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ON ERGODIC SINGULAR INTEGRAL OPERATORS

BY

A. MICHAEL ALPHONSE AND SHOBHA MADAN (KANPUR)

1. Introduction. In [11] Petersen gave a direct proof of Cotlar's [8] result on the existence a.e. and boundedness of the ergodic Hilbert transform defined by a measure-preserving invertible transformation on a probability space (X, μ). Petersen's proof consists in proving ℓ^p-inequalities for the maximal discrete Hilbert transform on sequence spaces and then applying Calderón's transference principle ([6], also [7]).

In this paper we study a class of operators, called singular series operators (Definition 2.1), which are discrete analogues of singular integral operators on $\mathbb{R}$ ([13], [14]). By transference, we then consider the corresponding ergodic operators on L^p-spaces of Banach space valued functions on X, for suitable Banach spaces B.

In Section 2, the singular series operators are defined as convolution operators on the sequence spaces ℓ^p, $1 \le p < \infty$, and we show that the associated maximal operator is bounded on ℓ^p for $1 < p < \infty$ and is of weak type $(1, 1)$. This result can be proved by standard real variable methods using Calderón–Zygmund decomposition. We prove the maximal operator inequalities by transferring these from the corresponding inequalities on $\mathbb{R}$. This transference works for Banach space valued sequence spaces ℓ^p_B where B is a Banach space. The singular integral operators are well behaved on the Banach space valued function spaces $L^p_B(\mathbb{R})$ if B is a UMD space (unconditional martingale differences) with an unconditional basis. UMD spaces were discovered by Burkholder ([4]). A Banach space B is a UMD space iff the Hilbert transform is a bounded operator on $L^p_B(\mathbb{R})$, $1 < p < \infty$ ([4] and [2]). A UMD space can also be characterized by the boundedness of the discrete Hilbert transform on ℓ^p_B ([1]) . For a geometric characterization, we refer to [5].

In Section 3, we define the ergodic singular operators and using the transference principle as in [11], we prove the existence a.e. and boundedness of these operators on the spaces $L^p_B(X)$ consisting of B-valued (strongly)

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measurable functions on X with $\int_X \|f(x)\|^p\,d\mu(x) < \infty$, where B is as in the above paragraph.

Throughout we write C_1, C_p, C' etc. for positive constants which may vary from one line to the next. $\mathbb{Z}$ denotes the additive group of integers, $\mathbb{T}$ the circle group. We write $\|\ \|_p$ for the norm in ℓ^p as well as in L^p_B, as the case may be. For a subset E of $\mathbb{Z}$, $\operatorname{card} E$ denotes the cardinality of E.

2. Singular series operators on sequence spaces

2.1. DEFINITION. A sequence $\phi = \{\phi(n)\}_{n\in\mathbb{Z}}$ is said to be a *singular kernel* if there exist constants C_1, $C_2 > 0$ such that

$$\text{(S1)}\qquad \sum_{n=-N}^{N} \phi(n) \text{ converges as } N \to \infty\,,$$

$$\text{(S2)}\qquad \phi(0) = 0 \quad\text{and}\quad |\phi(n)| \le C_1/|n|, \quad n \ne 0\,,$$

$$\text{(S3)}\qquad |\phi(n+1) - \phi(n)| \le C_2/n^2, \quad n \ne 0\,.$$

Clearly a singular kernel ϕ is in ℓ^r for all $1 < r \le \infty$. Hence the convolution

$$\phi * a(n) = \sum_{k\in\mathbb{Z}} \phi(n-k)a(k) \equiv T_\phi a(n)$$

is defined for all $a = \{a(n)\}_{n\in\mathbb{Z}}$ in ℓ^p, $1 \le p < \infty$ (in fact also for $a \in \ell^p_B$ where B is any Banach space). The kernel of the discrete Hilbert transform, $\phi(n) = 1/n$, is an example, as is also $\phi(n) = 1/(n\log|n|)$, $n \ne 0, \pm 1$.

It is not difficult to see that if ϕ is a singular kernel, then the truncations ϕ_N, $N \ge 1$, defined as $\phi_N(n) = \phi(n)$ for $|n| \le N$ and 0 otherwise, satisfy

$$\text{(S3)}'\qquad \sup_n \sum_{|k|>2|n|} |\phi_N(k-n) - \phi_N(k)| \le C_2$$

where C_2 does not depend on N. This fact is needed in §2.3. We remark that the $\{\phi_N\}$ need not satisfy (S3) uniformly in N (take $\phi(n) = 1/n$ as an example).

2.2. The following proposition along with the Plancherel theorem shows that the operator T_ϕ , where $T_\phi a(n) = \phi * a(n)$, is bounded on ℓ^2.

