\Omega \frac{\partial}{\partial x_i}K(x,y)h(y)\,dy - h(x)\int_{S^{n-1}} c(x,\theta)\theta_i\,dS_\theta$$

where dS_θ is the area element on the sphere S^{n-1}.

LEMMA 3. *Let $h(x,y) = g(x, x-y)$ where $g(x,z)$ is smooth with respect to x and homogeneous of order $-(n-1)$ with respect to z. Then* *for every* $x \in \mathbb{R}^n$,

$$\int_{S^{n-1}} \frac{\partial}{\partial y_i} h(x,y)\, dS_y = 0\,.$$

P r o o f. Define $P(r_1, r_2) = \{y : r_1 \le |x-y| \le r_2\}$. Since $\frac{\partial}{\partial y_i} h(x,y)$ is homogeneous of order $-n$ with respect to $x-y$, we have

$$\int_{P(r_1,r_2)} \frac{\partial}{\partial y_i} h(x,y)\, dy = \ln\left(\frac{r_1}{r_2}\right) \int_{S^{n-1}} \frac{\partial}{\partial y_i} h(x,y)\, dS_y\,.$$

But the left hand side is zero since by Green's formula

$$\int_{P(r_1,r_2)} \frac{\partial}{\partial y_i} h(x,y)\, dy$$

$$= \int_{|x-y|=r_2} h(x,y) \frac{y_i - x_i}{|x-y|}\, dS_y - \int_{|x-y|=r_1} h(x,y) \frac{y_i - x_i}{|x-y|}\, dS_y$$

and h is homogeneous. ■

LEMMA 4. *Let Ω be a bounded domain, $\varrho \in A_p$, $1 < p < \infty$, and let α, β, γ be multiindices such that $|\alpha| = |\beta| = |\gamma| = m$. Then the operator*

$$h \mapsto D^\gamma \int_\Omega D_y^\beta K_\alpha(x,y) h(y)\, dy$$

is bounded in $L^p_\varrho(\Omega)$.

P r o o f. Let

$$r(x,y) = |y-x|, \qquad \theta(x,y) = \frac{y-x}{|y-x|}\,,$$

$$A_\alpha(x,y) = \frac{\theta^\alpha}{r^{n-m}} \int_0^\infty \omega(x+t\theta) t^{n-1}\, dt\,,$$

$$B_\alpha(x,y) = \frac{\theta^\alpha}{r^{n-m}} \int_0^r \omega(x+t\theta) t^{n-1}\, dt\,.$$

Since $K_\alpha = A_\alpha - B_\alpha$ it is sufficient to prove that the corresponding operators with K_α replaced by A_α and B_α respectively are bounded in $L^p_\varrho(\Omega)$.

Let us look at the function $A_\alpha(x,y)$. Its first $m-1$ derivatives are sums of functions of the form $c(x,\theta)/r^{n-k}$ for $k \ge 1$, smooth with respect to x and θ. If $k > 1$ then by Lemma 2 and Corollary 1 the operator

$$h \mapsto \frac{\partial}{\partial x_i} \int_\Omega \frac{c(x,\theta)}{r^{n-k}} h(y)\, dy$$

is bounded in $L^p_\varrho(\Omega)$. If $k = 1$ then by Lemma 2,

$$\frac{\partial}{\partial x_i}\int\limits_{\Omega}\frac{c(x,\theta)}{r^{n-1}}h(y)\,dy=\int\limits_{\Omega}\frac{\partial}{\partial x_i}\left(\frac{c(x,\theta)}{r^{n-1}}\right)h(y)\,dy-h(x)\int\limits_{S^{n-1}}c(x,\theta)\theta_i\,dS_\theta\,.$$

And

$$\frac{\partial}{\partial x_i}\left(\frac{c(x,\theta)}{r^{n-1}}\right)=-\frac{\partial}{\partial y_i}\left(\frac{c(x,\theta)}{r^{n-1}}\right)+\frac{c'(x,\theta)}{r^{n-1}}$$

with $c' \in C^\infty(\mathbb{R}^n \times S^{n-1})$. To obtain the boundedness of the operator $h \mapsto D^\gamma \int_\Omega D^\beta_y A_\alpha(x,y)h(y)\,dy$ it is enough to show that the operator

$$h \mapsto Th=\int\limits_{\Omega}\frac{\partial}{\partial y_i}\left(\frac{c(x,\theta)}{r^{n-1}}\right)h(y)\,dy$$

is bounded in $L^p_\varrho(\Omega)$.

