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A NOTE ON F.P.P. AND F.P.P.*

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In [3], Kinoshita defined the notion of f*.p.p. and he proved that each compact AR has f*.p.p. In [4], Yonezawa gave some examples of not locally connected continua with f.p.p., but without f*.p.p. In general, for each $n = 1, 2, \dots$, there is an n-dimensional continuum X_n with f.p.p., but without f*.p.p. such that X_n is locally $(n-2)$-connected (see [4, Addendum]). In this note, we show that for each n-dimensional continuum X which is locally $(n-1)$-connected, X has f.p.p. if and only if X has f*.p.p.

1. Introduction. Let X be a compact connected metric space (= *continuum*). If every map $f : X \to X$ has a fixed point x_0 of f, i.e., $f(x_0) = x_0$, X is said to have the *fixed point property* (= f.p.p.). Let $f : X \to X$ be a map of a continuum X. A component C of the fixed point set $F(f) = \{x \in X \mid f(x) = x\}$ of f is called *essential* [3] if for any $\varepsilon > 0$ there is a $\delta > 0$ such that each map $f' : X \to X$ with $d(f, f') = \sup\{d(f(x), f'(x)) \mid x \in X\} < \delta$ has a fixed point in the ε-neighborhood $U_\varepsilon(C)$ of C in X. A continuum X has f*.p.p. if it has f.p.p. and the fixed point set of every map $f : X \to X$ has an essential component. Put $C(X, X) = \{f : X \to X \mid f \text{ is a (continuous) map}\}$. Then $C(X, X)$ has the *supremum metric* $d(f, f') = \sup\{d(f(x), f'(x)) \mid x \in X\}$ $(f, f' \in C(X, X))$.

In [3], Kinoshita proved that every compact absolute retract (=AR) has f*.p.p. In [4], Yonezawa gave several examples of not locally connected continua with f.p.p., but without f*.p.p. Also, he showed that there is a not locally connected 2-dimensional continuum Z which has f*.p.p. In this note, we prove that for an n-dimensional continuum X which is locally $(n-1)$-connected, X has f.p.p. if and only if X has f*.p.p. It is known that for each $n \geq 1$, there is a continuum X such that $\dim X = n$, X is locally $(n-2)$-connected and X has f.p.p., but X does not have f*.p.p. (see [4, Addendum]). This implies that Yonezawa's examples are best possible for dimension and local m-connectedness.

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2. Locally $(n-1)$-connected and n-dimensional continua with f.p.p. Let X be a space. Then X is said to be *locally m-connected* ($m = 0, 1, \ldots$) if for any $x \in X$ and any neighborhood U of x in X there is a neighborhood V ($V \subset U$) of x in X such that if $f : S^i \to V$ is a map from the i-sphere S^i ($0 \le i \le m$) to V, then f is null-homotopic in U.

The following theorem is the main result of this note.

(2.1) THEOREM. *Let X be an n-dimensional continuum which is locally $(n-1)$-connected. Let $\mathbf{F} = \{f \in C(X,X) \mid F(f) \neq \emptyset\}$ and suppose that $f \in \mathbf{F}$. Then f has a neighborhood $\mathbf{U}$ in $C(X,X)$ with $\mathbf{U} \subset \mathbf{F}$ if and only if the fixed point set of f has an essential component. In particular*, *X has f.p.p. if and only if X has f.*p.p.*

P r o o f. The idea of the proof comes from [3]. Suppose that f has a neighborhood $\mathbf{U}$ as in the theorem. Suppose, on the contrary, that the fixed point set of f has no essential component. For each component C of $F(f)$, there is a neighborhood $U(C)$ of C in X such that $\mathrm{Bd}(U(C)) \cap F(f) = \emptyset$ and there are maps in $C(X,X)$ arbitrarily close to f which have no fixed points in $U(C)$.

Choose a finite covering $\{U(C_1), \ldots, U(C_k)\}$ of $F(f)$. Put $H_1 = F(f) \cap U(C_1)$, and $H_i = (F(f) \cap U(C_i)) - \bigcup_{j=1}^{i-1} H_j$ ($i = 2, \ldots, k$). Then $\{H_i\}$ is a finite closed covering of $F(f)$ such that $H_i \cap H_j = \emptyset$ ($i \neq j$). Choose open sets V_i ($i = 1, \ldots, k$) such that $H_i \subset V_i \subset \overline{V}_i \subset U(C_i)$ and $\overline{V}_i \cap \overline{V}_j = \emptyset$ ($i \neq j$). Put $\varepsilon_0 = \inf\{d(x, f(x)) \mid x \in X - \bigcup_{i=1}^{k} V_i\} > 0$. Since X is locally $(n-1)$-connected, X satisfies the following condition $(*)$:

$(*)$ For any $\varepsilon > 0$ there is a $\delta > 0$ such that if $h : |L| \to X$ is a map from a subpolyhedron $|L|$ of an n-dimensional polyhedron $|K|$ such that the 0-skeleton K^0 of K is contained in L, $\operatorname{diam} h(\Delta^0) < \delta$ for each simplex Δ of K and $\operatorname{diam} h(\Delta) < \delta$ for each simplex Δ of L, then there is an extension $h' : |K| \to X$ of h such that $\operatorname{diam} h'(\Delta) < \varepsilon$ for each simplex Δ of K.