PROPOSITION. *If* $\phi = \{\phi(n)\}_{n\in\mathbb{Z}}$ *is a singular kernel*, *then* $\widehat{\phi} \in L^\infty(\mathbb{T})$.

Proof. Observe that

$$\widehat{\phi}(t) = \lim_{j\to\infty} \sum_{k=-N_j}^{N_j} \phi(k)e^{-ikt} = \lim_{j\to\infty} \widehat{\phi}_{N_j}(t) \qquad \text{a.e.}\,,$$

for some subsequence N_j, so that it is enough to prove that $\sup_N \|\widehat{\phi}_N\|_\infty < \infty$. Fix $N \ge 1$ and $t \in \mathbb{T}$. We will choose m, depending on N and t, to estimate $\widehat{\phi}_N(t)$:

$$\begin{aligned}|\widehat{\phi}_N(t)| &= \Big|\sum_{k=-N}^{N}\phi(k)e^{-ikt}\Big| \\ &\le \Big|\sum_{|k|\le m}\phi(k)e^{-ikt}\Big| + \Big|\sum_{m<|k|\le N}\phi(k)e^{-ikt}\Big| = A_1 + A_2\,.\end{aligned}$$

Let $m=\min(N,[\pi/|t|])$, where for a non-negative real number α, $[\alpha]$ denotes the largest integer less than or equal to α. Then

$$\begin{aligned}A_1 &\le \Big|\sum_{|k|\le m}\phi(k)(e^{-ikt}-1)\Big| + \Big|\sum_{|k|\le m}\phi(k)\Big| \\ &\le \sum_{|k|\le m}|\phi(k)||kt| + C \le C_1 2m|t| + C \le C\,,\end{aligned}$$

using (S1), (S2) and the choice of m.

For estimating A_2 we have $m<N$, and so

$$A_2 \le \Big|\sum_{k=m+1}^{N}\phi(k)e^{-ikt}\Big| + \Big|\sum_{k=m+1}^{N}\phi(-k)e^{ikt}\Big| = A_2' + A_2''$$

and

$$\begin{aligned}A_2' &= \Big|\sum_{k=m+1}^{N-1}(\phi(k)-\phi(k+1))\sum_{j=m+1}^{k}e^{-ijt} + \phi(N)\sum_{j=m+1}^{N}e^{-ijt}\Big| \\ &\le \Big|\sum_{k=m+1}^{N-1}(\phi(k)-\phi(k+1))\Big||1/\sin t/2| + |\phi(N)||1/\sin t/2| \\ &\le C|1/\sin t/2|\Big\{\sum_{k=m+1}^{N-1}1/k^2 + 1/N\Big\} \le \frac{C\pi}{|t|m} \le C\end{aligned}$$

since $|1/\sin t/2| \le \pi/|t|$ for $t\in[-\pi,\pi]$ and by the choice of m, we have $m\ge(m+1)/2\ge\pi/(2|t|)$. The estimate for A_2'' is similar. This completes the proof of the proposition.

2.3. With Proposition 2.2 and (S3)$'$, the kernels $\{\phi_N\}$ satisfy the hypothesis of Corollary 2.4.5 of [10], so that the operator T_ϕ ($T_\phi^\wedge$ in the notation of [10]) is bounded on ℓ^p for $1<p<\infty$ and is of weak type $(1,1)$.

In the following theorem we show that in fact the maximal operator defined as

$$T_\phi^* a(n) = \sup_N \Big|\sum_{k=-N}^{N}\phi(k)a(n-k)\Big|$$

is bounded on ℓ^p for $1<p<\infty$ and is of weak type $(1,1)$.

2.4. THEOREM. *Let ϕ be a singular kernel and $1 \le p < \infty$. Then there exists a constant $C_p > 0$ such that*

(i) $$\|T_\phi^* a\|_p \le C_p \|a\|_p, \quad \forall a \in \ell^p, \text{ if } 1 < p < \infty\,,$$

(ii) $$\operatorname{card}\{j \in \mathbb{Z} : T_\phi^* a(j) > \lambda\} \le \frac{C_1}{\lambda}\|a\|_1, \quad \forall \lambda > 0 \text{ and } a \in \ell^1\,.$$

Before proving the theorem we observe that if ϕ is a singular kernel and we let K be the linear extension of ϕ to $\mathbb{R}$, then K is locally integrable and satisfies :

(K1) $$\int_{\varepsilon<|x|<1/\varepsilon} K(x)\,dx \text{ converges as } \varepsilon \to 0\,,$$

