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MOST MONOTHETIC EXTENSIONS ARE RANK-1

BY

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Introduction. Let T be an ergodic automorphism of a standard probability space $(X,\mathcal{B},\mu)$ and G be a compact metrizable abelian group. For any measurable mapping $\phi : X \to G$ (a cocycle) we define an automorphism

$$T_\phi(x,g) = (Tx, g+\phi(x))$$

of $X\times G$, called a G-extension of T. The investigation of ergodic properties of such skew products goes back to Anzai [A] who studied the case of $X = G = \mathbb{T}$, the circle group, with T an irrational rotation.

In [R2], E. A. Robinson, Jr. proved that typically the G-extensions have simple spectrum. More specifically, if T admits a "good cyclic approximation" then most (in the sense of category for the L^1-distance in the space of cocycles) G-extensions have simple spectrum. In Section 2 of the present paper we show that if G is a monothetic group and T admits a cyclic approximation with speed $o(1/n)$, a condition implied by the existence of a "good cyclic approximation", then most G-extensions are in fact rank-1. (Recall that rank-1 implies simple spectrum by Baxter [Ba].) In particular, if $Tz = e^{2\pi i\alpha}z$ is an irrational rotation where α has unbounded partial quotients then most Anzai extensions of T are rank-1. Note that the set of such α's is large in the sense of both measure and category.

It is well known that any discrete spectrum ergodic automorphism is rank-1 (see [J]). To make sure that the discrete spectrum extensions are not generic we prove in Section 3 that in fact a typical G-extension of any ergodic T has no eigenfunctions other than those of T. In other words, a generic cocycle is weakly mixing (Theorem 2). This extends an old result of Jones and Parry [J-P] where the same is proved assuming T to be weakly mixing. In particular, we may now conclude that a typical Anzai cocycle is both weakly mixing and rank-1.

In Section 4 we focus on continuous Anzai cocycles $\phi : \mathbb{T} \to \mathbb{T}$ of topological degree zero. Such cocycles play an important role in the theory of

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Anzai skew products (see e.g. [G-L-L]). Endowed with the uniform metric they form a Polish space so Baire category considerations are still meaningful. We prove that, as in the measurable case, a typical continuous Anzai cocycle of topological degree zero is both rank-1 and weakly mixing.

1. Definitions and basic facts. Let $(X, \mathcal{B}, \mu)$ be a nonatomic standard probability space and let G be a compact metrizable abelian group endowed with the Borel σ-algebra $\mathcal{B}_G$ and normalized Haar measure ν. Denote by d_G an invariant metric on G. For any measurable functions $\phi, \psi : X \to G$ we define the L^1-*distance*

$$d(\phi, \psi) = \int_X d_G(\phi(x), \psi(x))\, d\mu(x)\,.$$

We identify functions that are equal μ-a.e., so it is clear that the set $\Phi = \Phi(X, G)$ of all (equivalence classes of) measurable functions $\phi : X \to G$ forms a Polish group with the complete invariant metric d and pointwise operations. The elements of Φ will be referred to as *cocycles.*

Given any automorphism (i.e. an invertible measure preserving transformation) T of $(X, \mathcal{B}, \mu)$ and a cocycle $\phi \in \Phi$ we define a *group extension* T_ϕ of T by letting

$$T_\phi(x, g) = (Tx, g + \phi(x))\,.$$

The mapping T_ϕ is an automorphism of the product space $(X \times G, \mathcal{B} \times \mathcal{B}_G, \mu \times \nu)$.

It should be noted that for a fixed T the set Φ of all cocycles can be identified with the set of all extensions T_ϕ, $\phi \in \Phi$. By a standard verification the topology determined by the metric d coincides with the weak topology inherited from the group of automorphisms on $X \times G$. In other words, $d(\phi_n, \phi) \to 0$ iff $U_{T_{\phi_n}} \to U_{T_\phi}$ in the weak, or equivalently, strong operator topology on $L^2(X \times G)$ (here $U_S f = f \circ S$ denotes the unitary operator determined by an automorphism S).

Let $n \geq 1$ and $C_0^{(n)}, \ldots, C_{k_n-1}^{(n)}$ be disjoint measurable subsets of X. Define $\zeta_n = \{C_0^{(n)}, \ldots, C_{k_n-1}^{(n)}\}$. We write

$$\zeta_n \to \varepsilon_X$$

if for every $A \in \mathcal{B}$ and every $\delta > 0$ there exists $n_0 \geq 1$ such that for every $n \geq n_0$ we can find a union A_n of some of the sets $C_j^{(n)}$ $(j = 0, \ldots, k_n - 1)$ satisfying $\mu(A \triangle A_n) < \delta$. By a *Rokhlin tower* we mean a family ζ_n as above with $TC_{j-1}^{(n)} = C_j^{(n)}$, $j = 1, \ldots, k_n - 1$. An automorphism T is said to be *rank*-1 if there exists a sequence of Rokhlin towers $\zeta_n \to \varepsilon_X$. It is well known that if T is rank-1 then the unitary operator U_T has simple spectrum [Ba].

To prove that certain group extensions are rank-1 we will apply the method of the Katok–Stepin approximation theory [K-S]. A similar approach has been exploited by E. A. Robinson, Jr. in [R1] to prove certain genericity results concerning spectral multiplicity and continuity of the spectrum of cyclic group extensions and in [R2] to show that typically a group extension has simple spectrum.

