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A CLASS OF NONLOCAL PARABOLIC PROBLEMS OCCURRING IN STATISTICAL MECHANICS

BY

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We consider parabolic equations with nonlocal coefficients obtained from the Vlasov–Fokker–Planck equations with potentials. This class of equations includes the classical Debye system from electrochemistry as well as an evolution model of self-attracting clusters under friction and fluctuations. The local in time existence of solutions to these equations (with no-flux boundary conditions) and properties of stationary solutions are studied.

1. Introduction. Our aim in this paper is to prove the existence of solutions of the initial-boundary value as well as the stationary problems for a class of parabolic equations with nonlocal coefficients. The equations under study read

$$u_t = \Delta u + \nabla \cdot (uX(u)) \tag{1}$$

where the drift term is determined by a vector field $X = X(u)$ which may depend on u in a nonlocal way, e.g. via a (linear weakly singular) integral operator. These equations are supplemented with the no-flux conditions

$$(\nabla u + uX(u)) \cdot \nu = 0 \tag{2}$$

imposed at the boundary of a bounded open set $\Omega \subset \mathbb{R}^n$. Here ν denotes the unit normal vector to $\partial\Omega$. Moreover, the initial condition is

$$u(x, 0) = u_0(x)\,. \tag{3}$$

There are several physical motivations to study such diffusion equations for the density functions $u = u(x,t) \geq 0$. The physical models fall, roughly speaking, into two classes. The first one deals with charge carriers (e.g. electrons and holes in semiconductors, ions in electrolytes) interacting by

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Coulomb forces. The second one describes gravitational attraction of particles.

The density dependent vector field

$$X = X(u) = X(u,x,t) = \mathcal{K}(u) + \nabla V(x,t) \tag{4}$$

describes not only the interaction of particles with the field $\mathcal{K}(u)$ generated by them but also includes the external potential forces derived from a potential $V = V(x,t)$.

Note that (2) is the simplest physically relevant (no-flux) boundary condition which guarantees the conservation of the integral $\int_\Omega u$ in time.

Besides equations of Fokker–Planck type with $X = \nabla V$ independent of u, concrete examples such as the Van Roosbroeck equations from the theory of semiconductors, the Debye system from electrochemistry, a parabolic-elliptic system for gravitational interaction of particles, and evolution versions of the so-called generalized Lane–Emden equations have been studied in [7], [2, Sec. 1.6], [3] and [4], [5], [15], respectively. The characteristic feature of these models, except for the last one, is an elliptic differential relation connecting u and $X(u)$ like $u = \mp\Delta\varphi$, $X(u) = \nabla\varphi(u)$ in Ω, with φ or $X(u)\cdot\nu$ prescribed on $\partial\Omega$ (the minus sign corresponds to the repulsion case, and the plus sign to the attraction case). The equations (1.5)–(1.6) in [15] are more general and usually cannot be reduced to a parabolic-elliptic system of partial differential equations. Wolansky gives in [15] a physical derivation of them from kinetic equations of Vlasov–Fokker–Planck type:

$$f_t = -\nabla_x\cdot(vf) + \nabla_v\cdot(Xf) + \beta\nabla_v\cdot(vf + \nabla_v f)$$

describing the evolution of densities $f = f(x,v,t)$ (in the phase space $\{(x,v)\in\Omega\times\mathbb{R}^n\}$) of particles subjected to a frictional, velocity dependent force with a random fluctuation. Equations of this type go back to Jeans (1915) for the gravitational attraction of particles in astrophysics, and to Vlasov (1938) for the electrostatic interaction in plasma. Suppose the initial distribution is Maxwellian in velocities: $f(x,v,0) = cu_0(x)\exp(-v^2/2)$. It is expected (see [15]) that in the adiabatic limit of large friction $\beta\to\infty$ the distribution function $f(x,v,t)$ converges in probability to $cu(x,t)\exp(-v^2/2)$ with $u = u(x,t) = \int_\Omega f(x,v,t)\,dv$ satisfying (1). The natural reflection condition for $f(x,v,t)$ at $x\in\partial\Omega$ is then translated into the no-flux boundary condition (2).