Let $\gamma > 0$ be a sufficiently small positive number. Choose an open set W_i such that $\overline{V}_i \subset W_i$ ($i = 1, \ldots, k$), $\overline{W}_i \cap \overline{W}_j = \emptyset$ ($i \neq j$) and $d(x, \overline{V}_i) < \gamma$ for $x \in \overline{W}_i - \overline{V}_i$. Let $h_i : X \to X$ be a map such that h_i has no fixed points in $\overline{V}_i$ and $d(h_i, f) < \gamma$. Consider the map $f_i : \overline{V}_i \cup (X - W_i) \to X$ defined by $f_i|\overline{V}_i = h_i$, $f_i|(X - W_i) = f|X - W_i$. Let $1 \le i \le k$ be fixed. Choose a canonical covering $\mathcal{U}$ of $W_i - \overline{V}_i$ such that $\operatorname{diam} \mathrm{St}(U, \mathcal{U}) < \gamma$ for each $U \in \mathcal{U}$ (see [1, Chapter III, 9]). Let $N(\mathcal{U})$ be the nerve of $\mathcal{U}$. For each $U \in \mathcal{U}$, choose a point $a(U) \in A = (X - W_i) \cup \overline{V}_i$ such that $d(U, a(U)) < 2d(U, A)$. Consider the map $F' : A \cup |N(\mathcal{U})^0| \to X$ defined by $F'|A = f_i$ and $F'(U) = f_i(a(U))$ for $U \in N(\mathcal{U})^0$, where $A \cup |N(\mathcal{U})|$ is the space such that the neighborhoods of a point $z \in A \cup |N(\mathcal{U})|$ are defined to be subsets U_z of z such that

(a) if $z \in A - \overline{X - A}$, then $U_z \cap A$ is a neighborhood of z in X,

(b) if $z \in |N(\mathcal{U})|$, then $U_z \cap |N(\mathcal{U})|$ is a neighborhood of z in $|N(\mathcal{U})|$,

(c) if $z \in A \cap \overline{X - A}$, then $U_z \cap A$ is a neighborhood of z in A and there is a neighborhood V of z in X such that every simplex $\langle U_1, \ldots, U_t \rangle$ in $N(U)$ with $\bigcup_{j=1}^{t} U_j \subset V$ is contained in U_z (see [1]).

By choosing a sufficiently small positive number $\gamma > 0$ and using $(*)$, we obtain a map $F : A \cup |N(\mathcal{U})| \to X$ which is an extension of F'. Put $g_i = F \circ \varphi|\overline{W}_i : \overline{W}_i \to X$, where $\varphi : X \to A \cup |N(\mathcal{U})|$ is a canonical map (see [1, p. 82]). Also, by the definition of the canonical map φ, we assume that g_i is arbitrarily close to $f|\overline{W}_i$. Note that $g_i|\overline{V}_i = h_i$, and we assume $d(x, g_i(x)) \geq d(x, f(x)) - d(f(x), g_i(x)) \geq \varepsilon_0 - d(f(x), g_i(x)) > 0$ for $x \in \overline{W}_i - V_i$. Hence we see that g_i has no fixed points. Put

$$f'(x) = \begin{cases} f(x) & \text{if } x \in X - \bigcup_{i=1}^{k} W_i, \\ g_i(x) & \text{if } x \in \overline{W}_i \ (i = 1, \ldots, k). \end{cases}$$

Then we may assume that $f' \in \mathbf{U}$ and f' has no fixed points. This is a contradiction. Hence the fixed point set of f has an essential component. The converse assertion is obvious.

Let $\delta > 0$ be a positive number. A homotopy $H : X \times I \to X$ is a *δ-homotopy* from f to g if $H_0 = f$, $H_1 = g$ and $\operatorname{diam} H(\{x\} \times I) < \delta$ for each $x \in X$. For any maps $f, g : X \to X$, define $d_{\mathrm{h}}(f, g) = \inf\{\delta \mid$ there is a δ-homotopy H from f to $g\}$ if f and g are homotopic.

By a similar argument to one of [3], we can prove the following proposition.

(2.2) Proposition. *Let X be a continuum with f.p.p., and let $f : X \to X$ be a map. Then there is a component C of the fixed point set of f such that for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $g : X \to X$ is a map with $d_{\mathrm{h}}(f, g) < \delta$, then g has a fixed point in $U_\varepsilon(C)$.*

The following fact for ANR's is well-known.

(2.3) Lemma ([1]). *Let X be a compact ANR. For any $\varepsilon > 0$ there is a $\delta > 0$ such that if $f, g \in C(X, X)$ with $d(f, g) < \delta$, then f is ε-homotopic to g.*

By using (2.2) and (2.3), we have the following.

(2.4) Proposition. *Let X be a compact ANR.*

(1) *X has f.p.p. if and only if X has f*.p.p.*

(2) *If $f : X \to X$ is a map with $\Lambda_f \neq 0$, then the fixed point set of f has an essential component, where Λ_f is the Lefschetz number of f.*

(2.5) Corollary. *If X has f.p.p., and $f : X \to X$ is a map such that $f(X)$ is contained in an ANR-neighborhood in X, then the fixed point set of f has an essential component.*

(2.6) Remark. In [4], Yonezawa showed that there is a not locally connected continuum X with $\dim X = 2$ which has f*.p.p. Naturally, the following question arises: Is there a 1-dimensional continuum Z such that Z is not locally connected and Z has f*.p.p. ? In [2, Theorem 11], Cook constructed a 1-dimensional continuum C such that the identity is the only mapping of C onto a nondegenerate subcontinuum of C. Clearly, the Cook continuum C is not locally connected and C has f*.p.p.

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