We say that T admits a *cyclic approximation with speed* $f(n)$ if there exists a sequence of measurable partitions $\xi_n = \{C_0^{(n)}, \ldots, C_{q_n-1}^{(n)}\}$ satisfying $\xi_n \to \varepsilon_X$ and a sequence of automorphisms T_n satisfying $T_n C_{i-1}^{(n)} = C_i^{(n)}$ $(i = 1, \ldots, q_n - 1)$ and $T_n C_{q_n-1}^{(n)} = C_0^{(n)}$ such that

$$\sum_{i=0}^{q_n-1} \mu(TC_i^{(n)} \triangle T_n C_i^{(n)}) < f(q_n) .$$

It should be remarked that the existence of a "good cyclic approximation" as assumed in [R2] implies a cyclic approximation with speed $o(1/n)$. On the other hand, there exists a "good cyclic approximation" of T whenever T admits a cyclic approximation with speed $o(1/n^2)$ (cf. [K-S], (2.4)).

The following lemma seems to be well known but the authors have not been able to locate a reference.

LEMMA 1. *If T admits a cyclic approximation with speed $o(1/n)$ then T is rank-*1.

Proof. Let ξ_n and T_n be as above with $f(n) = o(1/n)$. We are going to construct a sequence of Rokhlin towers $\zeta_n \to \varepsilon_X$ for T. Clearly ξ_n is a Rokhlin tower for T_n. We let

$$D = \bigcup_{i=0}^{q_n-1} T^{-(i+1)}(TC_i \triangle T_n C_i)$$

and $E_0 = C_0 \setminus D$ (we omit the superscript n). Observe that

$$\mu(D) \le \sum_{i=0}^{q_n-1} \mu(TC_i \triangle T_n C_i) = o(1/q_n) ,$$

so E_0 approximates C_0 within an error that is small relative to $\mu(C_0)$. Now we show that

$$T^j E_0 \subset C_j \quad (j = 0, \ldots, q_n - 1) .$$

Indeed, the inclusion is obvious for $j = 0$. Suppose $0 \le j < q_n - 1$ and $T^j E_0 \subset C_j$. Since $E_0 \cap D = \emptyset$, we have

$$T^{j+1} E_0 \cap T^{j+1} D = \emptyset ,$$

T being an automorphism. Note that

$$T^{j+1} D \supset TC_j \triangle T_n C_j .$$

On the other hand,

$$T^{j+1}E_0 = T(T^j E_0) \subset TC_j\,.$$

Combining the last three formulas we get

$$T^{j+1}E_0 \subset T_n C_j = C_{j+1}$$

by the definition of the symmetric difference. By induction we have shown $T^j E_0 \subset C_j$ for $j = 0, \ldots, q_n - 1$. Consequently,

$$\zeta_n = \{E_0, TE_0, \ldots, T^{q_n-1}E_0\}$$

is a Rokhlin tower for T. In view of $\mu(D) = o(1/q_n)$ we obtain $\mu(E_0) = 1/q_n - o(1/q_n)$ and $\mu(\bigcup_{j=0}^{q_n-1} T^j E_0) \to 1$. Since $\xi_n \to \varepsilon_X$ by assumption, we easily deduce $\zeta_n \to \varepsilon_X$ as required. ■

Let G be a compact metrizable abelian group. The following "cyclicity" property will play an important role in Section 2.

(C) There exists a sequence $\zeta_n = \{G_0, \ldots, G_{r_n-1}\} \to \varepsilon_G$ of measurable partitions of G and a sequence of elements $g_n \in G$ such that $g_n + G_i = G_{i+1}$ $(i = 0, \ldots, r_n - 2)$ and $g_n + G_{r_n-1} = G_0$ for every $n \geq 1$.

Clearly there exist groups that do not satisfy (C): the simplest example is $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. Also it is easy to see that if G is infinite and the order of its elements is uniformly bounded then G cannot satisfy (C). On the other hand, the cyclic groups, the tori, and the counting machines do satisfy (C). The idea of the proof of the following proposition was kindly suggested to the authors by Michael Keane.

PROPOSITION 1. *Let G be a compact metrizable monothetic group. Then G has property (C).*

Proof. The dual group $\widehat{G}$ can be identified with a (discrete) countable subgroup of $\mathbb{T}$, so it is the union of an increasing sequence of finitely generated groups H_k. By the basic structure theorem for finitely generated abelian groups, each H_k is a finite direct product of cyclic groups. Since every finite subgroup of $\mathbb{T}$ is cyclic,

$$H_k = \mathbb{Z}^{d_k} \times \mathbb{Z}_{m_k}$$

where $d_k \geq 0,\ m_k \geq 1$. Let $G^{(k)} = \mathbb{T}^{d_k} \times \mathbb{Z}_{m_k}$. By duality, $\widehat{H}_k = G^{(k)}$ and G can be viewed as the projective limit of the compact groups $G^{(k)}$, the natural topological homomorphism $G^{(k+1)} \to G^{(k)}$ being the dual of the imbedding $H_k \to H_{k+1}$. Since property (C) is easily seen to be preserved by projective limits, it suffices to show that $G^{(k)}$ satisfies (C). To this end, for each $n \geq 1$ choose pairwise relatively prime natural numbers $l_1^{(n)}, \ldots, l_{d_k}^{(n)}$, all relatively prime to m_k, such that $\min\{l_1^{(n)}, \ldots, l_{d_k}^{(n)}\} \to \infty$ as $n \to \infty$.

