SOME REMARKS ON SECOND CATEGORY SETS

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0. Introduction. In this paper we give some loosely connected consistency results concerning second category sets. In [4], the author proved, extending some earlier results of Mycielski, that $\text{MA}_\kappa$ implies (see [2]) that there exists a measure zero, first category set $X$ such that if $Y$ is a set of reals of size $\leq \kappa$ then $Y \subseteq X + c$ for some real $c$. K. Muthuvel [7] gave some applications of this construction and asked if it follows already from ZFC that such an $X$ exists for every measure zero, first category set $Y$ of size $< 2^\omega$. We show that this is not the case. Namely, it is consistent that $2^\omega = \omega_2$ and for no first category $X$ is it true that for every first category, measure zero set $Y$ of size $\omega_1$ we have $Y \subseteq X + c$ for some $c$.

In [1], U. Abraham proved that it is consistent that $2^\omega = \omega_2$ and there is a set mapping $f: \mathbb{R} \rightarrow P(\mathbb{R})$ such that $f(x)$ is nowhere dense for $x \in \mathbb{R}$, and there is no uncountable free set for $f$. Also, $\text{MA}_{\omega_1}$ is consistent with the statement that every set mapping as above has an uncountable free set. For set mappings where the images are first category sets, the situation is not as easy. If $2^\omega = \omega_2$, an easy well-ordering argument gives that there is a set mapping $f: \mathbb{R} \rightarrow [\mathbb{R}]^{\omega_1}$ with no free sets of size 2. If $\text{MA}_{\omega_1}$ holds, the images are of first category. Here we prove that it is consistent that $\text{MA}_{\omega_1}$ holds, and there is a set $X$ of size $2^\omega = \omega_2$ such that if $Y \subseteq X$ is of first category, then $|Y| \leq \omega_1$. This implies, by a result of Ruziewicz (see [1]), that if $f$ is a set mapping on $\mathbb{R}$ with first category images, then there is a (second category) free set of size $\aleph_3$.

The last problem we address is the following. Is it true that there is an almost disjoint family of second category sets which is of cardinality $> 2^\omega$? Here, almost disjoint means that the intersection of any two members in the family is of first category. Sierpiński [8] proved that if CH holds, then there are $\aleph_2$ second category sets with pairwise countable intersection. It is easy to show that there exist $2^\omega$ disjoint second category sets. We show that it is consistent that $2^\omega = \omega_2$ and there are no $\aleph_3$ almost disjoint second category sets.

1991 Mathematics Subject Classification: 03E35, 28A05.

Research partially supported by the Hungarian National Science Fund No. 1908.
Acknowledgments. The author is grateful for the referees’ remarks.

1. Notation, preliminaries. We use the standard notions and notation of axiomatic set theory (see [3, 5]). Cardinals are identified with initial ordinals. If $X$ is a set and $\kappa$ is a cardinal, then

$$[X]^\kappa = \{Y \subseteq X : |Y| = \kappa\}, \quad [X]^{<\kappa} = \{Y \subseteq X : |Y| < \kappa\}.$$

If $X$ is a set, then $P(X)$ is its power set. A set mapping is a function $f : X \to P(X)$. A subset $Y \subseteq X$ is free for $f$ if for $x, y \in Y$, $x \neq y$, $y \notin f(x)$ holds.

For the definitions of forcing and Cohen reals, see [3, 5, 6]. If $X \subseteq \mathbb{R}$ is a set, the canonical notion of forcing making $X$ of first category is defined as follows. $p \in P$ iff $p = (s, f, F)$, where $s \in [X]^{<\omega}$, $f : s \to \omega$, $F$ is a function with $\text{Dom}(F) \subseteq [\omega]^{<\omega}$, for $n \in \text{Dom}(F)$, $F(n)$ is a finite set of rational open intervals, and if $y \in s$, $n = f(y)$ then $y \notin \bigcup F(n)$. $(s', f', F') \leq (s, f, F)$ if $s' \supseteq s$, $f' \supseteq f$, $F'(n) \supseteq F(n)$ for $n \in \text{Dom}(F)$ (see [2]).

Lemma 1. If $p(\xi) \in P (\xi < \omega_1)$, then there is a set of indices $Z \in [\omega_1]^{\omega_1}$ such that if $\xi_1, \ldots, \xi_n \in Z$, then there is a $q \leq p(\xi_1), \ldots, p(\xi_n)$.