The time independent solutions of (1)–(2) can be obtained from the equation

$$\varphi = M\Big(\int_\Omega \exp(-(\varphi+V))\Big)^{-1} J(\exp(-(\varphi+V)))\,, \tag{5}$$

which is called the *generalized Lane–Emden equation.* Here $\varphi(x) = J(u)(x) =$

$\int_\Omega K(x,y)u(y)\,dy$ is the potential induced by u, $\int_\Omega u = M > 0$ is the total mass confined to Ω, and $\mathcal{K}(u) = \nabla J(u)$ in the representation (4) of $X(u)$.

The nonlinear integral equation (5) is equivalent to (1)–(2) when $u = u(x)$ does not depend on t. In particular, the no-flux condition (2) is encoded in the proper normalization $\lambda = M(\int_\Omega \exp(-(\varphi+V)))^{-1}$ in $u = \lambda\exp(-(\varphi+V))$, leading of course to (5) (for details see Section 3, (11), (12)).

The equation (5) is studied in [15] in the one-dimensional case mainly. The phenomenon of nonexistence of solutions for certain $M > 0$ in the two- and three-dimensional case is called there the gravitational collapse. There is a conjecture (6.2) in [15] that in the two-dimensional case for small $M > 0$ there exists a unique solution of (5). The existence and regularity of solutions to the evolution problem (1)–(3) have not been treated in [15].

In the papers [3], [4], [10], [11], [12] the questions of existence of solutions of parabolic-elliptic systems (both local and global in time), their uniqueness, regularity, and convergence to steady states as time tends to infinity have been studied, together with the existence and multiplicity of these steady solutions. Moreover, it has been shown that the nonexistence of stationary solutions for large mass may lead to a finite time blow-up phenomenon for the evolution problem (cf. [5]).

In this note, we develop some ideas (mainly from [3], [4], [10], [12]) to prove the *local* in time existence and regularity of solutions to (1)–(3), and the existence of the stationary solutions of (5) for small $M > 0$ in the n-dimensional case. The uniqueness of solutions to (5) with sufficiently small $M > 0$ is shown in the two-dimensional case. Nonexistence of solutions to (5) is proved for the n-dimensional gravitational case in a ball, with $M > 0$ large enough. Thus, we give in particular an affirmative answer to Conjecture 6.2 in [15], a generalization (for all $n \ge 2$ and certain potentials V) of the result in [15, Corollary 4.1], and we establish rigorously the existence of nonstationary solutions taken for granted in [15]. Of course, the cases $n = 2$, $n = 3$ deserve more attention as they correspond to a direct physical interpretation. In our framework we will not distinguish the type of interaction (repulsion–attraction) described by $X(u)$ (only average size conditions will be imposed on $X(u)$), hence the *global* in time existence of solutions—expected (and proved in certain cases in [3], [4], [11]) in the repulsion case, and generally not expected in the attraction case (cf. [5])—will not be considered here. We will touch neither on the questions of the existence of stationary solutions for *all* $M > 0$, nor on their multiplicity for some $M > 0$, because of similar reasons.

We note that the boundary condition (2) causes some technical difficulties (cf. e.g. [7] where only linear boundary conditions are treated), since some maximum principle arguments may fail for solutions of (1)–(2). We also remark that the stationary solutions (5) (in a particular case when the kernel K of J is symmetric: $K(x,y)=K(y,x)$) are studied in [15] using heavily variational arguments involving the free energy functional

$$\mathcal{E}(u)=\int_\Omega u\log u+\frac{1}{2}\int_\Omega J(u)u+\int_\Omega uV\,, \tag{6}$$

while our approach is based on compactness properties of the right hand side of (5). Modifications of (6) play the important role of Lyapunov functionals controlling efficiently the size of solutions in the repulsion case (cf. [7], [3], [4]). In the attraction case the contribution of the second term on the right hand side of (6) is very negative, hence in general $\mathcal{E}(u)$ cannot control the quantity u in a reasonable way.