Now let $G_0 = A_1 \times \ldots \times A_{d_k} \times \{0\}$ where

$$A_j = \{e^{2\pi ix} : 0 \le x < 1/l_j^{(n)}\}\,.$$

It is easy to verify that (C) is satisfied by $G^{(k)}$ with $g_n = (e^{2\pi i l_1^{(n)}}, \ldots, e^{2\pi i l_{d_k}^{(n)}}, 1)$, $r_n = l_1^{(n)} \ldots l_{d_k}^{(n)} m_k$ and $G_j = jg_n + G_0$ in the additive notation. ∎

2. Most cocycles are rank-1. For two positive functions $f(x)$ and $h(x)$ $(x > 0)$ we write

$$f_h(x) = f(xh(x))\,.$$

THEOREM 1. *Let $f(x)$ and $h(x)$ be positive monotone functions converging to* 0 *as $x \to \infty$. Assume G has property (C). If T admits a cyclic approximation with speed $f(n)$ then the set of cocycles ϕ such that T_ϕ admits a cyclic approximation with speed $f_h(n)$ is residual in $\Phi(X, G)$.*

Proof. By assumption, for any $n \ge 1$ there exists a cyclic approximation T_n of T such that

$$\sum_{i=0}^{q_n-1} \mu(T_n A_i \bigtriangleup TA_i) = f_0(q_n) < f(q_n)$$

where $\xi_n = \{A_0, A_1, \ldots, A_{q_n-1}\}$ is a cyclic partition for T_n and $\xi_n \to \varepsilon_X$. Fix monotone positive functions f_1, f_2 converging to zero with $f_0(q_n) < f_1(q_n)$ and $f_1(q_n) + f_2(q_n) < f(q_n)$. Without loss of generality we may also assume that the sequence r_n in (C) satisfies $r_n h(q_n) \le 1$.

Denote by Φ_n the cocycles in Φ that are ξ_n-measurable. For $\phi \in \Phi_n$ we write

$$\phi^{(k)}(x) = \phi(x) + \phi(T_n x) + \ldots + \phi(T_n^{k-1} x)\,.$$

By altering the value of ϕ on a single cell of ξ_n we obtain $\widetilde{\phi} \in \Phi_n$ such that $\widetilde{\phi}^{(q_n)} = g_n$. Denote by $\widetilde{\Phi}_n$ the set of all cocycles thus modified in Φ_n. Since $\xi_n \to \varepsilon_X$, the union $\bigcup_{n \ge N} \widetilde{\Phi}_n$ is dense in Φ for every $N \ge 1$. For any $\widetilde{\phi} \in \widetilde{\Phi}_n$ we let

$$T_{n,\tilde{\phi}}(x, g) = (T_n x, g + \widetilde{\phi}(x))\,.$$

This formula defines a $q_n r_n$-periodic automorphism which cyclically permutes the partition $\eta_n = \{C_0, C_1, \ldots, C_{q_n r_n - 1}\}$ of $X \times G$ into the measurable rectangles $C_0 = A_0 \times G_0$ and $C_k = T_{n,\tilde{\phi}} C_{k-1}$, $k = 1, \ldots, q_n r_n - 1$. Note that $C_k = A_i \times (G_j + a_k)$ for some i, j and $a_k \in G$. Since $\xi_n \to \varepsilon_X$ and $\zeta_n \to \varepsilon_G$, it is easy to see that $\eta_n \to \varepsilon_{X \times G}$.

Now we produce a dense G_δ-subset as in [R2], p. 165. Given $\theta > 0$ consider the open neighbourhood

$$N_\theta(\phi) = \{\psi \in \Phi : d(\phi, \psi) < \theta\}$$

of ϕ in Φ. For any $n \geq 1$ fix $\theta_n > 0$ (to be determined later) and let

$$\Psi = \bigcap_{N \geq 1} \bigcup_{n \geq N} \bigcup_{\phi \in \tilde{\Phi}_n} N_{\theta_n^2}(\phi) .$$

By the Baire theorem Ψ is a dense G_δ-subset of Φ, hence residual. It remains to prove that, with a right choice of the θ_n's, the automorphism T_ψ admits a cyclic approximation with speed $f_h(n)$ for every $\psi \in \Psi$.

Let $\psi \in \Psi$. For infinitely many n's there exists $\widetilde{\phi} \in \widetilde{\Phi}_n$ such that $d(\psi, \widetilde{\phi}) < \theta_n^2$. We are going to estimate the error

$$S = \sum_{i=0}^{q_n r_n - 1} (\mu \times \nu)(T_\psi C_i \bigtriangleup T_{n,\tilde{\phi}} C_i)$$

of the cyclic approximation of T_ψ by $T_{n,\tilde{\phi}}$. In view of $d(\psi, \widetilde{\phi}) < \theta_n^2$, there exists a measurable set $B_n \subset X$ such that $\mu(B_n) < \theta_n$ and $d_G(\psi(x), \widetilde{\phi}(x)) < \theta_n$ off B_n. We compare the action of T_ψ with that of $T_{n,\tilde{\phi}}$ on any $C_k \in \eta_n$. We have

$$\begin{aligned}
&(\mu \times \nu)(T_\psi C_k \bigtriangleup T_{n,\tilde{\phi}} C_k) \\
&\quad \leq \frac{1}{r_n} \mu(TA_i \bigtriangleup T_n A_i) + \int_{A_i} \nu((G_j + a_k + \psi(x)) \bigtriangleup (G_j + a_k + \widetilde{\phi}(x)))\, d\mu(x) \\
&\quad = \frac{1}{r_n} \mu(TA_i \bigtriangleup T_n A_i) + \int_{A_i} \nu((G_j + \psi(x)) \bigtriangleup (G_j + \widetilde{\phi}(x)))\, d\mu(x) .
\end{aligned}$$

