C O L L O Q U I U M   M A T H E M A T I C U M

VOL. LXVI 1993 FASC. 1

SOME REMARKS ON SECOND CATEGORY SETS

BY

PÉTER K O M J Á T H (BUDAPEST)

0. Introduction. In this paper we give some loosely connected consistency results concerning second category sets. In [4], the author proved, extending some earlier results of Mycielski, that MA_κ implies (see [2]) that there exists a measure zero, first category set X such that if Y is a set of reals of size $\le \kappa$ then $Y \subseteq X + c$ for some real c. K. Muthuvel [7] gave some applications of this construction and asked if it follows already from ZFC that such an X exists for every measure zero, first category set Y of size $< 2^\omega$. We show that this is not the case. Namely, it is consistent that $2^\omega = \omega_2$ and for no first category X is it true that for every first category, measure zero set Y of size ω_1 we have $Y \subseteq X + c$ for some c.

In [1], U. Abraham proved that it is consistent that $2^\omega = \omega_2$ and there is a set mapping $f: \mathbb{R} \to P(\mathbb{R})$ such that $f(x)$ is nowhere dense for $x \in \mathbb{R}$, and there is no uncountable free set for f. Also, MA_{ω_1} is consistent with the statement that every set mapping as above has an uncountable free set. For set mappings where the images are first category sets, the situation is not as easy. If $2^\omega = \omega_2$, an easy well-ordering argument gives that there is a set mapping $f: \mathbb{R} \to [\mathbb{R}]^{\omega_1}$ with no free sets of size 2. If MA_{ω_1} holds, the images are of first category. Here we prove that it is consistent that MA_{ω_1} holds, and there is a set X of size $2^\omega = \omega_3$ such that if $Y \subseteq X$ is of first category, then $|Y| \le \omega_1$. This implies, by a result of Ruziewicz (see [1]), that if f is a set mapping on $\mathbb{R}$ with first category images, then there is a (second category) free set of size $\aleph_3$.

The last problem we address is the following. Is it true that there is an almost disjoint family of second category sets which is of cardinality $> 2^\omega$? Here, almost disjoint means that the intersection of any two members in the family is of first category. Sierpiński [8] proved that if CH holds, then therei are $\aleph_2$ second category sets with pairwise *countable* intersection. It is easy to show that there exist 2^ω disjoint second category sets. We show that it is consistent that $2^\omega = \omega_2$ and there are no $\aleph_3$ almost disjoint second category sets.

1991 *Mathematics Subject Classification*: 03E35, 28A05.

Research partially supported by the Hungarian National Science Fund No. 1908.

Acknowledgments. The author is grateful for the referees' remarks.

1. Notation, preliminaries. We use the standard notions and notation of axiomatic set theory (see [3, 5]). Cardinals are identified with initial ordinals. If X is a set and κ is a cardinal, then

$$[X]^\kappa = \{Y \subseteq X : |Y| = \kappa\}, \quad [X]^{<\kappa} = \{Y \subseteq X : |Y| < \kappa\}.$$

If X is a set, then $P(X)$ is its power set. A *set mapping* is a function $f : X \to P(X)$. A subset $Y \subseteq X$ is *free* for f if for $x, y \in Y$, $x \neq y$, $y \notin f(x)$ holds.

For the definitions of forcing and Cohen reals, see [3, 5, 6]. If $X \subseteq \mathbb{R}$ is a set, the canonical notion of forcing making X of first category is defined as follows. $p \in P$ iff $p = (s, f, F)$, where $s \in [X]^{<\omega}$, $f : s \to \omega$, F is a function with $\mathrm{Dom}(F) \in [\omega]^{<\omega}$, for $n \in \mathrm{Dom}(F)$, $F(n)$ is a finite set of rational open intervals, and if $y \in s$, $n = f(y)$ then $y \notin \bigcup F(n)$. $(s', f', F') \le (s, f, F)$ if $s' \supseteq s$, $f' \supseteq f$, $F'(n) \supseteq F(n)$ for $n \in \mathrm{Dom}(F)$ (see [2]).

Lemma 1. *If $p(\xi) \in P$ $(\xi < \omega_1)$, then there is a set of indices $Z \in [\omega_1]^{\omega_1}$ such that if $\xi_1, \ldots, \xi_n \in Z$, then there is a $q \le p(\xi_1), \ldots, p(\xi_n)$.*

Proof. By the Δ-system lemma (p. 225 in [3]), and the pigeon-hole principle, we may find $Z \in [\omega_1]^{\omega_1}$ such that $p(\xi) = (s \cup s_\xi, f_\xi, F)$ where $\{s, s_\xi : \xi \in Z\}$ are disjoint and $f_\xi|s = f$ for $\xi \in Z$. But then, if $\xi_1, \ldots, \xi_n \in Z$, then the coordinatewise union of $p(\xi_1), \ldots, p(\xi_n)$ is a condition extending them.