Proof. By the $\Delta$-system lemma (p. 225 in [3]), and the pigeon-hole principle, we may find $Z \in [\omega_1]^{\omega_1}$ such that $p(\xi) = (s \cup s_\xi, f_\xi, F)$ where $\{s, s_\xi : \xi \in Z\}$ are disjoint and $f_\xi | s = f$ for $\xi \in Z$. But then, if $\xi_1, \ldots, \xi_n \in Z$, then the coordinatewise union of $p(\xi_1), \ldots, p(\xi_n)$ is a condition extending them.

2. Translations of first category sets

Theorem 1. (CH) If $\kappa > \omega_1$ is a regular cardinal then in some forcing extension, $2^{\omega_1} = \kappa$ holds, and no first category set $X$ has the property that for every first category, measure zero set $Y$ of size $\omega_1$ there exists a real $d$ such that $Y + d \subseteq X$.

Proof. Let $V$ be a model of CH with $\kappa = \text{cf}(\kappa) > \omega_1$ in $V$. We are going to define a finite support forcing iteration of length $\kappa$, $(P_\alpha, Q_\alpha : \alpha < \kappa)$. If $\alpha < \kappa$, let $Q_{2\alpha}$ add $\omega_1$ Cohen reals, say $Y_\alpha = \{r_\xi^\alpha : \xi < \omega_1\}$, and then let $Q_{2\alpha+1}$ be the canonical poset making $Y_\alpha$ of first category. For simplicity, we assume that the underlying set of $Q_{2\alpha+1}$ is $\omega_1$.

We call a condition $p \in P_\alpha (\alpha < \kappa)$ full if for every $\beta < \alpha$, $p|\beta$ determines what $p(\beta)$ is.

Lemma 2. The full conditions are dense in $P_\alpha$.

Proof. Straightforward, by induction on $\alpha \leq \kappa$.

Lemma 3. If $\alpha \leq \kappa$, $p_\xi \in P_\alpha (\xi < \omega_1)$, then for some $Z \in [\omega_1]^{\omega_1}$, if $\xi_1, \ldots, \xi_n \in Z$, there is a common extension of $p_{\xi_1}, \ldots, p_{\xi_n}$.
Proof. We may assume that the conditions are full. By the $\Delta$-system lemma, we can assume that the supports of the conditions form a $\Delta$-system, $\text{supp}(p) = s \cup s_i$, where $\{s, s_i : \xi < \omega_1\}$ are disjoint. We can then use Lemma 1 and the counterpart of it for the Cohen forcing to thin out the system to $p_\xi (\xi \in Z)$, such that if $\xi_1, \ldots, \xi_n \in Z$ then the coordinatewise union of $p_{\xi_1}, \ldots, p_{\xi_n}$ is a condition, therefore extending them.

Towards proving the theorem, assume that $X$ has the property described in the theorem. We can assume that $X = \bigcup \{X_i : i < \omega\}$ where $X_i$ is a compact nowhere dense set. As every closed set can be identified with a countable collection of dyadic intervals, we can code $X$ by a real, which, by ccc, appears in $V^{P_{2\alpha}}$, for some $\alpha < \kappa$. The set $Y_\alpha$ added by $Q_{2\alpha}$ is a measure zero Lusin set in $V^{P_{2\alpha+1}}$, i.e., every uncountable subset of it is somewhere dense (see [6]).

We work in $V^{P_{2\alpha+1}}$. Assume that over $V^{P_{2\alpha+1}}$, $1 \models Y_\alpha + d \subseteq X$ holds for some name of a real $d$. Select, for $\xi < \omega_1$, a condition $q_\xi$ such that $q_\xi \forces r^\alpha_\xi + d \in X_i$ for some $i < \omega$. By Lemma 3 and the pigeonhole principle we can find a $Z \in [\omega_1]^{\omega_1}$ such that for $\xi \in Z$ the $i$ as above is the same and finite subsets of $\{q_\xi : \xi \in Z\}$ have common lower bounds.

If $\xi_1, \ldots, \xi_n \in Z$, $q \subseteq q_{\xi_1}, \ldots, q_{\xi_n}$, $q \forces d \in (X_i - r^\alpha_{\xi_1}) \cap \ldots \cap (X_i - r^\alpha_{\xi_n})$, i.e., $q$ forces that the compact sets $X_i - r^\alpha_{\xi_1}, \ldots, X_i - r^\alpha_{\xi_n}$, when redefined in $V^{P_{\alpha}}$, have a common point. This property is absolute (see [5, 6]), so we argue that $(X_i - r^\alpha_{\xi_1}) \cap \ldots \cap (X_i - r^\alpha_{\xi_n})$ is nonempty in $V^{P_{2\alpha+1}}$. By compactness, the intersection $\bigcap \{X_i - r^\alpha_{\xi} : \xi \in Z\}$ is nonempty, and if $c$ is in the intersection, then $\{c + r^\alpha_{\xi} : \xi \in Z\} \subseteq X_i$ so the nowhere dense $X_i - c$ has an uncountable intersection with $Y_\alpha$, which is impossible, as the latter is a Lusin set.