Therefore

$$\begin{aligned}
S &\leq \sum_{j=0}^{r_n - 1} \sum_{i=0}^{q_n - 1} \left(\frac{1}{r_n} \mu(TA_i \bigtriangleup T_n A_i) + \int_{A_i} \nu((G_j + \psi(x)) \bigtriangleup (G_j + \widetilde{\phi}(x)))\, d\mu(x) \right) \\
&= \sum_{i=0}^{q_n - 1} \mu(TA_i \bigtriangleup T_n A_i) + r_n \int_X \nu((G_0 + \psi(x)) \bigtriangleup (G_0 + \widetilde{\phi}(x)))\, d\mu(x) \\
&= S_1 + r_n \int_X \nu((G_0 + \psi(x)) \bigtriangleup (G_0 + \widetilde{\phi}(x)))\, d\mu(x) .
\end{aligned}$$

We let

$$S_2 = r_n \int_{B_n} \nu((G_0 + \psi(x)) \bigtriangleup (G_0 + \widetilde{\phi}(x)))\, d\mu(x) ,$$

$$S_3 = r_n \int_{X \setminus B_n} \nu((G_0 + \psi(x)) \bigtriangleup (G_0 + \widetilde{\phi}(x)))\, d\mu(x) ,$$

so that $S \leq S_1 + S_2 + S_3$. By the beginning of the proof we have

$S_1 \le f_0(q_n) < f_1(q_n)$. On the other hand,

$$S_2 \le r_n \mu(B_n) < r_n \theta_n$$

and

$$\begin{aligned} S_3 &= r_n \int\limits_{X \setminus B_n} \nu((G_0 + \psi(x) - \widetilde{\phi}(x)) \bigtriangleup G_0)\, d\mu(x) \\ &\le r_n \sup\{\nu((G_0 + g) \bigtriangleup G_0) : d_G(g, 0) < \theta_n\}\,. \end{aligned}$$

By the continuity of translation in $L^1(G)$, it is possible to find the θ_n small enough so that $S_2 + S_3 < f_2(q_n r_n)$. Therefore, by monotonicity,

$$\begin{aligned} S &< f_1(q_n) + f_2(q_n r_n) < f(q_n) \le f(q_n r_n h(q_n)) \\ &\le f(q_n r_n h(q_n r_n)) = f_h(q_n r_n)\,, \end{aligned}$$

which ends the proof of the theorem. ∎

Under an additional assumption on the sequence $f(n)$, Theorem 1 can be restated in the following more symmetric version.

COROLLARY 1. *Assume G satisfies (C) and let $f(n)$ be monotone and converge to* 0 *with* $\sup f(n)/f(2n) < \infty$. *If T admits a cyclic approximation with speed $o(f(n))$ then a generic G-extension T_ϕ also admits a cyclic approximation with speed $o(f(n))$.*

Proof. By the definition of the symbol o there exists a monotone sequence $1 \le a(n) \to \infty$ such that T admits a cyclic approximation with speed $f(n)/a(n)^2$. Since $f(n)/f(2n) \le M_2 < \infty$, we can easily deduce that $f(n)/f(kn) \le M_k < \infty$ for any $k \ge 1$. Moreover, we may extend $f(n)$ to a monotone function $f(x)$ on $[1, \infty)$ with $\sup f(x)/f(kx) < \infty$ for every $k \ge 1$. Consequently, there exists a monotone function $1 \le k(x) \to \infty$ on $[1, \infty)$ such that

$$f(x)/a(x) \le f(k(x)x)\,,$$

where $a(x)$ is a monotone extension of the sequence $a(n)$. Let $\widetilde{f}(x) = f(x)/a(x)^2$ and choose a monotone function $h(x) \to 0$ such that $xh(x) \to \infty$ as well as $h(x)k(xh(x)) \ge 1$. Since T admits a cyclic approximation with speed $\widetilde{f}(n)$, in view of Theorem 1 a generic extension T_ϕ admits a cyclic approximation with speed $\widetilde{f}_h(n)$. By monotonicity,

$$\widetilde{f}_h(n) = \frac{f(nh(n))}{a(nh(n))^2} \le \frac{f(k(nh(n))nh(n))}{a(nh(n))} \le \frac{f(n)}{a(nh(n))} = o(f(n))\,. \ \blacksquare$$

The irrational numbers with unbounded partial quotients form a set which is residual as well as of Lebesgue measure 1 in the unit interval. Accordingly, the next corollary tells us that most Anzai skew products are rank-1.

COROLLARY 2. *Let $Tz = e^{2\pi i\alpha}z$ be an irrational rotation of $\mathbb{T}$ where α has unbounded partial quotients in its continued fraction expansion. The set of cocycles $\phi \in \Phi(\mathbb{T},\mathbb{T})$ such that T_ϕ is rank-1 is residual.*

Proof. By assumption there exists a sequence of rational numbers $\alpha_n = p_n/q_n$ such that $(p_n, q_n) = 1$ and $|\alpha - \alpha_n| = o(1/q_n^2)$. It is now clear that the rational rotation $T_n z = e^{2\pi i\alpha_n} z$ is a cyclic approximation of T with

$$S_1 = \sum_{j=0}^{q_n-1} \mu(T_n A_j \triangle T A_j) = o(1/q_n)\,,$$

where the sets $A_j = \{e^{2\pi ix} : j/q_n \le x < (j+1)/q_n\}$ form a q_n-cyclic partition. Consequently, T admits a cyclic approximation with speed $o(1/n)$. By Corollary 1, a generic extension also admits a cyclic approximation with speed $o(1/n)$. Now apply Lemma 1. ∎

3. Weakly mixing cocycles. Let T be an ergodic automorphism of $(X, \mathcal{B}, \mu)$ and G be any compact metrizable abelian group. A cocycle $\phi \in \Phi(X,G)$ will be called *weakly mixing* if there exist no $\gamma \in \widehat{G} \setminus \{1\}$, $\lambda \in \gamma(G)$, and $\psi \in \Phi(X, \gamma(G))$ with

$$\gamma(\phi(x)) = \lambda\psi(Tx)/\psi(x) \quad \text{a.e.}$$

It is well known that ϕ is weakly mixing iff there are no eigenfunctions in the orthocomplement of the Hilbert subspace $L^2(X)$ in $L^2(X \times G)$.