(K2) $$|K(x)| \le C/|x|, \quad x \ne 0\,,$$

(K3) $$|K(x) - K(x-y)| \le C'|y|/x^2 \quad \text{for } |x| > 2|y|\,.$$

Then K is a Calderón–Zygmund singular kernel on $\mathbb{R}$ ([14], Ch. XI, §5). The principal value integral

$$T_K f(x) = \text{p.v.} \int K(x-y) f(y)\,dy$$

is defined a.e. for $f \in L^p(\mathbb{R})$, $1 \le p < \infty$, and the maximal operator

$$T_K^* f(x) = \sup_{\varepsilon>0} \Big| \int_{\varepsilon<|x-y|<1/\varepsilon} K(x-y) f(y)\,dy \Big|$$

satisfies

(i) $$\|T_K^* f\|_p \le C_p \|f\|_p \quad \text{for } 1 < p < \infty\,,$$

(ii) $$m\{x \in \mathbb{R} : T_K^* f(x) > \lambda\} \le \frac{C_1}{\lambda}\|f\|_1$$

(where m denotes the Lebesgue measure on $\mathbb{R}$).

For these results we refer to [14] , Theorems 6.2 and 6.3 in Ch. XI.

Proof of Theorem 2.4. Let $a = \{a(k)\}_{k\in\mathbb{Z}} \in \ell^p$ and define

$$f(x) = \sum_{k\in\mathbb{Z}} a(k) \chi_{I_k}(x)$$

where $I_k = [k - 1/4, k + 1/4]$ and χ_I denotes the characteristic function of I. Clearly $f \in L^p(\mathbb{R})$ with $\|f\|_p = \frac{1}{2}\|a\|_p$. Fix $n \ge 0$; then for $j \in \mathbb{Z}$ and $x \in I_j$, we have

$$\sum_{|k-j|>n} a(k)\phi(j-k) - 2 \int_{|x-y|>n+1/2} K(x-y) f(y)\,dy$$
$$= 2 \sum_{|k-j|>n} \int_{y\in I_k} (\phi(j-k) - K(x-y)) f(y)\,dy\,.$$

If $x \in I_j$, $y \in I_k$ and $j \neq k$, we have $|x-y-j+k| \leq \frac{1}{2} \leq \frac{1}{2}|j-k|$ so that by (K3),

$$|\phi(j-k) - K(x-y)| \leq \frac{C}{(x-y)^2}\,.$$

Hence

$$\Big|\sum_{|k-j|>n} a(k)\phi(j-k) - 2\int_{|x-y|>n+1/2} K(x-y)f(y)\,dy\Big|$$
$$\leq C\sum_{|k-j|>n}\int_{y\in I_k}\frac{|f(y)|}{(x-y)^2}\,dy \leq C\int_{|x-y|>1/2}\frac{|f(y)|}{(x-y)^2}\,dy \equiv CSf(x)\,.$$

Therefore, if $x \in I_j$ then

$$\Big|\sum_{|k-j|>n} a(k)\phi(j-k)\Big| \leq C(T_K^*f(x) + Sf(x))\,.$$

In particular, putting $n = 0$,

$$|T_\phi a(j)| \leq C(T_K^*f(x) + Sf(x)) \qquad \text{for } x \in I_j$$

and

$$\begin{aligned} T_\phi^* a(j) &= \sup_n \Big|\sum_{|k-j|\leq n} a(k)\phi(j-k)\Big| \\ &\leq T_\phi^* a(j) + \sup_n \Big|\sum_{|k-j|>n} a(k)\phi(j-k)\Big| \\ &\leq C(T_K^*f(x) + Sf(x)), \qquad x \in I_j\,. \end{aligned}$$

From this the conclusion of Theorem 2.4 follows, since using Hölder's inequality we have $\|Sf\|_p \leq C\|f\|_p$ for $1 \leq p < \infty$.

2.5. If B is a Banach space for which the maximal singular integral operators

$$T_K^*f(x) = \sup_{\varepsilon>0}\Big\|\int_{\varepsilon<|x-y|<1/\varepsilon} K(x-y)f(y)\,dy\Big\|, \qquad f \in L_B^p(\mathbb{R})\,,$$

satisfy inequalities (i) and (ii) of Section 2.4, where now $\|\cdot\|_p$ denotes the norm in $L_B^p(\mathbb{R})$, then the above proof works and Theorem 2.4 holds for the sequence spaces ℓ_B^p. In particular, this is the case if B is a UMD space with an unconditional basis ([3] and [12]). We remark that the geometric properties of the Banach space do not play any role in the above transference.

