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THE PRODUCT OF A FUNCTION AND A BOEHMIAN

BY

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Let $\mathcal{A}$ be the class of all real-analytic functions and β the class of all Boehmians. We show that there is no continuous operation on β which is ordinary multiplication when restricted to $\mathcal{A}$.

1. Introduction and preliminaries. The study of generalized functions has been a major area of research for more than forty years. Most classes of generalized functions are constructed analytically ([2], [3], [11]), that is, starting with a class of functions A (called test functions) and a convergence structure on A, elements of the dual A' (space of continuous linear functionals on A) are called generalized functions.

The most well-known space of generalized functions is the space of distributions [11], denoted by $D'(\mathbb{R})$. The construction of $D'(\mathbb{R})$ is as follows. Let $D(\mathbb{R})$ be the set of all complex-valued infinitely smooth functions on $\mathbb{R}$ having compact support. A sequence $\{\varphi_n\}$ in $D(\mathbb{R})$ is said to converge to 0 if (i) there exists a compact set K such that the support of φ_n is contained in K for all n, and (ii) for $k = 0, 1, 2, \ldots$ the sequence $\{\varphi_n^{(k)}\}$ converges to 0 uniformly on $\mathbb{R}$ as $n \to \infty$. Then $D'(\mathbb{R})$ is the collection of all continuous linear functionals on $D(\mathbb{R})$.

Another approach to generalized functions is Mikusiński's operational calculus [5]. Mikusiński's approach is algebraic. It involves the quotient field of the ring of all continuous functions which vanish for $x \le 0$ under addition and convolution. One problem which arises is that Mikusiński operators are defined globally and their local properties are unknown. Another problem is that the convergence structure, called type I convergence, on the space of Mikusiński operators is not topological.

Recently, using an algebraic approach similar to the construction of Mikusiński operators, a new class of generalized functions β, called Boehmians, was constructed by P. Mikusiński. This class of generalized functions is very general. Indeed, by considering a special case, the space of distribu-

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tions can be viewed as a proper subspace of the space of Boehmians. Moreover, there are Boehmians, which are not functions, that satisfy Laplace's equation $u_{xx} + u_{yy} = 0$ [8]. The problems, stated above, with Mikusiński operators no longer exist with Boehmians. That is, some local properties of Boehmians are known. For example, a definition can be given for the equality of two Boehmians on an open set. Also, the convergence structure given to β is topological. Indeed, β with this convergence structure is a complete metric topological vector space [6].

In this note, we will investigate the possibility of defining a pointwise product of a function and a Boehmian which extends the notion of the product of two functions.

The product of an element from a class of functions and an element from a class of generalized functions is an important notion for applications. One possible area of application is in the area of differential equations (see [4], [12], and [13]).

For any continuous function g, let M_g be the mapping from $C(\mathbb{R})$ into $C(\mathbb{R})$ given by

$$M_g(f) = gf \quad \text{(i.e. ordinary multiplication)}. \tag{1.1}$$

If g is infinitely differentiable, then M_g has a unique continuous extension to the space of distributions [11]. If g is real-analytic, then M_g has a unique continuous extension to the space of hyperfunctions [3]. That is, a continuous product can be defined between elements of the class of infinitely differentiable functions (real-analytic functions) and the space of distributions (hyperfunctions).

If the function g in (1.1) is a polynomial, then M_g has a unique continuous extension to the space of Boehmians. This gives rise to the natural question: can a continuous product be defined between elements of the class of real-analytic functions and the class of Boehmians? The purpose of this note is to show that the answer to this question is no.

The collection of all continuous complex-valued functions on $\mathbb{R}$ will be denoted by $C(\mathbb{R})$. The support of a continuous function f, denoted by $\operatorname{supp} f$, is the complement of the largest open set on which f is zero.

The *convolution* of two continuous functions, where at least one has compact support, is given by $(f * g)(x) = \int_{\mathbb{R}} f(x-t)g(t)\,dt$.