In [J-P], Jones and Parry proved among other things that if T is weakly mixing then a generic cocycle ϕ in $\Phi(X,G)$ is also weakly mixing (in which case T_ϕ is weakly mixing itself). Using some ideas of [J-P] we shall prove the same without assuming T to be weakly mixing. In view of Corollary 2 this will imply that a generic Anzai cocycle ϕ is weakly mixing with T_ϕ rank-1.

THEOREM 2. *Let T be ergodic. The set of weakly mixing cocycles $\phi \in \Phi(X,G)$ is residual.*

First we prove a lemma.

LEMMA 2. *Let $a \in \mathbb{T} \setminus \{1\}$. There exists an array of numbers $a_{nj} \in \{1, a, a^2, \ldots\}$ $(n \ge 1,\ j = 0, \ldots, 2n-1)$ such that*

$$\limsup \frac{1}{2n}\Big|\sum_{j=0}^{2n-1} a_{nj}\lambda_n^j\Big| < 1$$

for any sequence $\lambda_n \in \mathbb{T}$.

Proof. Define $a_{n,2j} = a_{n,2j+1} = a^j$ for $j = 0, \ldots, 2n-1$. We have

$$\sum_{j=0}^{2n-1} a_{nj}\lambda_n^j = (1+\lambda_n)\sum_{j=0}^{n-1} a^j\lambda_n^{2j}\,,$$

so the equality $\lim(2n_k)^{-1}|\sum_{j=0}^{2n_k-1} a_{n_kj}\lambda_{n_k}^j| = 1$ for a subsequence n_k would readily imply $\lim|1+\lambda_{n_k}|/2 = 1$ whence $\lambda_{n_k} \to 1$. But then $a\lambda_{n_k}^2 \to a \neq 1$ and

$$\frac{1}{2n_k}\Big|\sum_{j=0}^{2n_k-1} a_{n_kj}\lambda_{n_k}^j\Big| = \frac{|1+\lambda_{n_k}|}{2} \cdot \frac{|1-a^{n_k}\lambda_{n_k}^{2n_k}|}{n_k|1-a\lambda_{n_k}^2|} \to 0\,,$$

a contradiction. Consequently, the assertion follows. ■

Proof of Theorem 2. Let $1 \le r_n \to \infty$ and $0 < \varepsilon_n \to 0$. For fixed n choose a Rokhlin tower $\{C_0, \ldots, C_{2nr_n-1}\}$ for T such that $\mu(X \setminus \bigcup C_j) < \varepsilon_n$. Now define a sequence ϕ_n in $\Phi(X,\mathbb{T})$ by letting $\phi_n(x) = a_{nj}$ if $x \in C_{jr_n} \cup \ldots \ldots \cup C_{(j+1)r_n-1}$ $(j = 0, \ldots, 2n-1)$, and $\phi_n(x) = 1$ if $x \in X \setminus \bigcup C_j$.

Note that the set of $x \in X$ such that $\phi_n(Tx) \neq \phi_n(x)$ is contained in $\bigcup_{j=1}^{2n} C_{jr_n-1} \cup (X \setminus \bigcup C_j)$ so its measure does not exceed

$$2n\frac{1}{2nr_n} + \varepsilon_n = 1/r_n + \varepsilon_n\,.$$

Consequently, $\phi_n \circ T/\phi_n \to 1$ in $\Phi(X,\mathbb{T})$. We denote by Δ the subgroup in $\Phi(X,\mathbb{T})$ consisting of the unimodular eigenfunctions of T ($\Delta = \mathbb{T}$ if T is weakly mixing). For any $h_n \in \Delta$ with $h_n \circ T = \lambda_n h_n$ we have

$$\Big|\int_X \phi_n h_n\,d\mu\Big| \le \varepsilon_n + \Big|\int_{\bigcup C_j} \phi_n h_n\,d\mu\Big|\,.$$

We shall prove that

$$\limsup\Big|\int_X \phi_n h_n\,d\mu\Big| < 1\,.$$

Indeed,

$$\begin{aligned}\int_{\bigcup C_j} \phi_n h_n\,d\mu &= \sum_{j=0}^{2n-1}\sum_{k=0}^{r_n-1}\int_{C_{jr_n+k}} \phi_n h_n\,d\mu = \sum_{j=0}^{2n-1}\sum_{k=0}^{r_n-1} a_{nj}\int_{C_{jr_n+k}} h_n\,d\mu\\ &= \sum_{j=0}^{2n-1} a_{nj}\sum_{k=0}^{r_n-1}\lambda_n^{jr_n+k}\int_{C_0} h_n\,d\mu\\ &= \sum_{j=0}^{2n-1} a_{nj}(\lambda_n^{r_n})^j(1+\lambda_n+\ldots+\lambda_n^{r_n-1})\int_{C_0} h_n\,d\mu\end{aligned}$$

which in view of

$$\Big|\int_{C_0} h_n\,d\mu\Big| \le \mu(C_0) \le 1/(2nr_n)$$

implies

$$\Big|\int\limits_{\cup C_j} \phi_n h_n\, d\mu\Big| \le \frac{1}{2n}\Big|\sum_{j=0}^{2n-1} a_{nj}(\lambda_n^{r_n})^j\Big|$$

so

$$\limsup \Big|\int\limits_X \phi_n h_n\, d\mu\Big| = \limsup \Big|\int\limits_{\cup C_j} \phi_n h_n\, d\mu\Big| < 1$$

by Lemma 2.