We use largely the notation and results of papers [3], [4] relevant to the study of the system (1)–(3). In particular, we use the standard notation $|u|_p$ for the $L^p(\Omega)$ norms of functions, and $\|u\|_s$ for the $H^s(\Omega)$ norms. The constants independent of functions defined on Ω will be denoted generically by C, even if they may vary from line to line. For various Sobolev imbeddings interpolation inequalities we refer to [1], [6] and [8].

2. Evolution problem. This section is devoted to a proof of the local existence of solutions for the problem (1)–(3) in the case when $X(u)$ is a sublinear vector field, i.e. the nonlinearity in the equation (1) is at most quadratic. Our assumptions read:

(A) Ω is a bounded open subset of $\mathbb{R}^n$ with $C^{1+\varepsilon}$ boundary $\partial\Omega$ for some $\varepsilon>0$.

If $n=2,3$ the vector field $X(u)$ and its derivative satisfy estimates of the form

$$|X(u)|_1\le C(|u|_2+1)\quad\text{and}\quad |DX(u)|_2\le C(|u|_2+1) \tag{B_2}$$

with C independent of $u\in L^2(\Omega)$, and more generally for $n\ge 2$,

$$|X(u)|_1\le C(|u|_p+1)\quad\text{and}\quad |DX(u)|_p\le C(|u|_p+1) \tag{B_p}$$

with some $n/2<p\le n$, and C independent of $u\in L^p(\Omega)$.

Alternatively, for $n\ge 2$ and some $n<p<\infty$ we may assume

$$|X(u)|_\infty\le C(|u|_p+1). \tag{$C_{p,\infty}$}$$

Observe that an immediate consequence of (B_p) is the condition

$$|X(u)|_q\le C(|u|_p+1)\quad\text{with } 1/q=1/p-1/n\,, \tag{$C_{p,q}$}$$

a counterpart of $(C_{p,\infty})$ above for small p. $(C_{p,q})$ follows from the Sobolev imbedding theorem combined with the Poincaré inequality (applicable since the average value of $X(u)$ is controlled by $|u|_p$ in (B_p)).

For the uniqueness of solutions assume either

$$(\varDelta C_2)\qquad |X(u)-X(v)|_6 \le C|u-v|_2 \quad \text{for } n\le 3\,,$$

or

$$(\varDelta C_p)\qquad |X(u)-X(v)|_\infty \le C|u-v|_p \quad \text{for } n\ge 2,\ p>n\,.$$

We define a *weak $H^1(\Omega)$ solution* of the problem (1)–(3) on $\Omega\times(0,T)$ to be a function $u\in L^\infty((0,T);L^2(\Omega))\cap L^2((0,T);H^1(\Omega))$ satisfying for each test function $\eta\in H^1(\Omega\times(0,T))$ and for a.e. $t\in(0,T)$ the integral identity

$$(D)\qquad \int_\Omega u(x,t)\eta(x,t)\,dx - \int_0^t\int_\Omega u\eta_t + \int_0^t\int_\Omega (\nabla u + uX(u))\cdot\nabla\eta = \int_\Omega u_0(x)\eta(x,0)\,dx\,.$$

This definition coincides with standard definitions of weak solutions of (linear) initial-boundary value problems in [13, Ch. III, Secs. 1, 4, 5] when the no-flux condition (2) is to be satisfied, and the vector field $X=X(u)$ is determined by u itself (i.e. X is a *self-consistent field* in physical terminology). Observe that this definition can be modified to that of *weak $W^{1,p}(\Omega)$ solutions* of (1)–(3) (with $p>n$), when stronger conditions on u are imposed: $u\in L^\infty((0,T);L^p(\Omega))\cap L^p((0,T);\,W^{1,p}(\Omega))$, and a larger set of test functions is admitted: $\eta\in W^{1,p'}(\Omega\times(0,T))$, $1/p+1/p'=1$, similarly to the case of parabolic equations and systems in the framework of [2].

