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A NOTE ON A CONJECTURE OF D. OBERLIN

BY

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1. Let N be a positive integer, P_N be the set of all real-valued polynomials on $\mathbb{R}$ of degree at most N. In [1], D. Oberlin stated the following conjecture concerning uniform estimates for oscillatory integrals with polynomial phases:

Conjecture. *Let n, N be two positive integers. Then there is a constant $C(N,n)$ such that*

$$\Big|\int_a^b e^{iP(x)}|P^{(n)}(x)|^{1/n+is}\,dx\Big| \le C(N,n)(1+|s|)^{1/n}, \tag{1.1}$$

for $P\in P_N$, $a<b$ and $s\in\mathbb{R}$.

For the significance of such estimates in Fourier analysis, we refer the reader to [1] and [2].

Clearly, (1.1) holds if $n \ge N$ (for $n > N$ it is trivial; for $n = N$ it follows from van der Corput's lemma). Hence we need to be concerned with $n = 1, \ldots, N-1$ only. For $n = 1$ or 2, the conjecture has been proved by Oberlin ([1], Theorem 2). The purpose of this note is to prove the conjectured estimate (1.1) in the case $n = N - 1$.

2. We state our result as the following theorem.

Theorem. *Let $N \ge 2$ be an integer. Then there exists a constant $C(N) > 0$ such that*

$$\Big|\int_a^b e^{iP(x)}|P^{(N-1)}(x)|^{1/(N-1)+is}\,dx\Big| \le C(N)(1+|s|)^{1/(N-1)}, \tag{2.1}$$

for $P\in P_N$, $a<b$ and $s\in\mathbb{R}$.

First we state a simple lemma whose proof is deferred until the end of this note.

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LEMMA 2.1. *Let n be a positive integer, $Q(x)$ be a monic polynomial with real coefficients and degree n. Suppose that the coefficient of the x^{n-1} term in $Q(x)$ is zero. Then there are m $(m \le n)$ disjoint intervals $J_1, \ldots, J_m$ and $r_1, \ldots, r_m \in \mathbb{R}$ such that $\bigcup_{k=1}^m J_k = \mathbb{R}$ and*

$$|Q(x)| \ge |x||x - r_k|^{n-1}, \tag{2.2}$$

for $x \in J_k$, $k = 1, \ldots, m$.

We shall need the following lemma which is due to van der Corput.

LEMMA 2.2 ([3], p. 197). *Suppose φ and ψ are smooth on $[a,b]$ and φ is real-valued. If $|\varphi'(x)| \ge \lambda$, and φ' is monotone on $[a,b]$, then*

$$\Big|\int_a^b e^{i\varphi(x)}\psi(x)\,dx\Big| \le 4\lambda^{-1}\Big(|\psi(b)| + \int_a^b |\psi'(x)|\,dx\Big).$$

Proof of the Theorem. Let $P \in P_N$, and $\deg(P) = N$. By a change of variable $x \to cx + d$, for suitable c and d, we may assume that $P(x)$ is of the form

$$P(x) = x^N + R(x), \tag{2.3}$$

with $\deg(R) \le N - 2$. To prove the Theorem, it suffices to prove that, for $a < b$ and $s \in \mathbb{R}$,

$$\Big|\int_a^b e^{iP(x)}|x|^{1/(N-1)+is}\,dx\Big| \le C(N)(1 + |s|)^{1/(N-1)}. \tag{2.4}$$