2. Translations of first category sets

Theorem 1. (CH) *If $\kappa > \omega_1$ is a regular cardinal then in some forcing extension, $2^\omega = \kappa$ holds, and no first category set X has the property that for every first category, measure zero set Y of size ω_1 there exists a real d such that $Y + d \subseteq X$.*

Proof. Let V be a model of CH with $\kappa = \mathrm{cf}(\kappa) > \omega_1$ in V. We are going to define a finite support forcing iteration of length κ, $(P_\alpha, Q_\alpha : \alpha \le \kappa)$. If $\alpha < \kappa$, let $Q_{2\alpha}$ add ω_1 Cohen reals, say $Y_\alpha = \{r^\alpha_\xi : \xi < \omega_1\}$, and then let $Q_{2\alpha+1}$ be the canonical poset making Y_α of first category. For simplicity, we assume that the underlying set of $Q_{2\alpha+1}$ is ω_1.

We call a condition $p \in P_\alpha$ $(\alpha < \kappa)$ *full* if for every $\beta < \alpha$, $p|\beta$ determines what $p(\beta)$ is.

Lemma 2. *The full conditions are dense in P_α.*

Proof. Straightforward, by induction on $\alpha \le \kappa$.

Lemma 3. *If $\alpha \le \kappa$, $p_\xi \in P_\alpha$ $(\xi < \omega_1)$, then for some $Z \in [\omega_1]^{\omega_1}$, if $\xi_1, \ldots, \xi_n \in Z$, there is a common extension of $p_{\xi_1}, \ldots, p_{\xi_n}$.*

Proof. We may assume that the conditions are full. By the Δ-system lemma, we can assume that the supports of the conditions form a Δ-system, $\mathrm{supp}(p_\xi) = s \cup s_\xi$, where $\{s, s_\xi : \xi < \omega_1\}$ are disjoint. We can then use Lemma 1 and the counterpart of it for the Cohen forcing to thin out the system to p_ξ ($\xi \in Z$), such that if $\xi_1, \ldots, \xi_n \in Z$ then the coordinatewise union of $p_{\xi_1}, \ldots, p_{\xi_n}$ is a condition, therefore extending them.

Towards proving the theorem, assume that X has the property described in the theorem. We can assume that $X = \bigcup\{X_i : i < \omega\}$ where X_i is a compact nowhere dense set. As every closed set can be identified with a countable collection of dyadic intervals, we can code X by a real, which, by ccc, appears in $V^{P_{2\alpha}}$, for some $\alpha < \kappa$. The set Y_α added by $Q_{2\alpha}$ is a measure zero Lusin set in $V^{P_{2\alpha+1}}$, i.e., every uncountable subset of it is somewhere dense (see [6]).

We work in $V^{P_{2\alpha+1}}$. Assume that over $V^{P_{2\alpha+1}}$, $1 \Vdash Y_\alpha + \underline{d} \subseteq X$ holds for some name of a real $\underline{d}$. Select, for $\xi < \omega_1$, a condition q_ξ such that $q_\xi \Vdash r^\alpha_\xi + \underline{d} \in X_i$ for some $i < \omega$. By Lemma 3 and the pigeonhole principle we can find a $Z \in [\omega_1]^{\omega_1}$ such that for $\xi \in Z$ the i as above is the same and finite subsets of $\{q_\xi : \xi \in Z\}$ have common lower bounds.

If $\xi_1, \ldots, \xi_n \in Z$, $q \le q_{\xi_1}, \ldots, q_{\xi_n}$, $q \Vdash \underline{d} \in (X_i - r^\alpha_{\xi_1}) \cap \ldots \cap (X_i - r^\alpha_{\xi_n})$, i.e., q forces that the compact sets $X_i - r^\alpha_{\xi_1}, \ldots, X_i - r^\alpha_{\xi_n}$, when redefined in V^{P_κ}, have a common point. This property is absolute (see [5, 6]), so we argue that $(X_i - r^\alpha_{\xi_1}) \cap \ldots \cap (X_i - r^\alpha_{\xi_n})$ is nonempty in $V^{P_{2\alpha+1}}$. By compactness, the intersection $\bigcap\{X_i - r^\alpha_\xi : \xi \in Z\}$ is nonempty, and if c is in the intersection, then $\{c + r^\alpha_\xi : \xi \in Z\} \subseteq X_i$ so the nowhere dense $X_i - c$ has an uncountable intersection with Y_α, which is impossible, as the latter is a Lusin set.