Lemma 3 yields that T is a Calderón–Zygmund operator (see [T], Sec. XI, Remark 8.11 and [CZ], Th. 2). Now it is enough to apply a version of the Calderón–Zygmund theorem stating that Calderón-Zygmund operators are bounded in L^p_ϱ for $p > 1$ and $\varrho \in A_p$ ([TJ], see also [T], Sec. XIII, Remark 4.5).

The result for B_α follows by similar methods and the observation that there exists a constant C such that $\frac{1}{r}\int_0^r \omega(x+t\theta)t^l\,dt \le C$ for all $r < \operatorname{diam}(\Omega)$, $\theta \in S^{n-1}$. That property follows from Lebesgue's differentiation theorem (see e.g. [T]). ∎

THEOREM 4. *Let Ω be a bounded, starshaped domain, $\varrho \in A_p$, $1 < p < \infty$, and let $\{P_j\}_{j=1,\ldots,N}$ be a family of differential operators acting on vector-valued functions $f = (f_1,\ldots,f_k)$ and satisfying*

- *the P_j are homogeneous of order m and have constant coefficients,*
- *the matrix $\{P_{ji}(\xi)\}^{j=1,\ldots,N}_{i=1,\ldots,k}$ has rank k for any $\xi \neq (0,\ldots,0)$ with complex ξ_i.*

Then there exists a constant C such that for any $f \in L^{\{P_j\},p}_\varrho(\Omega)$,

$$\|\nabla^m f\|_{L^p_\varrho(\Omega)} \le C\Big\{\|f\|_{L^p_\varrho(\Omega)}+\sum_{j=1}^N\|P_jf\|_{L^p_\varrho(\Omega)}\Big\}\,.$$

Proof. Since every domain with the cone property is a finite union of starshaped domains ([Ma], Lemma 1.1.9/1), we may assume that Ω is starshaped. Now the assertion is an immediate consequence of Theorem 2 and Lemma 4. ∎

Remarks. 1. Using Lemma 2 we can also prove that if Ω is a bounded, starshaped domain, $\phi \in C^\infty_0(\mathbb{R}^n), \phi \equiv 1$ in a neighbourhood of Ω, and $1 < p < \infty$, then the function $\bar{f}(x)$ given by

$$\bar{f}(x) = \phi(x)\Big\{P_\omega^{m-1} f(x) + \sum_{|\alpha|=m} \int_\Omega K_\alpha(x,y) D^\alpha f(y)\, dy\Big\}$$

satisfies $\bar{f} = f$ in Ω, and

$$\|\bar{f}\|_{W_\varrho^{m,p}} \le C\|f\|_{W_\varrho^{m,p}(\Omega)}$$

with a constant C independent of f.

It follows by the same arguments that if P_j, p and ϱ are as in Theorem 2 then there exists a bounded extension operator $L_\varrho^{\{P_j\},p}(\Omega) \to L_\varrho^{\{P_j\},p}$.

2. If Ω is a bounded, starshaped domain then functions smooth in a neighbourhood of Ω are dense in $W_\varrho^{m,p}(\Omega)$ and in $L_\varrho^{\{P_j\},p}(\Omega)$, provided P_j, p and ϱ are as in Theorem 2. We will show this for the space $W_\varrho^{m,p}(\Omega)$. By Remark 1 it is enough to prove that any $f \in W_\varrho^{m,p}$ with compact support can be approximated by smooth functions in $W_\varrho^{m,p}$. Choose a radial-decreasing function $\phi \in C_0^\infty(\mathbb{R}^n)$ such that $\phi \equiv 1$ in a neighbourhood of 0 and $\int \phi = 1$. Define $\phi_\varepsilon(x) = \varepsilon^{-n}\phi(x/\varepsilon)$ and $f_\varepsilon = \phi_\varepsilon * f$. Since $f \in W^{m,1}$ we have $D^\alpha f_\varepsilon(x) \to D^\alpha f(x)$ a.e. for $|\alpha| \le m$ and by Lemma 1, $|D^\alpha f_\varepsilon(x) - D^\alpha f(x)| \le 2M(D^\alpha f)(x)$ almost everywhere. Thus, by the Lebesgue dominated convergence theorem and Muckenhoupt's theorem we obtain $D^\alpha f_\varepsilon \to D^\alpha f$ in L_ϱ^p.