A sequence of continuous nonnegative functions $\{\delta_n\}$ will be called a *delta sequence* if (i) $\int_{\mathbb{R}} \delta_n(x)\,dx = 1$ for $n = 1, 2, \ldots$, and (ii) $\operatorname{supp} \delta_n \subset (-\varepsilon_n, \varepsilon_n)$, where $\varepsilon_n \to 0$ as $n \to \infty$.

The following easily proved result will be needed. If f is a continuous function and $\{\delta_n\}$ is a delta sequence, then $f * \delta_n \to f$ uniformly on compact sets as $n \to \infty$.

2. Boehmians. In this section we construct the class of generalized functions known as Boehmians. For other results concerning Boehmians see [6]–[10].

A pair of sequences (f_n, δ_n) is called a *quotient of sequences*, and denoted by f_n/δ_n, if $f_n \in C(\mathbb{R})$ $(n = 1, 2, \ldots)$, $\{\delta_n\}$ is a delta sequence, and $f_n * \delta_m = f_m * \delta_n$ for all m and n. Two quotients of sequences f_n/δ_n and g_n/σ_n are equivalent if $f_n * \sigma_m = g_m * \delta_n$ for all m and n. The equivalence classes are called *Boehmians*. The space of all Boehmians will be denoted by β, and a typical element of β will be written as $F = f_n/\delta_n$. By defining a natural addition and scalar multiplication on β, i.e. $f_n/\delta_n + g_n/\sigma_n = (f_n * \sigma_n + g_n * \delta_n)/(\delta_n * \sigma_n)$ and $\alpha(f_n/\delta_n) = \alpha f_n/\delta_n$, where α is a complex number, β becomes a vector space.

R e m a r k s. (1) It follows that if $(f * \delta_n)/\delta_n = (g * \delta_n)/\delta_n$, then $f = g$. Thus, $C(\mathbb{R})$ can be identified with a subspace of β by identifying f with $(f * \delta_n)/\delta_n$, where $\{\delta_n\}$ is any delta sequence.

(2) Let $\{\delta_n\}$ be an infinitely differentiable delta sequence (i.e. $\delta_n \in C^\infty(\mathbb{R})$ for all n). Then for each $T \in D'(\mathbb{R})$ (the space of Schwartz distributions [11]), $T * \delta_n$ converges weakly to T. So, as above, $D'(\mathbb{R})$ can be identified with a subspace of β. Thus, we may view $D'(\mathbb{R})$ as a subspace of β. Moreover, this inclusion is proper. That is, there are Boehmians which are not distributions [6].

In a more general construction of Boehmians, P. Mikusiński [6] defines a convergence, called Δ-convergence, and shows that β with Δ-convergence is an F-space (a complete topological vector space in which the topology is induced by an invariant metric).

Before we define Δ-convergence, we will define a related convergence, called δ-convergence.

Let $F_n, F \in \beta$ for $n = 1, 2, \ldots$ We say that the sequence $\{F_n\}$ is *δ-convergent* to F if there exists a delta sequence $\{\delta_n\}$ such that for each n and j, $F_n * \delta_j, F * \delta_j \in C(\mathbb{R})$, and for each j, $F_n * \delta_j \to F * \delta_j$ uniformly on compact sets as $n \to \infty$. This will be denoted by $\delta\text{-}\lim F_n = F$.

DEFINITION 2.1. A sequence $\{F_n\}$ of Boehmians is said to be *Δ-convergent* to F, denoted by $\Delta\text{-}\lim F_n = F$, if there exists a delta sequence $\{\delta_n\}$ such that for each n, $(F_n - F) * \delta_n \in C(\mathbb{R})$ and $(F_n - F) * \delta_n \to 0$ uniformly on compact sets as $n \to \infty$.

R e m a r k. A sequence of Boehmians $\{F_n\}$ is Δ-convergent to F if and only if each subsequence of $\{F_n\}$ contains a subsequence which is δ-convergent to F [6].