3. A Lusin-like set

THEOREM 2. (GCH) *If $\kappa = \mathrm{cf}(\kappa) > \omega_1$, then it is consistent that MA_{ω_1} holds, and there is a set $X \subseteq \mathbb{R}$ of size $2^\omega = \kappa$ such that every first category subset of X is of size $\le \omega_1$.*

Proof. We give a finite support iteration. Let P_0 add κ Cohen reals, $X = \{r_\xi : \xi < \kappa\}$, and for $\alpha < \kappa$ let $Q_\alpha = (\omega_1, <_\alpha)$ be some ccc poset on ω_1. By an appropriate bookkeeping we can achieve MA_{ω_1} in V^{P_κ}.

We call a condition $p \in P_\alpha$ *full* if for every $1 \le \beta < \alpha$, $p|\beta$ determines $p(\beta) < \omega_1$ (which is the maximal element of $(\omega_1, <_\beta)$ for all but finitely many $\beta < \alpha$).

LEMMA 4. *The set of full conditions is dense in P_α.*

P r o o f. By induction on α. The limit case is trivial. For $\alpha = \beta + 1$, first extend $p|\beta$ to a p' determining $p(\beta)$, then extend p' to a full $p'' \in P_\beta$; then $p''{}^\wedge p(\beta)$ is full.

For $p \in P_0$, let $\mathrm{Dom}(p)$ be the set of those coordinates $< \kappa$ where p is not the trivial condition. For $X \subseteq \kappa$, we say that a $p \in P_0$ is in $P_0|X$ if $\mathrm{Dom}(p) \subseteq X$. For $p \in P_\alpha$, $p|X$ is the condition where one removes from the first coordinate the part outside X.

LEMMA 5. *If τ is a P_α-name for a subset of ω_1, then there is a set $X \in [\kappa]^{\omega_1}$ such that if p is a full condition and $p \Vdash i \in \tau$ then also $p|X \Vdash i \in \tau$.*

P r o o f. By induction on α. We first notice that it suffices to prove the result for $\tau \subseteq 2$.

If $\alpha = 0$ and no countable X satisfies the lemma, one can get by transfinite induction an increasing sequence of countable sets $X(\xi) \subseteq \kappa$ and conditions $p_\xi \in P_0$, $p_\xi \Vdash i \in \tau$ $(\xi < \omega_1)$ such that $p_\xi|X(\xi)$ does not force $i \in \tau$, with $p_\zeta \in P_0|X(\xi)$ $(\zeta < \xi)$. We can assume that $\{\mathrm{Dom}(p_\xi) : \xi < \omega_1\}$ forms a Δ-system with kernel contained in $X(\xi_0)$, and the p_ξ's are the same on the kernel. But then we can find a $q \le p_{\xi_0}|X(\xi_0)$, $q \Vdash i \notin \tau$, and therefore we can find a p_{ξ_1} such that $\mathrm{Dom}(p_{\xi_1}) \cap \mathrm{Dom}(q) = \mathrm{Dom}(p_{\xi_1}) \cap \mathrm{Dom}(p_{\xi_0})$. The conditions p_{ξ_1} and q are compatible, which is a contradiction as they force contradictory statements.

If we know the result for α and τ is a $P_{\alpha+1}$-name, let θ be the following P_α-name: $p \Vdash \langle \gamma, i\rangle \in \theta$ iff $p^\wedge\gamma \Vdash i \in \tau$. Now apply the inductive hypothesis for P_α and θ, and get an appropriate set $X \in [\kappa]^{\omega_1}$.

If $\mathrm{cf}(\alpha) \le \omega_1$, we can apply the inductive hypothesis, as if $\alpha = \sup\{\alpha_\xi : \xi < \omega_1\}$, $\tau = \bigcup\{\tau_\xi : \xi < \omega_1\}$, where $p \Vdash i \in \tau_\xi$ iff $p \Vdash i \in \tau$ and $p \in P_{\alpha_\xi}$, so τ_ξ is a P_{α_ξ}-name.

Assume finally that $\mathrm{cf}(\alpha) \ge \omega_2$. If τ is a P_β-name for some $\beta < \alpha$, we are done. If not, we can choose ω_2 conditions p_ξ $(\xi < \omega_2)$ such that $p_\xi \Vdash i \in \tau$ but $p_\xi|\alpha_\xi \not\Vdash i \in \tau$, where $\alpha_\xi < \alpha$ is increasing and $\alpha_\xi > \mathrm{supp}(p_\zeta)$ $(\zeta < \xi)$. Without loss of generality, $\mathrm{supp}(p_\xi) = s \cup s_\xi$, and we can assume that the $p_\xi|s$ are identical. But then, if $\xi \ge \omega$, $p_\xi|\alpha_\xi$ has an extension forcing $i \notin \tau$, which in turn is compatible with some p_ζ $(\zeta < \omega)$, a contradiction.