The rest of the proof is similar to the argument in [J-P], pp. 141–142. Denote by $\phi \to \phi'$ the canonical quotient homomorphism from $\Phi(X,\mathbb{T})$ onto the quotient group $\Phi(X,\mathbb{T})/\mathbb{T}$ and let $\varrho : \Phi(X,\mathbb{T}) \to \Phi(X,\mathbb{T})$ be given by $\varrho(\phi) = \phi\circ T/\phi$. The composed map $\varrho' : \Phi(X,\mathbb{T}) \to \Phi(X,\mathbb{T})/\mathbb{T}$ is a continuous homomorphism. Since $\ker \varrho' = \Delta$, the mapping ϱ' defines a continuous one-to-one homomorphism $\varrho''([\phi]) = \varrho'(\phi) = (\phi\circ T/\phi)'$ of $\Phi(X,\mathbb{T})/\Delta$ into $\Phi(X,\mathbb{T})/\mathbb{T}$, where $[\phi]$ denotes the coset of ϕ mod Δ.

In the first part of the proof we have shown that $[\phi_n]$ does not converge to 1 in $\Phi(X,\mathbb{T})/\Delta$, yet $\phi_n\circ T/\phi_n \to 1$ so $\varrho(\phi_n) \to 1$ in $\Phi(X,\mathbb{T})$ and consequently $\varrho''([\phi_n]) \to 1$ in $\Phi(X,\mathbb{T})/\mathbb{T}$. This implies that ϱ'' is not an open map. By the open mapping theorem for topological groups (see [P], Thm. 7) the set $\varrho''(\Phi(X,\mathbb{T})/\Delta) = \varrho'(\Phi(X,\mathbb{T}))$ is of the first category in $\Phi(X,\mathbb{T})/\mathbb{T}$. Since the quotient homomorphism is open, this implies that the inverse image

$$\{\lambda\phi\circ T/\phi : \lambda\in\mathbb{T},\ \phi\in\Phi(X,\mathbb{T})\}$$

of $\varrho'(\Phi(X,\mathbb{T})) \subset \Phi(X,\mathbb{T})/\mathbb{T}$ is of the first category in $\Phi(X,\mathbb{T})$.

Clearly the above argument is also valid for any finite subgroup $F \ne \{1\}$ of $\mathbb{T}$ in place of $\mathbb{T}$ (choose $a \in F\setminus\{1\}$ in Lemma 2). In particular, we may apply it to $\Phi(X,\gamma(G))$, $\gamma\in\widehat{G}\setminus\{1\}$, so for a fixed γ the set

$$\Phi_\gamma = \{\psi\in\Phi(X,G) : (\exists\lambda\in\gamma(G))(\exists\phi\in\Phi(X,\gamma(G)))\ \gamma\circ\psi = \lambda\phi\circ T/\phi\}$$

is of the first category. This follows from the fact that

$$\Phi_\gamma = \Pi_\gamma^{-1}\{\lambda\phi\circ T/\phi : \lambda\in\gamma(G),\ \phi\in\Phi(X,\gamma(G))\},$$

where $\Pi_\gamma : \Phi(X,G) \to \Phi(X,\gamma(G))$ is the continuous open homomorphism given by $\Pi_\gamma\psi = \gamma\circ\psi$. Since $\widehat{G}$ is countable, we obtain the desired result. ∎

4. Continuous Anzai cocycles of topological degree zero. In the present section we consider an irrational rotation $Tz = e^{2\pi i\alpha}z$ of the circle group $\mathbb{T}$. We let $G = \mathbb{T}$ (with multiplicative notation) and denote by Φ_0 the set of all continuous Anzai cocycles $\phi : \mathbb{T}\to\mathbb{T}$ that have topological degree zero. In other words, $\phi\in\Phi_0$ iff there exists a continuous 1-periodic function $f:\mathbb{R}\to\mathbb{R}$ such that

$$\phi(e^{2\pi ix}) = e^{2\pi i f(x)}, \qquad x\in\mathbb{R}.$$

Certain ergodic properties of the group extensions T_ϕ with $\phi \in \Phi_0$ have been studied in [G-L-L]. Below we prove two genericity results which can be viewed as a counterpart of Theorems 1 and 2 above.

We endow Φ_0 with the *uniform metric* d_0, i.e.

$$d_0(\phi,\psi) = \sup\{|\phi(z) - \psi(z)| : z \in \mathbb{T}\}\,.$$

It is clear that d_0 is a complete metric so that Φ_0 becomes a Polish space and the category considerations are meaningful as in the case of $\Phi(\mathbb{T},\mathbb{T})$. The idea of the proof of our next theorem is similar to that of Theorem 1.

THEOREM 3. *Let α be an irrational number with unbounded partial quotients and let $Tz = e^{2\pi i\alpha}z$. The set of cocycles $\phi \in \Phi_0$ such that the extension T_ϕ is rank-*1 *is residual in Φ_0 endowed with the uniform metric.*

Proof. By assumption we have $|\alpha - \alpha_n| = \delta_n = o(1/q_n^2)$ for some $\alpha_n = p_n/q_n$, $(p_n, q_n) = 1$, $q_n \to \infty$. Fix a sequence of integers $r_n \to \infty$ such that

$$\delta_n = o\bigg(\frac{1}{r_n q_n^2}\bigg)$$

and let $0 < \eta_n < 1/q_n$, $\eta_n = o(1/(r_n q_n^2))$. For every $n \ge 1$ we denote by $\Phi_{0,n}$ the cocycles $\phi \in \Phi_0$ that are constant on the arcs $A_j = \{e^{2\pi i x} : j/q_n \le x < (j+1)/q_n - \eta_n\}$, $j = 0,\ldots,q_n-1$, taking value $z_j = e^{2\pi i y_j}$ (depending on ϕ) on A_j and

$$\phi(e^{2\pi i x}) = e^{2\pi i (y_{j+1} + (y_{j+1}-y_j)(x-(j+1)/q_n)/\eta_n)}$$

if $(j+1)/q_n - \eta_n \le x < (j+1)/q_n$ ($j = 0,\ldots,q_n-1$), where we let $y_{q_n} = y_0$. In other words,

$$\phi(e^{2\pi i x}) = e^{2\pi i f(x)}\,,$$

where $f(x)$ is constant with values y_j on the intervals $[j/q_n, (j+1)/q_n - \eta_n)$ and linear between them.