It will be seen from the proof of Theorem 1 that $u_t\in L^2((0,T);H^{-1}(\Omega))$, hence the energy (in)equality

$$(7)\qquad \frac12\int_\Omega u^2(x,t)\,dx + \int_0^t\int_\Omega (\nabla u + uX(u))\cdot\nabla u = \frac12\int_\Omega u_0^2(x)\,dx$$

holds for all $t\in[0,T]$. Its proof begins with showing (7) for a.e. $t\in(0,T)$, and then by the continuity of $u\in C([0,T];L^2(\Omega))$ (cf. [13, Ch. III, Th. 5.1]) for all $t\in[0,T]$ (see [4, (9), (10)]). By abuse of language we will write (7) in the differential form

$$(8)\qquad \frac12\frac{d}{dt}|u|_2^2 + |\nabla u|_2^2 = -\int_\Omega uX(u)\cdot\nabla u$$

whose *formal* derivation consists in multiplying (1) by u and integrating by parts. In the sequel certain integral inequalities following from (D) will be written formally as differential inequalities, but we will understand them properly, in integral form.

THEOREM 1. *Assume* (A) *and either* (B_2) *for* (i), *or* $(C_{p,\infty})$ *for* (ii), *or* (B_p) *for* (iii), *or* (B_p) *and* $(C_{p,\infty})$ *for* (iv).

(i) *If* $n = 2, 3$, $p = 2$ *and* $u_0 \in L^2(\Omega)$ *then there exist* $T = T(|u_0|_2) > 0$ *and a weak solution* u *of* (1)–(3) *belonging to* $L^\infty((0, T); L^2(\Omega)) \cap L^2((0, T); H^1(\Omega))$.

(ii) *If* $n \ge 2$ *and* $u_0 \in L^p(\Omega)$, $p > n$, *then there is* $T = T(p, |u_0|_p) > 0$ *and a weak solution* u *such that* $u \in L^\infty((0, T); L^p(\Omega))$, *moreover*, $u^{p/2} \in L^2((0, T); H^1(\Omega))$.

If additionally either (ΔC_2) *in* (i), *or* (ΔC_p) *in* (ii) *is assumed*, *then these solutions are unique.*

(iii) *If* $n \ge 2$, $p > n/2$ *and* $u_0 \in L^p(\Omega)$, *then the conclusion of* (ii) *holds true with this* p *and some* $T = T(p, |u_0|_p) > 0$.

(iv) *If* $u_0(x) \ge 0$ *then* $u(x, t) \ge 0$ *a.e.*

The weak solutions in (i)–(iii) *are regular in the sense that* $u \in L^\infty_{\mathrm{loc}}((0, T); L^\infty(\Omega))$.

Before proving this theorem we give some examples that satisfy our set of assumptions.

EXAMPLES. 1. Suppose that $X(u) = \nabla\varphi$ where $\varphi = \varphi(u)$ satisfies the Poisson equation $\mp\Delta\varphi = u$ and either the Dirichlet condition $\varphi = \phi_1$, or the Neumann condition $\partial\varphi/\partial\nu = \phi_2$ (with a proper normalization of ϕ_2), or the Robin condition $\partial\varphi/\partial\nu + \sigma\varphi = \phi_3$ at the boundary $\partial\Omega$, and ϕ_1, resp. ϕ_2, resp. σ, ϕ_3 are bounded in time (in particular, $\phi_1 = \text{const}$ is admissible). Problems like these are studied in [3]–[5], [7].

In this case the validity of conditions (B_p), $(C_{p,q})$, $(C_{p,\infty})$ results from the properties of weak solutions of the Poisson equation (cf. [6, Vol. 1, Ch. 2]). Namely, $|D^2\varphi|_p \le C|\Delta\varphi|_p \le C|u|_p$ is the Calderón–Zygmund inequality, $|\nabla\varphi|_1 \le C(|u|_p + 1)$ follows from the boundary condition imposed, and $|\nabla\varphi|_q \le C(|u|_p + 1)$ with $1/q = 1/p - 1/n$ is a consequence of the Sobolev theorem.

2. The same applies, of course, to $\varphi(u) = E * u$, where E is the fundamental solution of the Laplacian: $E = E_n(x) = -((n-2)\sigma_n)^{-1}|x|^{2-n}$ for $n > 2$, $E_2(x) = (2\pi)^{-1}\log|x|$. This corresponds to equations considered in [15].