3. A Lusin-like set

Theorem 2. (GCH) If $\kappa = \text{cf}(\kappa) > \omega_1$, then it is consistent that $\text{MA}_{\omega_1}$ holds, and there is a set $X \subseteq \mathbb{R}$ of size $2^\omega = \kappa$ such that every first category subset of $X$ is of size $\leq \omega_1$.

Proof. We give a finite support iteration. Let $P_\alpha$ add $\kappa$ Cohen reals, $X = \{r_\xi : \xi < \kappa\}$, and for $\alpha < \kappa$ let $Q_\alpha = (\omega_1, <_\alpha)$ be some ccc poset on $\omega_1$. By an appropriate bookkeeping we can achieve $\text{MA}_{\omega_1}$ in $V^{P_\alpha}$.

We call a condition $p \in P_\alpha$ full if for every $1 \leq \beta < \alpha$, $p|\beta$ determines $p(\beta) < \omega_1$ (which is the maximal element of $(\omega_1, <_\beta)$ for all but finitely many $\beta < \alpha$).

Lemma 4. The set of full conditions is dense in $P_\alpha$. 

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Proof. By induction on \( \alpha \). The limit case is trivial. For \( \alpha = \beta + 1 \), first extend \( p|\beta \) to a \( p' \) determining \( p(\beta) \), then extend \( p' \) to a full \( p'' \in P_\beta \); then \( p'' \vdash p(\beta) \) is full.

For \( p \in P_0 \), let \( \text{Dom}(p) \) be the set of those coordinates \( < \kappa \) where \( p \) is not the trivial condition. For \( X \subseteq \kappa \), we say that a \( p \in P_0 \) is in \( P_0 | X \) if \( \text{Dom}(p) \subseteq X \). For \( p \in P_\alpha \), \( p|X \) is the condition where one removes from the first coordinate the part outside \( X \).

Lemma 5. If \( \tau \) is a \( P_\alpha \)-name for a subset of \( \omega_1 \), then there is a set \( X \in [\kappa]^{\omega_1} \) such that if \( p \) is a full condition and \( p \vdash i \in \tau \) then also \( p|X \vdash i \in \tau \).

Proof. By induction on \( \alpha \). We first notice that it suffices to prove the result for \( \alpha \leq 2 \).

If \( \alpha = 0 \) and no countable \( X \) satisfies the lemma, one can get by transfinite induction an increasing sequence of countable sets \( X(\xi) \subseteq \kappa \) and conditions \( p_\xi \in P_0 \), \( p_\xi \vdash i \in \tau \) such that \( p_\xi | X(\xi) \) does not force \( i \in \tau \), with \( p_\xi \in P_0 | X(\xi) \) \( (\xi < \omega) \). We can assume that \( \{ \text{Dom}(p_\xi) : \xi < \omega_1 \} \) forms a \( \Delta \)-system with kernel contained in \( X(\xi_0) \), and the \( p_\xi \)'s are the same on the kernel. But then we can find a \( q \leq p_\xi \upharpoonright X(\xi_0) \), \( q \vdash i \notin \tau \), and therefore we can find a \( p_\xi \) such that \( \text{Dom}(p_\xi) \cap \text{Dom}(q) = \text{Dom}(p_\xi) \cap \text{Dom}(p_\xi) \). The conditions \( p_\xi \) and \( q \) are compatible, which is a contradiction as they force contradictory statements.

If we know the result for \( \alpha \) and \( \tau \) is a \( P_{\alpha+1} \)-name, let \( \theta \) be the following \( P_\alpha \)-name: \( p \vdash (\gamma, i) \in \theta \) iff \( p^\gamma \vdash i \in \tau \). Now apply the inductive hypothesis for \( P_\alpha \) and \( \theta \), and get an appropriate set \( X \in [\kappa]^{\omega_1} \).

If \( \text{cf}(\alpha) \leq \omega_1 \), we can apply the inductive hypothesis, as if \( \alpha = \sup \{ \alpha_\xi : \xi < \omega_1 \} \), \( \tau = \bigcup \{ \tau_\xi : \xi < \omega_1 \} \), where \( p \vdash i \in \tau_\xi \) iff \( p \vdash i \in \tau \) and \( p \in P_{\alpha_\xi} \), so \( \tau_\xi \) is a \( P_{\alpha_\xi} \)-name.