3. Ergodic operators

3.1. Let (X,μ) be a probability space and U an invertible measure-preserving transformation on X. In this section, we assume that B is a

UMD space with an unconditional basis. Then by 2.5 above, Theorem 2.4 holds for the B-valued sequence spaces ℓ^p_B.

If $\phi = \{\phi(n)\}_{n\in\mathbb{Z}}$ is a singular kernel and $f \in L^p_B(X)$, consider the operator defined, a priori, as

$$\widetilde{T}_\phi f(x) = \sum_{j\in\mathbb{Z}} \phi(j) f(U^{-j}x)\,.$$

We observe that for $f \in L^p_B(X)$, the sequences $\{f(U^{-j}x)\}_{j\in\mathbb{Z}}$ need not be in the sequence spaces ℓ^p_B, so that nothing can be said about the convergence of the series defining $\widetilde{T}_\phi f$. We show below that if B is reflexive then the operator is well defined for f belonging to a dense subspace of $L^p_B(X)$. In particular, this is so for a UMD space, since such a space is always reflexive [5].

3.2. Lemma. *Let B be a reflexive Banach space, and U an invertible measure-preserving transformation on a probability space (X,μ). Let $1 \le p \le 2$ and*

$$\mathcal{D} = \{f : f = g - g\circ U \text{ with } g \in L^\infty_B(X)\} + \{f \in L^p_B(X) : f = f\circ U\}\,.$$

Then $\mathcal{D}$ is dense in $L^p_B(X)$.

Proof. For a reflexive Banach space B with dual B^*, the dual of $L^p_B(X)$ is $L^{p'}_{B^*}(X)$, where $1/p+1/p' = 1$ ([9]). Suppose $h^* \in L^{p'}_{B^*}(X)$ and $\langle f, h^*\rangle = 0$, $\forall f \in \mathcal{D}$.

First, if $f = g - g\circ U$, $g \in L^\infty_B(X)$, we have

$$\begin{aligned} 0 = \langle f, h^*\rangle &= \int_X \langle g(x) - g(Ux), h^*(x)\rangle\, d\mu(x) \\ &= \int_X \langle g(x), h^*(x) - h^*(U^{-1}x)\rangle\, d\mu(x) \\ &= \langle g, h^* - h^*\circ U^{-1}\rangle\,. \end{aligned}$$

Since this holds for all $g \in L^\infty_B(X)$, a dense subset of $L^p_B(X)$, we conclude

$$h^*(x) = h^*(U^{-1}x) \quad \text{a.e.}$$

Now observe that h^* is almost separably valued, i.e. the space $M = (\text{ess. range } h^*)^-$ is separable and reflexive, so that M^* is also separable.

Let $\{\overline{b}_j\}$ be a countable dense set in $M^* \approx B/M^\perp$ and put $f_j(x) = \overline{\langle b_j, h^*(x)\rangle} b_j$. Then $f_j \in L^p_B(X)$ if $1 \le p \le 2$, since $h^* \in L^{p'}_{B^*}(X)$. Also if $h^* = h^*\circ U$, we have $f_j = f_j\circ U$ so that $f_j \in \mathcal{D}$ and $\langle f_j, h^*\rangle = 0$ implies

$$\int_X |\langle b_j, h^*(x)\rangle|^2\, d\mu(x) = 0\,.$$

Hence $\langle b_j, h^*(x)\rangle = 0$ a.e., $\forall\, \overline{b}_j$. But then $h^* = 0$ a.e.

3.3. LEMMA. *Let $\phi = \{\phi(n)\}_{n\in\mathbb{Z}}$ be a singular kernel. Then the series $\sum_{k=-N}^{N} \phi(k) f(U^{-k}x)$ converges a.e. for all $f \in \mathcal{D}$.*

Proof. If $f = f\circ U$, this is obvious by (S1). If $f = g - g\circ U$, $g \in L^\infty_B(X)$, we use a partial summation formula:

$$\begin{aligned}\Big\| \sum_{N\le|k|\le M} & \phi(k) f(U^{-k}x)\Big\| \\ &= \Big\| \sum_{N\le|k|\le M} \phi(k)\{g(U^{-k}x) - g(U^{-k+1}x)\}\Big\| \\ &\le \|g\|_\infty \Big(\sum_{k=N}^{M-1} + \sum_{k=-M}^{-N-1}\Big)|\phi(k)-\phi(k+1)| \\ &\quad + \|g\|_\infty(|\phi(N)| + |\phi(-N)| + |\phi(M)| + |\phi(-M)|) \\ &\le \|g\|_\infty\Big(C\sum_{k=N}^{M} 1/k^2 + \frac{C}{\min(M,N)}\Big) \to 0 \quad \text{as } N, M \to \infty\,.\end{aligned}$$

3.4. In the following theorem we prove the boundedness of the maximal ergodic singular operator by transferring the result of Theorem 2.4. This transference works exactly as it does for the ergodic Hilbert transform ([11]). The details are given below for completeness.