3. The main result. If the function g in (1.1) is a polynomial then M_g has a unique continuous extension to β. This follows from observing that the product of a polynomial and a Boehmian can be defined using the algebraic derivative introduced by J. Mikusiński [5]. The product of $-x$ and the Mikusiński operator f/g is given by

$$-x(f/g) = (Df * g - f * Dg)/(g * g)\,, \quad \text{where} \quad Df = -xf\,.$$

Then $(-x)^n(f/g)$ $(n = 1, 2, \ldots)$ is defined inductively. Using the same idea we can define the product of a polynomial and a Boehmian. Moreover, it is not difficult to show that multiplication by a polynomial is a continuous operation on β. That is, if $P(x)$ is a polynomial and $\Delta\text{-}\lim F_n = F$, then $\Delta\text{-}\lim P(x)F_n = P(x)F$. Finally, the uniqueness follows from the fact that $C(\mathbb{R})$ is dense in β (see [6]).

Our goal is now to prove Theorem 3.6 which shows that multiplication cannot be extended, as a continuous operation, to the class of real-analytic functions. A function $\varphi : \mathbb{R} \to \mathbb{C}$ is said to be *real-analytic* if for each $x_0 \in \mathbb{R}$, φ can be represented, in some neighborhood of x_0, by its Taylor series about x_0.

If either f is a periodic function of period 2π or $\operatorname{supp} f \subset (-\pi, \pi)$, the nth *Fourier coefficient* $\widehat{f}(n)$ of f is defined as $\widehat{f}(n) = (2\pi)^{-1}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$, for $n = 0, \pm1, \pm2, \ldots$ By a simple calculation we see that $(\widehat{f * \delta})(n) = 2\pi\widehat{f}(n)\widehat{\delta}(n)$ for all n.

Let $P = \{F \in \beta : F = \sum_{n=-\infty}^{\infty} a_n e^{inx}$, for some sequence $\{a_n\}$ of complex numbers$\}$. That is, $F = \Delta\text{-}\lim_n \sum_{k=-n}^{n} a_k e^{ikx}$.

Definition 3.1. For $F \in P$ such that $F = \sum_{n=-\infty}^{\infty} a_n e^{inx}$, the nth *Fourier coefficient* of F, denoted by $\widehat{F}(n)$, is $\widehat{F}(n) = a_n$.

A useful representation for elements of P is given in the following theorem.

Theorem 3.2. *The following three statements are equivalent.*

(i) *$F \in P$.*

(ii) *There exists a representation of the Boehmian F, say f_n/δ_n, where, for all n, f_n is a periodic function of period 2π.*

(iii) *For every representation f_n/δ_n of F, f_n is periodic of period 2π for all n.*

Proof. (i)⇒(ii). Suppose $F \in P$. That is, $F = \Delta\text{-}\lim_n \sum_{k=-n}^{n} a_k e^{ikx}$. Because of the remark following Definition 2.1, we may assume that $\delta\text{-}\lim F_n = F$, where $F_n = \sum_{k=-n}^{n} a_k e^{ikx}$ for $n = 1, 2, \ldots$ Thus, there exists a delta sequence $\{\delta_n\}$ such that for each m, $F_n * \delta_m = \sum_{k=-n}^{n} 2\pi a_k \widehat{\delta}_m(k)e^{ikx} \to f_m$ uniformly on compact sets as $n \to \infty$ (for some $f_m \in C(\mathbb{R})$). Since for each m and n the continuous function $F_n * \delta_m$ has period 2π, thus f_m has period 2π for all m. Moreover, $\delta\text{-}\lim F_n = f_m/\delta_m$. Hence $F = f_m/\delta_m$.

The proof that (ii)⇒(iii) is straightforward and thus omitted.