Assume finally that \( \text{cf}(\alpha) \geq \omega_2 \). If \( \tau \) is a \( P_\beta \)-name for some \( \beta < \alpha \), we are done. If not, we can choose \( \omega_2 \) conditions \( p_\xi \) \( (\xi < \omega_2) \) such that \( p_\xi \vdash i \in \tau \) but \( p_\xi |\alpha_\xi \vdash i \in \tau \), where \( \alpha_\xi < \alpha \) is increasing and \( \alpha_\xi > \text{supp}(p_\xi) \) \( (\xi < \omega) \). Without loss of generality, \( \text{supp}(p_\xi) = s \cup s_\xi \), and we can assume that the \( p_\xi \)'s are identical. But then, if \( \xi \geq \omega \), \( p_\xi |\alpha_\xi \) has an extension forcing \( i \notin \tau \), which in turn is compatible with some \( p_\xi \) \( (\xi < \omega) \), a contradiction.

To prove the theorem, assume that \( \tau \) is a \( P_\alpha \)-name for a first category \( F_\sigma \) set \( Y \) (which again can be coded as a real). Let \( X \in [\kappa]^{\omega_1} \) be a set guaranteed by Lemma 5. If \( \xi \notin X \), we claim that \( r_\xi \notin Y \). If \( Y = \bigcup \{ Y_\xi : \xi < \omega \} \), where each \( Y_\xi \) is closed, nowhere dense and \( p \) is arbitrary, and if \( p \) gives the information that \( r_\xi \in I \) for some dyadic interval \( I \), let \( q \in P_0 | X \), \( q \leq p|X \) determine an interval \( J \subseteq I \) such that \( J \cap Y_\xi = \emptyset \), and then extend \( q \cup p \) to give \( r_\xi \in J \). This shows that \( p \vdash r_\xi \notin Y_\xi \), and we are done.
4. Almost disjoint sets

Theorem 3. It is consistent that $2^\omega = \omega_2$ and there does not exist a family of more than $2^{\omega_1}$ second category sets such that the intersection of any two of them is of first category.

Proof. It suffices to construct a model in which $2^\omega = 2^{\omega_1} = \omega_2$ and there does not exist a family of more than $2^{\omega_1}$ second category sets such that the intersection of any two of them is of first category.

Let $V$ be a model of GCH, and add $\omega_2$ Cohen reals, $\{r_\alpha : \alpha < \omega_2\}$. Assume that $1 \not\models X \subseteq R$ is of second category. Let $M$ be an elementary submodel of $(\mathcal{H}(2^{\omega_1}); \in, [\cdot])$ of size $\omega_1$ such that $[M]^\omega \subseteq M$. Put $\delta = M \cap \omega_2$.

We show that the intersection of $X$ with $V' = V[r_\alpha : \alpha < \delta]$ is of second category in $V'$. If $E = \bigcup \{F_i : i < \omega\}$, where $F_i$ is a nowhere dense closed set, and $E$ is (coded) in $V'$, then by the closure property of $M$, there is a name $\tau \in M$ of a real such that $1 \not\models \tau \in X - E$. Then $\tau$ gives rise to an element $y \in V'$ such that $y \in X - E$. The model $V[r_\alpha : \alpha < \omega_2]$ is obtained by a side-by-side Cohen extension from $V'$.

To conclude the proof, we need to show the following statement.

Lemma 6. If in $V$, $X$ is a second category set and $P$ adds some Cohen reals, then $V^P \models X$ is of second category.

Proof. Assume that $1 \not\models X \subseteq \bigcup \{F_i : i < \omega\}$, where $F_i$ is closed nowhere dense. Let $N$ be a countable elementary submodel of the model $(\mathcal{H}(2^\omega)^+; \in, [\cdot], F_i, \models)$. Select a $p(x) \in P$ for $x \in X$ such that $p(x) \models x \in F_i$ for some $i < \omega$. As $N$ is countable there are $i < \omega$, $p \in P \cap N$, and an interval $I$ such that $p(x) \cap N = p$ and $p(x) \models x \in F_i$ for a set of $x$ dense in $I$. Select a $q \leq p$, $q \in P \cap N$, and an interval $J \subseteq I$ with $q \models J \cap F_i = \emptyset$. For some $x \in J$, $p(x)$ and $q$ are compatible, and their common extension forces $x \in J \cap F_i$, a contradiction.

We notice that the same proof gives the result if an arbitrary number of Cohen reals are added. Also, adding random reals (see [6]), we get the measure variant of Theorem 3; the counterpart of Lemma 6 is a corollary of the Fubini theorem.

REFERENCES