3. More generally, if $\varphi(u) = \int_\Omega K(x, y)u(y)\,dy$, where the kernel K satisfies the condition that D^2K are sums of singular Calderón–Zygmund kernels (cf. [6, Vol. 4]) and weakly singular kernels, then $X(u) = \nabla\varphi(u)$ satisfies all the assumptions (B_p).

4. If the external potential $V = V(\cdot, t)$ is in $C^1(\overline{\Omega})$ for each t and satisfies $\sup_t |D^2V(t)|_p < \infty$, then ∇V can be added to $X(u)$ in each of the examples listed above (with the same p).

5. Equations with memory on $[0,t]$ where $X(u)=\int_0^t Y(\tau,u(\tau))\,d\tau$, and suitable conditions on $Y(\tau,u)$ are imposed, also fit into our framework.

6. The methods in this paper also extend to quasilinear parabolic systems with $u=(u_1,\dots,u_N)$, and $X(u)$ as in Example 1 above, similar to those considered in electrochemistry (cf. [2, Sec. 1.6] and [4]).

Proof of Theorem 1. (i) For fixed $T>0$, let the space $\mathcal{X}=L^4((0,T);\,L^2(\Omega))$ be endowed with the norm

$$|||u|||=\Big(\int_0^T\Big(\int_\Omega |u(x,t)|^2\,dx\Big)^2\,dt\Big)^{1/4}.$$

Beginning with an element $\overline{u}\in\mathcal{X}$ we consider the weak solution u of the auxiliary *linear* problem

$$\begin{gathered}u_t=\Delta u+\nabla\cdot(uX(\overline{u}))\quad \text{in } \Omega\times(0,T),\\ (\nabla u+uX(\overline{u}))\cdot\nu=0\quad \text{on } \partial\Omega\times(0,T),\\ u(x,0)=u_0(x),\end{gathered}$$

where $X(\overline{u})=X(\overline{u}(x,t))$ is defined for a.e. $t\in(0,T)$, $X(\overline{u})$ depends measurably on t, and since (B_2) implies the condition $(C_{p,q})$:

$$|X(\overline{u})|_q\le C(|\overline{u}|_2+1)\quad \text{with } q=6 \text{ if } n=3, \text{ and } q<\infty \text{ if } n=2,$$

we have

$$\int_0^T\Big(\int_\Omega |X(\overline{u})|^q\,dx\Big)^{4/q}\,dt\le C(|||\overline{u}|||^4+1),$$

so $X(\overline{u})\in L^4((0,T);\,L^q(\Omega))$.

The second part of (B_2) implies the existence of the trace $X(\overline{u})\cdot\nu$ on $\partial\Omega\times(0,T)$, which satisfies $X(\overline{u}(t))\cdot\nu\in H^{1/2}(\partial\Omega)\subset L^4(\partial\Omega)$ for a.e. $t\in(0,T)$. Moreover, by a similar argument to that for $X(\overline{u})$, $X(\overline{u})\cdot\nu\in L^4((0,T);\,L^4(\partial\Omega))$.

Now, the solvability conditions for linear equations in [13, Ch. III, Secs. 4, 5] are satisfied for the above problem since $X(\overline{u})\in L^r((0,T);\,L^q(\Omega))$ with $1/r+n/(2q)\le 1/2$ (here $n\le 3$, $q=6$, $r=4$), and $X(\overline{u})\cdot\nu\in L^{r'}((0,T);\,L^{q'}(\partial\Omega))$ with $1/r'+(n-1)/(2q')\le 1/2$ (here $n\le 3$, $q'=4$, $r'=4$).