(iii)⇒(i). Suppose that $F = f_n/\delta_n$, where f_n has period 2π. We may assume that, for each n, f_n is twice continuously differentiable. If not, let $\{\sigma_n\}$ be a twice continuously differentiable delta sequence (i.e. $\sigma_n \in C^2(\mathbb{R})$ for all n), and let $\psi_n = \delta_n * \sigma_n$ and $g_n = f_n * \sigma_n$ for all n. Then $F = g_n/\psi_n$ and for each n, $g_n \in C^2(\mathbb{R})$. Now, for $n = 1, 2, \ldots$ define $F_n = \sum_{k=-n}^{n} a_k e^{ikx}$, where $a_n = \widehat{f}_m(n)/(2\pi\widehat{\delta}_m(n))$ for all n. The a_n's are well-defined. This follows from the facts that $f_n * \delta_m = f_m * \delta_n$ for all m and n, and (as can be easily shown) that for each n, $\widehat{\delta}_m(n) \to 1/(2\pi)$ as $m \to \infty$.

Now, for each m and n,

$$F_n * \delta_m = \sum_{k=-n}^{n} 2\pi a_k \widehat{\delta}_m(k) e^{ikx} = \sum_{k=-n}^{n} \widehat{f}_m(k) e^{ikx}\,.$$

So, for each m, $F_n * \delta_m \to f_m$ uniformly on compact sets as $n \to \infty$ (see [1]). That is, δ-$\lim F_n = F$ and hence Δ-$\lim F_n = F$. Thus, the proof is complete.

THEOREM 3.3. *P is closed.*

Proof. It suffices to show that P is closed with respect to δ-convergence. For, by the remark following Definition 2.1, if Δ-$\lim F_n = F$ then there exists a subsequence $\{F_{n_k}\}$ of $\{F_n\}$ such that δ-$\lim_k F_{n_k} = F$. Thus, suppose that $F_n \in P$ for $n = 1, 2, \ldots$ and δ-$\lim F_n = F$. That is, there exists a delta sequence $\{\delta_n\}$ such that for each n and j, $F_n * \delta_j, F * \delta_j \in C(\mathbb{R})$ and for each j, $F_n * \delta_j \to F * \delta_j$ uniformly on compact sets as $n \to \infty$. Also, because of Theorem 3.2, we may assume that for each n and j, $F_n * \delta_j$ is periodic of period 2π. Thus, $F * \delta_j$ is periodic of period 2π for all j. Hence, $F = (F * \delta_n)/\delta_n \in P$. Therefore the theorem is established.

The proof of the next theorem is straightforward and hence is left to the reader.

THEOREM 3.4. *Suppose that $F_n \in P$ for $n = 1, 2, \ldots$ If Δ-$\lim F_n = F$, then $\lim_n \widehat{F}_n(k) = \widehat{F}(k)$ for each k.*

Before proving the main result, the following lemma is needed.

LEMMA 3.5. *Let $\{n_k\}$ be a subsequence of positive integers such that $\sum_{k=1}^{\infty} 1/n_k < \infty$. If $\{a_n\}$ is any sequence of complex numbers such that $a_n = 0$ for $n \neq n_k$ $(k = 1, 2, \ldots)$, then there is a Boehmian $F \in P$ such that $\widehat{F}(n) = a_n$ for all n.*

Proof. For $k = 1, 2, \ldots$ let $\varphi_k(x) = n_k/(2\pi)$ for $|x| \le \pi/n_k$ and zero otherwise. For $k = 1, 2, \ldots$ let $\delta_k = \prod_{j=k}^{\infty} \varphi_j$ (convolution product). Since $\sum_{k=1}^{\infty} 1/n_k < \infty$, $\{\delta_k\}$ is a delta sequence (see [6]). Now, for each k and n, $\widehat{\varphi}_k(n) = \alpha_{k,n} \sin(n\pi/n_k)$ ($\alpha_{k,n}$ constant) and hence $\widehat{\delta}_m(n_k) = \widehat{\delta}_m(-n_k)$