To prove the theorem, assume that τ is a P_κ-name for a first category F_σ set Y (which again can be coded as a real). Let $X \in [\kappa]^{\omega_1}$ be a set guaranteed by Lemma 5. If $\xi \notin X$, we claim that $r_\xi \notin Y$. If $Y = \bigcup\{Y_i : i < \omega\}$, where each Y_i is closed, nowhere dense and p is arbitrary, and if p gives the information that $r_\xi \in I$ for some dyadic interval I, let $q \in P_0|X$, $q \le p|X$ determine an interval $J \subseteq I$ such that $J \cap Y_i = \emptyset$, and then extend $q \cup p$ to give $r_\xi \in J$. This shows that $p \Vdash r_\xi \notin Y_i$, and we are done.

4. Almost disjoint sets

THEOREM 3. *It is consistent that $2^\omega = \omega_2$ and there does not exist a family of more than 2^ω second category sets such that the intersection of any two of them is of first category.*

Proof. It suffices to construct a model in which $2^\omega = 2^{\omega_1} = \omega_2$ and every second category set contains a second category subset of size $\aleph_1$. Let V be a model of GCH, and add ω_2 Cohen reals, $\{r_\alpha : \alpha < \omega_2\}$. Assume that $1 \Vdash \underline{X} \subseteq \mathbb{R}$ is of second category. Let M be an elementary submodel of $(H(2^{\omega_2}); \in, \Vdash, \underline{X})$ of size ω_1 such that $[M]^\omega \subseteq M$. Put $\delta = M \cap \omega_2$. We show that the intersection of X with $V' = V[r_\alpha : \alpha < \delta]$ is of second category in V' (note that $X \cap V' \in V'$). If $E = \bigcup\{F_i : i < \omega\}$, where F_i is a nowhere dense closed set, and E is (coded) in V', then by the closure property of M, there is a name $\tau \in M$ of a real such that $1 \Vdash \tau \in X - E$. Then τ gives rise to an element $y \in V'$ such that $y \in X - E$. The model $V[r_\alpha : \alpha < \omega_2]$ is obtained by a side-by-side Cohen extension from V'. To conclude the proof, we need to show the following statement.

LEMMA 6. *If in V, X is a second category set and P adds some Cohen reals, then $V^P \models X$ is of second category.*

Proof. Assume that $1 \Vdash$ "$X \subseteq \bigcup\{F_i : i < \omega\}$, where F_i is closed nowhere dense". Let N be a countable elementary submodel of the model $(H((2^{|P|})^+); \in, P, \underline{F}_i, \Vdash)$. Select a $p(x) \in P$ for $x \in X$ such that $p(x) \Vdash x \in \underline{F}_i$ for some $i < \omega$. As N is countable there are $i < \omega$, $p \in P \cap N$, and an interval I such that $p(x) \cap N = p$ and $p(x) \Vdash x \in \underline{F}_i$ for a set of x dense in I. Select a $q \le p$, $q \in P \cap N$, and an interval $J \subseteq I$ with $q \Vdash J \cap \underline{F}_i = \emptyset$. For some $x \in J$, $p(x)$ and q are compatible, and their common extension forces $x \in J \cap F_i$, a contradiction.

We notice that the same proof gives the result if an arbitrary number of Cohen reals are added. Also, adding random reals (see [6]), we get the measure variant of Theorem 3; the counterpart of Lemma 6 is a corollary of the Fubini theorem.

REFERENCES

[1] U. Abraham, *Free sets for nowhere-dense set mappings*, Israel J. Math. 39 (1981), 167–176.
[2] D. Fremlin, *Consequences of Martin's Axiom*, Cambridge University Press, 1984.
[3] T. Jech, *Set Theory*, Academic Press, 1978.
[4] P. Komjáth, *Large small sets*, Colloq. Math. 56 (1988), 231–233.
[5] K. Kunen, *Set Theory*, North-Holland, 1980.
[6] —, *Random and Cohen reals*, in: Handbook of Set-Theoretic Topology, K. Kunen and J. E. Vaughan (eds.), North-Holland, 1984, 887–911.

[7] K. Muthuvel, *On certain measure zero*, *first category sets*, Real Anal. Exchange 17 (1991–2), 771–774.
[8] W. Sierpiński, *Hypothèse du continu*, Monografie Mat. 4, Warszawa, 1934.

DEPARTMENT OF COMPUTER SCIENCE
R. EÖTVÖS UNIVERSITY
MÚZEUM KRT. 6-8
H-1088 BUDAPEST, HUNGARY

Reçu par la Rédaction le 1.7.1992;
en version modifiée le 16.2.1993