The energy (in)equality for the *linear* problem just solved reads (cf. [13, Ch. III, Sec. 2])

$$\begin{aligned}\text{(9)}\qquad \frac12\frac{d}{dt}|u|_2^2+|\nabla u|_2^2&\le\Big|\int_\Omega u\nabla u\cdot X(\overline{u})\Big|\le|\nabla u|_2|u|_3|X(\overline{u})|_6\\ &\le C|\nabla u|_2\|u\|_1^{1/2}|u|_1^{1/2}(|\overline{u}|_2+1)\\ &\le\frac12\|u\|_1^2+C|u|_2^2(|\overline{u}|_2+1)^4.\end{aligned}$$

As a consequence we obtain

$$|u(T)|_2^2 + \int_0^T |\nabla u(t)|_2^2\,dt \le |u_0|_2^2 \exp\Big(C\int_0^T (|\overline{u}(t)|_2+1)^4\,dt\Big)$$
$$\le |u_0|_2^2 \exp(C(|||\overline{u}|||^4+1))\,,$$

and $|||u||| \le T^{1/4}|u_0|_2 \exp(C(|||\overline{u}|||^4+1))$. Consequently, taking a sufficiently small $T>0$ and $R>0$ large enough, the image of the ball $B_R = \{\overline{u}\in\mathcal{X} : |||\overline{u}||| \le R\}$ under the operator $\mathcal{N}(\overline{u}) = u$ is contained in this ball. It is standard to verify (cf. [3], [4]) that $\mathcal{N}:\mathcal{X}\to\mathcal{X}$ is continuous, $\int_0^T \|\frac{d}{dt}u(t)\|_{-1}^2\,dt < \infty$, and the closure of $\mathcal{N}(B_R)$ is compact in $\mathcal{X}$ (by the Aubin–Lions lemma, [14]). The inequality above is proved by applying to (1) with $X = X(\overline{u})$ test functions from $H^1(\Omega)$ (independent of t).

The Schauder fixed point theorem assures that $\mathcal{N}(u) = u$ for some $u\in\mathcal{X}$, hence u solves (1)–(3) in the sense of the definition (D).

Concerning the uniqueness of weak solutions to (1)–(3) under the assumption (ΔC_2), consider two such solutions, say u and v. The difference $w = u - v$ satisfies

$$w_t = \Delta w + \nabla\cdot(uX(u)) - \nabla\cdot(vX(v))$$

so we obtain (analogously to (7))

$$\frac{1}{2}\frac{d}{dt}|w|_2^2 + |\nabla w|_2^2 \le \Big|\int_\Omega (uX(u)\cdot\nabla w - vX(v)\cdot\nabla w)\Big|$$
$$\le \int_\Omega |w\nabla w\cdot X(u)| + \int_\Omega |v\nabla w\cdot(X(u)-X(v))|$$
$$\le |\nabla w|_2(|w|_3|X(u)|_6 + |v|_3|u-v|_2)$$
$$\le \frac{1}{4}|\nabla w|_2^2 + C\|w\|_1|w|_2(|u|_2+1)^2 + C\|v\|_1|v|_2|w|_2^2$$
$$\le \frac{1}{2}|\nabla w|_2^2 + \alpha(t)|w|_2^2\,,$$

where $\int_0^T \alpha(t)\,dt < \infty$ from the properties of the solutions u, v. Consequently, this leads to a Gronwall type inequality $\frac{d}{dt}|w|_2^2 \le \alpha(t)|w|_2^2$, so the uniqueness of solutions follows since $w(0) = 0$.

(ii) The scheme of proof is completely analogous to that in (i) with (B_2) replaced by $(C_{p,\infty})$, and (ΔC_2) by (ΔC_p).

(iii) The proof follows by a standard approximation argument described in full details (for a related but slightly different problem) in [4]. Let us only recall the crucial estimate ([4, (19)])

$$(10)\qquad |u(t)|_p^p + \int_0^t |\nabla(|u|^{p/2}(\tau))|_2^2\,d\tau \le \exp\Big(Cp^2 \int_0^t |X(u(\tau))|_q^{2/(1-n/q)}\,d\tau\Big)|u_0|_p^p$$

valid for $p > n/2$ and any $q > n$. This implies that for $T > 0$ sufficiently small all the approximating solutions with initial data in $L^{p_*}(\Omega)$, $p_* > n$, which approach u_0 in $L^p(\Omega)$ norm, $p > n/2$, exist on the whole interval $[0,T]$, and they converge in $L^p(\Omega)$ to a weak solution of (1)–(3).