$= 0$ for all $k \geq m$. Let $\{\sigma_n\}$ be any delta sequence such that for each n, $\widehat{\sigma}_n(k) = O(k^{-2})$ as $|k| \to \infty$. Let $\{\psi_n\}$ be the delta sequence defined by $\psi_n = \delta_n * \sigma_n$ for $n = 1, 2, \ldots$ Now, define $f_n(x) = \sum_{j=-n}^{n} a_j e^{ijx}$ for $n = 1, 2, \ldots$ Then for each k and n,

$$(f_n * \psi_k)(x) = 2\pi \sum_{j=-n}^{n} a_j \widehat{\psi}_k(j) e^{ijx}\,.$$

Since for each k, $a_j \widehat{\psi}_k(j) = O(j^{-2})$ as $|j| \to \infty$, for each k the sequence of continuous functions $\{f_n * \psi_k\}_{n=1}^{\infty}$ converges uniformly as $n \to \infty$. Hence, $\Delta\text{-}\lim f_n = \Delta\text{-}\lim_n f_n * \psi_k/\psi_k = F \in P$. By Theorem 3.4, for each m, $\widehat{F}(m) = \lim_n \widehat{f}_n(m) = a_m$ and hence the lemma is established.

For a stronger version of Lemma 3.5 see Theorem 3.1 in [9].

THEOREM 3.6. *Let $\mathcal{A}$ be the class of all real-analytic functions and $T : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ be ordinary multiplication. If $\widetilde{T} : \mathcal{A} \times \beta \to \beta$ is a mapping such that $\widetilde{T}$ and T agree on $\mathcal{A} \times \mathcal{A}$, then $\widetilde{T}$ is not sequentially continuous in its second variable.*

Proof. Suppose that $\widetilde{T} : \mathcal{A} \times \beta \to \beta$ is any mapping such that $\widetilde{T}|_{\mathcal{A}\times\mathcal{A}}$ is ordinary multiplication. Assume that $\widetilde{T}$ is sequentially continuous in its second variable. Let $\varphi \in \mathcal{A} \cap P$ such that $\widehat{\varphi}(n) \neq 0$ for infinitely many $n \geq 1$. It is always possible to find such a φ since $\varphi \in \mathcal{A} \cap P$ if and only if $\widehat{\varphi}(n) = O(e^{-\varepsilon|n|})$ as $n \to \infty$ for some $\varepsilon > 0$ (see [1]).

Now, let $\{n_k\}$ be a subsequence of positive integers such that $\sum_{k=1}^{\infty} 1/n_k < \infty$ and $\widehat{\varphi}(n_k) \neq 0$ for all k. Let $\{a_n\}$ be the sequence of complex numbers defined by $a_{-n_k} = (\widehat{\varphi}(n_k))^{-1}$ and zero otherwise. By Lemma 3.5 there exists a Boehmian $F \in P$ such that $\widehat{F}(n) = a_n$ for all n. Since $\Delta\text{-}\lim F_n = F$, where $F_n = \sum_{k=-n}^{n} a_k e^{ikx}$, we obtain $\Delta\text{-}\lim \varphi F_n = \varphi F$. Using Theorem 3.4 we see that $\widehat{\varphi F}(m) = \lim_n \sum_{k=-n}^{n} a_k \widehat{\varphi}(m-k)$ for all m. In particular, $\widehat{\varphi F}(0) = \lim_n \sum_{k=-n}^{n} a_k \widehat{\varphi}(-k)$. But, because of the way the sequence $\{a_n\}$ is defined, the above limit does not exist. Hence, $\widetilde{T}$ cannot be sequentially continuous in its second variable and the proof is complete.

From the above proof we obtain a stronger result. That is, multiplication cannot be continuously extended to any class of functions which contains a periodic function with infinitely many nonzero Fourier coefficients. In particular, multiplication cannot be continuously extended to the class of real-analytic functions of slow growth. (A function φ is said to be of *slow growth* if $\varphi(x) = O((1+|x|)^m)$ as $|x| \to \infty$ for some integer m.)

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