It is clear that the set $\bigcup_{n\ge N} \Phi_{0,n}$ is dense in Φ_0 for every $N \ge 1$. For any $\phi \in \Phi_{0,n}$ and $z = e^{2\pi i x}$ we have

$$\begin{aligned}\phi^{(q_n)}(z) &= \phi(z)\phi(e^{2\pi i\alpha_n}z)\ldots\phi(e^{2\pi i(q_n-1)\alpha_n}z)\\ &= e^{2\pi i(f(x)+f(x+\alpha_n)+\ldots+f(x+(q_n-1)\alpha_n))}\,.\end{aligned}$$

Since $(p_n, q_n) = 1$ and $q_n\alpha_n = p_n$, the sum in the last exponent is a periodic function with period $1/q_n$. On the other hand, each term is constant on $[0, 1/q_n - \eta_n)$ and linear on $[1/q_n - \eta_n, 1/q_n)$ so the same must be true for the sum. As the functions are continuous, the sum must be constant with value $y_0 + \ldots + y_{q_n-1}$. In other words,

$$\phi^{(q_n)}(z) = z_0 \ldots z_{q_n-1}\,.$$

Consequently, by multiplying $\phi(z)$ by an appropriate constant $e^{2\pi i x_0}$, $0 \le$

$x_0 < 1/q_n$, we will obtain

$$\phi^* = e^{2\pi i x_0}\phi \in \Phi_{0,n}$$

with

$$\phi^{*(q_n)}(z) = z_0 \ldots z_{q_n-1} \cdot e^{2\pi i q_n x_0} = e^{2\pi i/r_n}\,.$$

Denote by $\Phi^*_{0,n}$ the set of all $\phi \in \Phi_{0,n}$ with $\phi^{(q_n)}(z) = e^{2\pi i/r_n}$. Since $0 \le x_0 < 1/q_n$, it is clear that the set $\bigcup_{n\ge N}\Phi^*_{0,n}$ is still dense in Φ_0. Now around each $\phi \in \Phi^*_{0,n}$ take the open ball of radius $\varrho_n = o(1/(r_n^2 q_n))$ in (Φ_0, d_0). Denote by U_N the union of all such balls for all $n \ge N$. The intersection

$$\Psi_0 = \bigcap_{N\ge 1} U_N$$

is a dense G_δ-set in Φ_0, hence residual.

To end the proof it suffices to show that for any $\psi \in \Psi_0$ the extension T_ψ admits a cyclic approximation with speed $o(1/n)$. To this end we choose $\phi^*_n \in \Phi^*_{0,n}$ with $d_0(\psi, \phi^*_n) < \varrho_n$ (this can be done for infinitely many n's) and denote by $\widetilde{\phi}_n$ the step cocycle such that

$$\widetilde{\phi}_n(e^{2\pi i x}) = \phi^*_n(e^{2\pi i j/q_n})$$

on $[j/q_n, (j+1)/q_n)$, $j = 0, \ldots, q_n - 1$. Now we have $\widetilde{\phi}^{(q_n)}_n = \phi^{*(q_n)}_n = e^{2\pi i/r_n}$ so that $\widetilde{T}_n(z,w) = (e^{2\pi i\alpha_n}z, w\widetilde{\phi}_n(z))$ is a cyclic automorphism of period $q_n r_n$ which permutes cyclically the rectangles C^j_n $(j = 0, \ldots, q_n r_n - 1)$, where

$$C^0_n = \{(e^{2\pi i x}, e^{2\pi i y}) : 0 \le x < 1/q_n,\ 0 \le y < 1/r_n\}$$

and $C^j_n = \widetilde{T}_n C^{j-1}_n$ $(j = 1, \ldots, q_n r_n - 1)$. To evaluate the error of the approximation of T_ψ by $\widetilde{T}_n$ we compare the sets $T_\psi C^j_n$, $T^*_n C^j_n$, and $\widetilde{T}_n C^j_n$, where $T^*_n(z,w) = (e^{2\pi i\alpha_n}z, w\phi^*_n(z))$. Note that

$$\sum_{j=0}^{q_n r_n - 1} (\mu\times\mu)(T^*_n C^j_n \triangle \widetilde{T}_n C^j_n) \le q_n\eta_n = o\left(\frac{1}{r_n q_n}\right)$$

as here the only errors occur outside the intervals $[j/q_n, (j+1)/q_n - \eta_n)$. Also,

$$\sum_{j=0}^{q_n r_n - 1} (\mu\times\mu)(T^*_n C^j_n \triangle T_\psi C^j_n) \le q_n\eta_n + 2\delta_n q_n + 2\varrho_n r_n = o\left(\frac{1}{r_n q_n}\right)$$

where the three parts of the error are caused by $\phi^*_n \ne$ const on $[j/q_n, (j+1)/q_n)$, $\alpha \ne \alpha_n$, and $d_0(\psi, \phi^*_n) \ne 0$, respectively. Since clearly $\xi_n = \{C^0_n, \ldots, C^{q_n r_n - 1}_n\} \to \varepsilon_{\mathbb{T}\times\mathbb{T}}$, by the last two inequalities we obtain

$$\sum_{j=0}^{q_n r_n - 1} (\mu\times\mu)(\widetilde{T}_n C^j_n \triangle T_\psi C^j_n) = o\left(\frac{1}{r_n q_n}\right),$$

which ends the proof of the theorem. ■

Our final theorem shows that the rank-1 extensions obtained in Theorem 3 are generically not of discrete spectrum.