(iv) The positivity of u under the assumption $u_0(x) \ge 0$ follows as in [3].

The regularity of weak solutions in (i), (ii), or those constructed in (iii), can be proved using the Moser iteration technique adapting with minor modifications the proof of Theorem 2(iii) of [4]. Here the condition $(C_{p,\infty})$ is used in the derivation of a counterpart of (10).

3. Stationary solutions. The assumptions on the vector field X in this section are weaker than those in Section 2, except for the structure of $X(u) = \nabla\varphi(u) + \nabla V$, i.e. X is derived from the potential $\varphi = \varphi(u)(x) = \int_\Omega K(x,y)u(y)\,dy$ generated by u and from the external potential $V = V(x)$.

As has been remarked in the introduction, stationary solutions U, $\Phi = \varphi(U)$ of (1)–(2) satisfy the integral equation (cf. (5))

$$(11)\qquad \Phi = M\Big(\int_\Omega \exp(-(\Phi+V))\Big)^{-1} J(\exp(-(\Phi+V)))\,.$$

Indeed, $\nabla\cdot(\nabla U + UX) = 0$ or $\nabla\cdot(\exp(-(\Phi+V))\nabla(\exp(\Phi+V)U)) = 0$ leads to $\int_\Omega \exp(-(\Phi+V))|\nabla(\exp(\Phi+V)U)|^2 = 0$, and

$$(12)\qquad U = \lambda\exp(-(\Phi+V))$$

with the normalizing constant $\lambda = M(\int_\Omega \exp(-(\Phi+V)))^{-1}$, $M = \int_\Omega U$. Applying to both sides of (12) the integral operator

$$J(U) = \int_\Omega K(x,y)U(y)\,dy$$

we obtain (11).

Note that if the kernel K is symmetric: $K(x,y) = K(y,x)$, the equation (12) is an immediate consequence of the following identity for the functional $\mathcal{E}$ of (6):

$$\frac{d}{dt}\mathcal{E} + \int_\Omega u|\nabla(\log u) + X|^2 = 0$$

valid for the weak solutions of (1)–(3) constructed in Theorem 1. To see this we calculate formally

$$\frac{d}{dt}\bigg(\frac{1}{2}\int_\Omega J(u)u\bigg)=\frac{d}{dt}\bigg(\frac{1}{2}\int\int_{\Omega\times\Omega} K(x,y)u(x)u(y)\,dx\,dy\bigg)=\int_\Omega \varphi(u)u_t$$

and $\frac{d}{dt}\mathcal{E}=\int_\Omega(\log u+\varphi+V)u_t$. This computation is made rigorous by approximating u by $u+\delta$ ($\delta>0$), and passing to the limit $\delta\to 0$ (cf. [3]). Then U satisfying (12) is a weak solution of the equation (1) (together with the boundary condition (2)) independent of time. We indicate that for symmetric positive definite kernels (generalizing the electrostatic case) the uniqueness of stationary solutions for arbitrary $M>0$ (a result of F. Bavaud cited in [15]) follows from the convexity of the functional $\mathcal{E}$ (see also Appendix). The gravitational case is more delicate, and uniqueness of solutions is expected for small $M>0$ only (see Theorem 2(iii) below).

THEOREM 2. *Let Ω be a bounded open subset of $\mathbb{R}^n$, $\ell=|V|_\infty<\infty$.*

(i) *Suppose that for some $r>1$,*

$$k=\sup_{x\in\Omega}\Big(\int_\Omega |K(x,y)|^r\,dy\Big)^{1/r}<\infty\,.$$

Then there exists $M_1>0$ such that for each $M\in(0,M_1)$ the equation (11) *has a solution $\Phi\in L^\infty(\Omega)$.*

(ii) *Suppose that $|\nabla V|_\infty<\infty$, and for some $r>1$ and $\beta\in(0,1]$ the kernel K satisfies*

$$\sup_{x\in\Omega}\int_\Omega (|K(x,y)|\,|x-y|^{-\beta})^r\,dy<\infty\,,$$

$$\sup_{x\in\Omega}\int_\Omega (|\nabla_x K(x,y)|\,|x-y|^{1-\beta})^r\,dy<\infty\,.$$

Then for each $M\in(0,M_2)$ with $M_2>0$ small enough, there exists a solution $\Phi\in C^\beta(\Omega)$ of (11) (*i.e. Φ satisfies the Hölder condition of order β; $\beta=1$ corresponds to the Lipschitz condition*: $\Phi\in\mathrm{Lip}(\Omega)$).