Let $n = N - 1$, $Q(x) = P'(x)$. We assume that $n \ge 2$ (for $n = 1$ is covered by Oberlin's result). By Lemma 2.1, there are disjoint intervals $J_1, \ldots, J_m$ and $r_1, \ldots, r_m \in \mathbb{R}$ (for some $m \le n$) such that $\bigcup_{k=1}^m J_k = \mathbb{R}$ and

$$|Q(x)| \ge |x||x - r_k|^{n-1},$$

for $x \in J_k$, $k = 1, \ldots, m$. To prove (2.4), we may assume that $P''(x)$ is of constant sign on $I = [a,b]$. As a further reduction, we shall consider the integral over each $I \cap J_k \cap (0,\infty)$ and $I \cap J_k \cap (-\infty, 0)$, for $k = 1, \ldots, m$. Without loss of generality, we pick $k = 1$, and consider the integral over $I \cap J_1 \cap (0,\infty)$. For the sake of convenience, we still denote $I \cap J_1 \cap (0,\infty)$ by I. Let $A = r_1$; we have

$$|P'(x)| \ge |x||x - A|^{n-1},$$

for $x \in I$. There are two cases.

Case I: $A > 0$. Let $\sigma > 0$ such that $\sigma^n(A + \sigma) = 1 + |s|$. Then,

$$\Big|\int_{I\cap[A,A+\sigma]} e^{iP(x)}|x|^{1/n+is}\,dx\Big| \le \sigma(A + \sigma)^{1/n} = (1 + |s|)^{1/n}. \tag{2.5}$$

On the other hand, by Lemma 2.2, we have, for $j \geq 0$,

$$\left| \int\limits_{I\cap[2^j(A+\sigma),2^{j+1}(A+\sigma)]} e^{iP(x)}|x|^{1/n+is}dx \right| \tag{2.6}$$

$$\leq \frac{C(N)}{2^j(A+\sigma)\sigma^{n-1}}\left(2^{(j+1)/n}(A+\sigma)^{1/n} + \left(\frac{1}{n}+|s|\right)\int\limits_{2^j(A+\sigma)}^{2^{j+1}(A+\sigma)} x^{1/n-1}\,dx\right)$$

$$\leq \frac{C(N)(1+|s|)}{(A+\sigma)^{(n-1)/n}\sigma^{n-1}}2^{(1/n-1)j} = C(N)(1+|s|)^{1/n}2^{(1/n-1)j}\,.$$

Hence we have

$$\left| \int\limits_{I\cap[A+\sigma,\infty)} e^{iP(x)}|x|^{1/n+is}\,dx \right| \leq C(N)(1+|s|)^{1/n}\sum_{j\geq 0}2^{(1/n-1)j} \leq C(N)(1+|s|)^{1/n}\,. \tag{2.7}$$

It remains for us to show that

$$\left| \int\limits_{I\cap[0,A]} e^{iP(x)}|x|^{1/n+is}\,dx \right| \leq C(N)(1+|s|)^{1/n}\,. \tag{2.8}$$

If $A \leq 4^{n/(n+1)}(1+|s|)^{1/(n+1)}$, then

$$\left| \int\limits_{I\cap[0,A]} e^{iP(x)}|x|^{1/n+is}\,dx \right| \leq \int_0^A x^{1/n}\,dx \leq C(N)(1+|s|)^{1/n}\,. \tag{2.9}$$

If $A > 4^{n/(n+1)}(1+|s|)^{1/(n+1)} = 8B$, we let $\sigma' = ((1+|s|)/A)^{1/n} \leq A/4$. Let n_1 and n_2 be two integers such that

$$2^{n_1} \leq B < 2^{n_1+1} \quad \text{and} \quad 2^{n_2} \leq A/4 < 2^{n_2+1}\,.$$

We write

$$\int\limits_{I\cap[0,A]} e^{iP(x)}|x|^{1/n+is}\,dx$$

$$= \int\limits_{I\cap[0,2^{n_1}]} e^{iP(x)}|x|^{1/n+is}\,dx + \sum_{j=n_1}^{n_2}\int\limits_{I\cap[2^j,2^{j+1}]} e^{iP(x)}|x|^{1/n+is}\,dx$$

$$+ \int\limits_{I\cap[2^{n_2+1},A-\sigma']} e^{iP(x)}|x|^{1/n+is}\,dx + \int\limits_{I\cap[A-\sigma',A]} e^{iP(x)}|x|^{1/n+is}\,dx\,.$$