3. If Ω, $\{P_j\}$, ϱ, and p are as in Theorem 4 then $L_\varrho^{\{P_j\},p}(\Omega) = W_\varrho^{m,p}(\Omega)$.

4. A norm equivalence condition for operators with non-constant coefficients

THEOREM 5. *Let Ω be a bounded domain with the cone property, $\varrho \in A_p$, $1 < p < \infty$, and let $\{P_j\}_{j=1,\ldots,N}$ be a family of differential operators of order m acting on vector-valued functions $f = (f_1, \ldots, f_k)$ and satisfying*

- *the coefficients of P_j^m are continuous in Ω, and those of P_j^0 are bounded in Ω,*
- *the matrix $\{P_{ji}(x,\xi)\}_{i=1,\ldots,k}^{j=1,\ldots,N}$ has rank k for any $\xi \ne (0,\ldots,0)$ with complex ξ_i and $x \in \Omega$.*

Then there exists a constant C such that for any $f \in W_\varrho^{m,p}(\Omega)$,

$$\|\nabla^m f\|_{L_\varrho^p(\Omega)} \le C\Big\{\|f\|_{L_\varrho^p(\Omega)} + \sum_{j=1}^N \|P_j f\|_{L_\varrho^p(\Omega)}\Big\}.$$

Proof. As in the proof of Theorem 4 we may assume that Ω is starshaped. We introduce the following notation:

- Q_j^x — the operator P_j^m evaluated at x:

$$Q_j^x f(y) = \sum_{i=1}^k \sum_{|\alpha|=m} a_{\alpha,j,i}(x) D^\alpha f_i(y)\,,$$

• $B(x,R)$ — the ball with center x and radius R,
• $\Omega(x,R)=\Omega\cap B(x,R)$,
• $a(x,R)=\sup_{y\in\Omega(x,R),j=1,\dots,N,|\alpha|=m,i=1,\dots,k}|a_{\alpha,j,i}(y)-a_{\alpha,j,i}(x)|$,
• $C(x)$ — the constant in the coercivity condition (i.e. the inequality of Theorem 4) for $\{Q_j^x\}$ on Ω.

Choosing R sufficiently small we may assume that

• $a(x,R)<\big(2C(x)Nk\binom{m}{n}\big)^{-1}$,
• $\Omega(x,R)$ is starshaped.

Since the balls $B(x,R)$ cover Ω, we can choose a finite subcover $\{B_k=B(x_k,R_k):k=1,\dots,K\}$ and a smooth partition of unity $\{\phi_k\}$ subordinate to this subcover. Set $\Omega_k=\Omega(x_k,R_k)$, $Q_j^k=Q_j^{x_k}$, $C_k=C(x_k)$.

Applying Theorem 4 to $\{Q_j^k\}_{j=1,\dots,N}$ we derive

$$\|\nabla^m(\phi_k f)\|_{L^p_\varrho(\Omega)}\le C_k\Big\{\|\phi_k f\|_{L^p_\varrho(\Omega)}+\sum_{j=1}^N\|Q_j^k(\phi_k f)\|_{L^p_\varrho(\Omega)}\Big\}$$

and

$$\begin{aligned}\|Q_j^k(\phi_k f)\|_{L^p_\varrho(\Omega)}\le{}&\|(Q_j^k-P_j^m)(\phi_k f)\|_{L^p_\varrho(\Omega_k)}\\&+\|P_j(\phi_k f)\|_{L^p_\varrho(\Omega_k)}+\|P_j^0(\phi_k f)\|_{L^p_\varrho(\Omega_k)}\,.\end{aligned}$$

Hence

$$\|\nabla^m(\phi_k f)\|_{L^p_\varrho(\Omega)}\le C\{\|f\|_{W^{m-1,p}_\varrho(\Omega)}+\|P_jf\|_{L^p_\varrho(\Omega)}\}+\tfrac12\|\nabla^m(\phi_k f)\|_{L^p_\varrho(\Omega)}\,.$$

Now the assertion follows easily from Theorem 3. ∎

Lemma 5. *Let $\{P_j\}_{j=1,\dots,N}$ be a family of differential operators of order m satisfying the following conditions*:

• *the P_j are homogeneous with constant coefficients*,
• *the matrix $\{P_{ji}(i\xi)\}_{i=1,\dots,k}^{j=1,\dots,N}$ has rank k for any $\xi\ne(0,\dots,0)$ with real ξ_i.*

Let $f=(f_1,\dots,f_k)$, $f_i\in C_0^\infty$. Then for every multiindex α of order m there exist functions $m_{\alpha,j}(\xi)$ such that