THEOREM 4. *Let α be any irrational number and $Tz = e^{2\pi i\alpha}z$. The set of weakly mixing cocycles in Φ_0 is a dense G_δ-subset of Φ_0 endowed with the uniform metric.*

Proof. We apply a theorem of D. Rudolph ([Ru], Thm. 12). Denote by $\mathcal{A}$ the uniformly closed algebra of all continuous functions $f : [0,1] \to \mathbb{R}$ satisfying $f(0) = f(1)$. It is clear that if $g \in L^1[0,1]$ and $|g| < M < \infty$ then there exists a sequence f_n in $\mathcal{A}$ such that $|f_n| < M$ and $\|f_n - g\|_1 \to 0$. Therefore, by Rudolph's theorem, for every $g \in L^1[0,1]$ there exist $f \in \mathcal{A}$ and a measurable function $h : [0,1] \to \mathbb{R}$ such that

$$g(x) - f(x) = h(Tx) - h(x) \quad \text{a.e.}$$

(here we identify T with the mapping $Tx = x + \alpha \pmod 1$ of the unit interval).

Now if $\psi \in \Phi(\mathbb{T},\mathbb{T})$ is any measurable cocycle then clearly

$$\psi(e^{2\pi ix}) = e^{2\pi ig(x)}$$

for some $g \in L^1[0,1]$. By the above there exist $\phi_0(e^{2\pi ix}) = e^{2\pi if(x)}$ in Φ_0 and $\phi(e^{2\pi ix}) = e^{2\pi ih(x)}$ in $\Phi(\mathbb{T},\mathbb{T})$ such that $\psi = \phi_0(\phi \circ T/\phi)$. Since ψ and ϕ_0 are cohomologous, the extensions they generate are isomorphic. In particular, there exists a cocycle $\phi_0 \in \Phi_0$ which is cohomologous to the cocycle $\psi(z) = z$, hence weakly mixing (see [A]).

In the rest of the proof we use some ideas of Baggett [B]. First note that if $\psi(e^{2\pi ix}) = e^{2\pi ip(x)}$ where $p(x)$ is a real-valued trigonometric polynomial then there exists a cocycle $\phi \in \Phi_0$ such that $\psi = \phi \circ T/\phi$ (see [B], Thm. 1). Now it follows by the Weierstrass theorem that the cocycles of the form $\psi = \phi \circ T/\phi$ are dense in Φ_0. Since Φ_0 is a topological group, the weakly mixing cocycles of the form $\phi_0(\phi \circ T/\phi)$ are dense, too. To end the proof it suffices to show that the cocycles ϕ that are not weakly mixing, i.e., those $\phi \in \Phi_0$ such that $\phi^m = c\psi \circ T/\psi$ for some $m \neq 0$, $c \in \mathbb{T}$, and $\psi \in \Phi(\mathbb{T},\mathbb{T})$, form an F_σ-subset of Φ_0. Define

$$\Phi^m_{j,k} = \{\phi \in \Phi_0 : (\exists c \in \mathbb{T})(\exists \psi \in \Phi(\mathbb{T},\mathbb{T}))\ \phi^m = c\psi \circ T/\psi,\ |\widehat{\psi}(j)| \geq 1/k\}$$

where $\widehat{\psi}(j)$ denotes the jth Fourier coefficient of the function $\psi : \mathbb{T} \to \mathbb{C}$. We prove that $\Phi^m_{j,k}$ is closed in Φ_0. Assume $\phi_n \to \phi$ in Φ_0 with $\phi_n \in \Phi^m_{j,k}$, $\phi^m_n = c_n\psi_n \circ T/\psi_n$. By the weak compactness of the unit ball in $L^2(\mathbb{T})$ there exist a cocycle $\psi \in \Phi(\mathbb{T},\mathbb{T})$ and a subsequence n' (we write n for simplicity) such that $\psi_n \to \psi$ weakly in $L^2(\mathbb{T})$. This clearly implies that $\phi^m_n\psi_n \to \phi^m\psi$ weakly. By choosing a further subsequence if necessary we may assume $c_n \to c$ in $\mathbb{T}$ so that $\phi^m_n\psi_n = c_n\psi_n \circ T \to c\psi \circ T$ weakly, whence $\phi^m\psi = c\psi \circ T$. Besides, $|\widehat{\psi}(j)| \geq 1/k > 0$, so $\psi \neq 0$. Since $|\psi| = |\phi^m\psi| = |c\psi \circ T| = |\psi \circ T|$, the function $|\psi|$ is constant by ergodicity. By multiplying ψ by the constant $1/|\psi| \geq 1$ we may assume without loss of generality that $\psi \in \Phi(\mathbb{T},\mathbb{T})$ so $\phi \in \Phi^m_{j,k}$ in view of the equality $\phi^m = c\psi \circ T/\psi$. It is now clear that the set $\bigcup_{j\geq 1} \bigcup_{k\geq 1} \bigcup_{m\neq 0} \Phi^m_{j,k}$ of all cocycles $\phi \in \Phi_0$ which are not weakly mixing is an F_σ-subset of Φ_0. ■

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Added in proof. By a recent result of the first author (*Cyclic approximation of irrational rotations*, Proc. Amer. Math. Soc., to appear), Corollary 2 is valid for all irrational numbers α.