The first term and fourth term are easily seen to be bounded by $(1+|s|)^{1/n}$. For the third term, one observes that

$$|P'(x)| \geq |x||x-A|^{n-1} \geq (A/4)(\sigma')^{n-1}\,,$$

for $x \in I \cap [2^{n_2+1}, A-\sigma']$, and the desired bound follows from van der Corput's lemma. To treat the second term, we use

$$|P'(x)| \geq |x||x-A|^{n-1} \geq (2^{1-n})2^j A^{n-1}$$

for $x \in I \cap [2^j, 2^{j+1}]$, $n_1 \leq j \leq n_2$. Then

$$\Big|\sum_{j=n_1}^{n_2} \int\limits_{I\cap[2^j,2^{j+1}]} e^{iP(x)}|x|^{1/n+is}\,dx\Big|$$
$$\leq C\sum_{j=n_1}^{n_2} \frac{(1+|s|)}{2^j A^{n-1}} 2^{j/n} \leq C(1+|s|)A^{1-n}2^{n_1(1/n-1)}$$
$$\leq C(1+|s|)^{1-\frac{n-1}{n+1}}(1+|s|)^{\frac{1}{n+1}(\frac{1}{n}-1)} = C(1+|s|)^{1/n}\,.$$

The above argument shows that (2.8) holds. Combining (2.5), (2.7) and (2.8), we see that case I is proved.

Case II: $A \leq 0$. This case is actually easier than the previous case. Now we have $|P'(x)| \geq |x|^n$ for $x \in I$. Let $\delta = (1+|s|)^{1/(n+1)}$; we decompose the integral as

$$\int\limits_I e^{iP(x)}|x|^{1/n+is}\,dx = \int\limits_{I\cap[0,\delta]} e^{iP(x)}|x|^{1/n+is}\,dx + \sum_{j=1}^{\infty} \int\limits_{I\cap[2^j\delta,2^{j+1}\delta]} e^{iP(x)}|x|^{1/n+is}\,dx\,.$$

While the first term is trivially bounded by $(1+|s|)^{1/n}$, an application of van der Corput's lemma shows that the second term is also bounded by $(1+|s|)^{1/n}$.

The proof of the theorem is now complete. ∎

Proof of Lemma 2.1. Let $z_1,\ldots,z_n$ be the n roots of $Q(x)$, and $\Delta = \{z_1,\ldots,z_n\}$. Then we have

$$Q(x) = \prod_{j=1}^{n}(x-z_j)\,. \tag{2.10}$$

Suppose

$$\{\mathrm{Re}\, z \mid z \in \Delta\} = \{r_1,\ldots,r_m\}\,,$$

and $r_1 < \ldots < r_m$. Define

$$J_1 = \Big(-\infty, \frac{r_1+r_2}{2}\Big], \qquad J_m = \Big(\frac{r_{m-1}+r_m}{2}, \infty\Big)\,,$$

and

$$J_k = \left(\frac{r_{k-1} + r_k}{2}, \frac{r_k + r_{k+1}}{2}\right], \qquad \text{for } k = 2, \ldots, m-1\,.$$

For $x \in I_k$, $1 \le k \le m$, we find

$$|x - z_j| \ge |x - \operatorname{Re} z_j| \ge |x - r_k|, \tag{2.11}$$

for all $j = 1, \ldots, n$. On the other hand, we have

$$\sum_{j=1}^{n} (x - z_j) = nx - \sum_{j=1}^{n} z_j = nx\,,$$

where we used the fact that the coefficient of the x^{n-1} term in $Q(x)$ is zero. Hence, for every $x \in \mathbb{R}$, there is a j_x, $1 \le j_x \le n$, such that

$$|x - z_{j_x}| \ge |x|\,. \tag{2.12}$$

(2.11) and (2.12) imply that

$$|Q(x)| \ge |x||x - r_k|^{n-1}\,,$$

for $x \in J_k$, $k = 1, \ldots, m$. ■

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