THEOREM. *Let $\phi = \{\phi(n)\}$ be a singular kernel, (X,μ) a probability space and U an invertible measure-preserving transformation on X. Then the maximal ergodic singular operator*

$$\widetilde{T}^*_\phi f(x) = \sup_N \Big|\sum_{k=-N}^{N} f(U^{-k}x)\phi(k)\Big|$$

satisfies

(i) $$\|\widetilde{T}^*_\phi f\|_p \le C_p\|f\|_p \quad \text{if } 1<p<\infty\,,$$

(ii) $$\mu\{x\in X : \widetilde{T}^*_\phi f(x) > \lambda\} \le \frac{C}{\lambda}\|f\|_1, \quad \forall f\in L^1(X) \text{ and } \lambda>0\,.$$

Proof. Fix $N > 0$ and let

$$\widetilde{T}^*_N f(x) = \sup_{1\le n\le N}\Big|\sum_{k=-n}^{n} f(U^{-k}x)\phi(k)\Big|\,.$$

It is enough to prove that $\widetilde{T}^*_N$ satisfies (i) and (ii) with constants not depending on N. Let $\lambda > 0$ and put

$$E_N = \{x\in X : \widetilde{T}^*_N f(x) > \lambda\}\,.$$

Since U is measure-preserving, we have

$$\begin{aligned}\mu(E_N) &= \mu(U^{-m}E_N) \qquad \forall m\\ &= \frac{1}{2M+1}\sum_{m=-M}^{M}\mu(U^{-m}E_N) \qquad \forall M\\ &= \frac{1}{2M+1}\sum_{m=-M}^{M}\mu\Big\{x : \sup_{1\le n\le N}\Big|\sum_{k=-n}^{n} f(U^{-k+m}x)\phi(k)\Big| > \lambda\Big\}\,.\end{aligned}$$

For x lying outside a μ-null set, we can define

$$a_x^M(k) = \begin{cases} f(U^k x) & \text{if } |k| \le M+N,\\ 0 & \text{otherwise.}\end{cases}$$

Then, using Theorem 2.4 and Fubini's theorem, we get

$$\begin{aligned}\mu(E_N) &= \frac{1}{2M+1}\sum_{m=-M}^{M}\mu\Big\{x : \sup_{1\le n\le N}\Big|\sum_{k=-n}^{n} a_x^M(m-k)\phi(k)\Big| > \lambda\Big\}\\ &\le \frac{1}{2M+1}(\mathrm{card}\times\mu)\Big\{(m,x) : \sup_{1\le n\le N}\Big|\sum_{k=-n}^{n} a_x^M(m-k)\phi(k)\Big| > \lambda\Big\}\\ &\le \frac{1}{2M+1}\int\limits_X \mathrm{card}\{m : T_\phi^* a_x^M(m) > \lambda\}\,d\mu(x)\\ &\le \frac{1}{2M+1}\frac{C_p}{\lambda^p}\int\limits_X \sum_{j\in\mathbb{Z}} |a_x^M(j)|^p\,d\mu(x)\\ &\le \frac{1}{2M+1}\frac{C_p}{\lambda^p}\sum_{j=-M-N}^{M+N}\int\limits_X |f(U^j x)|^p\,d\mu(x)\\ &= \frac{C_p}{\lambda^p}\frac{2(M+N)+1}{2M+1}\|f\|_p^p\end{aligned}$$

and so by choosing M large enough,

$$\mu(E_N) \le \frac{C_p}{\lambda^p}\|f\|_p^p, \qquad \forall\lambda > 0\,.$$

Conclusion (i) of the theorem now follows by using the Marcinkiewicz interpolation theorem.

3.5. As remarked earlier, if B is a UMD Banach space with an unconditional basis, then Theorem 2.4 holds for the sequence spaces ℓ_B^p. In that case Theorem 3.4 holds for $L_B^p(X)$ with the same proof upon replacing $|\cdot|$ by the norm in B wherever necessary. For such Banach spaces, $\widetilde{T}_\phi$ is defined on a dense subset of $L_B^p(X)$ by Lemmas 3.2 and 3.3. Then Theorem 3.4

and standard arguments show that $\widetilde{T}_\phi f$ is defined a.e. for all $f \in L^p_B(X)$, $1 \le p < \infty$.

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DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY, KANPUR
KANPUR-208016, INDIA

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