(i) *$m_{\alpha,j}(\xi)$ is smooth except at $\xi=0$,*
(ii) *$|m_{\alpha,j}(\xi)|\le C$ in $\mathbb{R}^n\setminus\{0\}$,*
(iii) *$R^{2|\alpha|-n}\int_{R<|\xi|<2R}|D^\alpha m_{\alpha,j}(\xi)|^2\,d\xi\le C$ for all $R>0,|\alpha|<n/2+1$,*
(iv) *$\widehat{D^\alpha f_i}(\xi)=\sum_j m_{\alpha,j}(\xi)\widehat{P_jf}(\xi)$ for any multiindex α of order m, where $\widehat{g}$ denotes the Fourier transform of g.*

The construction of $m_{\alpha,j}$ is given in [BIN], Theorem 11.6.

LEMMA 6. *Let Ω be a bounded domain with the cone property, and $\{P_j\}_{j=1,\ldots,N}$ be a family of differential operators of order m acting on vector-valued functions $f=(f_1,\ldots,f_k)$ and satisfying*

• *the coefficients of P_j^m are continuous in Ω, and those of P_j^0 are bounded in Ω,*

• *the matrix $\{P_{ji}(x,i\xi)\}_{i=1,\ldots,k}^{j=1,\ldots,N}$ has rank k for any $\xi\neq(0,\ldots,0)$ with real ξ_i and $x\in\Omega$.*

Let $f\in(C_0^\infty(\Omega))^k, 1<p<\infty$, and $\varrho\in A_p$. Then

$$\|\nabla^m f\|_{L^p_\varrho(\Omega)}\le C\Big\{\|f\|_{L^p_\varrho(\Omega)}+\sum_{j=1}^N\|P_jf\|_{L^p_\varrho(\Omega)}\Big\}$$

with a constant independent of f.

Proof. If the operators P_j are homogeneous with constant coefficients then Lemma 6 follows directly from Lemma 5 and Hörmander's multiplier theorem in a weighted version (see [T], Sec. XIII, Remark 4.3). In the general case we apply the above observation and the methods described in the proof of Theorem 5.

Now we can formulate the main theorem.

THEOREM 6. *Let Ω be a bounded domain with the cone property, $\varrho\in A_p$, $1<p<\infty$, and let $\{P_j\}_{j=1,\ldots,N}$ be a family of differential operators of order m acting on vector-valued functions $f=(f_1,\ldots,f_k)$ such that*

• *the coefficients of P_j^m are continuous in Ω, and those of P_j^0 are bounded in Ω,*

• *the matrix $\{P_{ji}(x,i\xi)\}_{i=1,\ldots,k}^{j=1,\ldots,N}$ has rank k for any $\xi\neq(0,\ldots,0)$ with real ξ_i and $x\in\Omega$, and for any $\xi\neq(0,\ldots,0)$ with complex ξ_i and $x\in\partial\Omega$.*

Then there exists a constant C such that for any $f\in W_\varrho^{m,p}(\Omega)$,

$$\|\nabla^m f\|_{L^p_\varrho(\Omega)}\le C\Big\{\|f\|_{L^p_\varrho(\Omega)}+\sum_{j=1}^N\|P_jf\|_{L^p_\varrho(\Omega)}\Big\}.$$

Proof. We may assume that Ω is starshaped. By Remark 2 of Section 3 it is enough to prove the inequality for $f\in(C^\infty(\Omega))^k$. It follows from the assumptions that there exists a set Ω_δ, a neighbourhood of $\partial\Omega$ in Ω, such that the matrix $\{P_{ji}(x,i\xi)\}_{i=1,\ldots,k}^{j=1,\ldots,N}$ has rank k for any nontrivial ξ with complex components and $x\in\Omega_\delta$. Choose any $\phi\in C_0^\infty(\Omega\setminus\Omega_\delta)$. We have $f=\phi f+(1-\phi)f$ and $\phi f\in(C_0^\infty(\Omega))^k$. Now it is enough to apply Lemma 6 to ϕf , Theorem 5 to $(1-\phi)f$ and add the resulting estimates. ■

Remarks. 1. Theorem 6 can be stated for differential operators acting between sections of bundles on differentiable manifolds.

2. Theorem 6 does not hold for $p = 1$ or $p = \infty$ (see [B], [O]).

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INSTITUTE OF MATHEMATICS
WARSAW UNIVERSITY
BANACHA 2
02-097 WARSZAWA, POLAND
E-mail: KALAMAJS@MIMUW.EDU.PL

Reçu par la Rédaction le 7.4.1993;
en version modifiée le 9.6.1993