(iii) *If $\Omega\subset\mathbb{R}^2$ is bounded, and the kernel K satisfies an estimate of the form*

$$|K(x,y)|\le k_1(|\log|x-y||+1)$$

for some $k_1>0$, then there exists $M_3>0$ such that for all $M\in(0,M_3)$ stationary solutions of (11) *are unique in $L^\infty(\Omega)$.*

(iv) *If $\Omega=B=B(0,R)$ is the ball of radius R in $\mathbb{R}^n$, $n\ge 2$, $V\in C^1(\overline{B})$, and either*

$K=E_n$ is the fundamental solution of the Laplacian in $\mathbb{R}^n$,
V and Φ are radial functions,

or

K is the Green function of B and V is not necessarily radial,

then for sufficiently large $M > 0$ stationary solutions of (11) *cannot exist in* $L^\infty(\Omega)$.

We remark that examples given after the formulation of Theorem 1 also satisfy the assumptions of Theorem 2(i), (ii).

EXAMPLES. I. If $|V|_\infty < \infty$ and $|K(x,y)| \le C|x-y|^{-n+\gamma}$ for some $\gamma > 0$, then the assumptions of (i) are obviously satisfied with $r \in (1, n/(n-\gamma))$. In classical situations (Examples 1–3 illustrating Theorem 1) we can take e.g. $\gamma \in (1,2)$ or even $\gamma = 2$ for $n \ge 3$.

II. If $|\nabla V|_\infty < \infty$ and $|\nabla_x K(x,y)| \le C|x-y|^{-n-1+\gamma}$ for some $\gamma > 0$, then the validity of the hypotheses of (ii) follows. Indeed, we obtain $|K(x,y)| \le C|x-y|^{-n+\gamma}$, so (ii) holds for $\beta < \gamma$, $\beta \in (0,1]$, and any $r \in (1, n/(n+\beta-\gamma))$. Again we have $\gamma \in (1,2)$ in Examples 1–3.

III. The assumptions in (iii) are satisfied e.g. in the two-dimensional gravitational case when either $K = E_2$ or K is any of the kernels corresponding to the boundary conditions in Example 1, $X(U) = \nabla\Phi + \nabla V$, $\Delta\Phi = U$. In fact, these kernels are bounded from above and have a singularity $(2\pi)^{-1}\log|x-y|$ as $x \to y$. Of course, part (i) of Theorem 2 can be applied in (iii) with an arbitrary $r < \infty$.

P r o o f o f T h e o r e m 2. (i) For $\Psi = -(\Phi + V)$, (11) assumes the form

$$\Psi = -M\Big(\int_\Omega \exp\Psi\Big)^{-1} J(\exp\Psi) - V =: \mathcal{T}(\Psi)\,. \tag{13}$$

The nonlinear integral operator $\mathcal{T}$ is well defined for $\Psi \in L^\infty(\Omega)$. It is easy to see that for $|\Psi|_\infty \le R$, $R > 0$, the estimate

$$|\mathcal{T}(\Psi)|_\infty \le Me^R|\Omega|^{-1}k|\Omega|^{1/r'}e^R + \ell$$

holds, where $|\Omega|$ = volume of Ω, $1/r + 1/r' = 1$. Taking $R > \ell$ and $M > 0$ small enough (e.g. $R = \ell + 1/2$, $M \in (0, M_1)$ with $M_1 = |\Omega|^{1/r}(2k)^{-1}e^{-2\ell-1}$) we obtain $|\mathcal{T}(\Psi)|_\infty \le R$.

Moreover, the operator J with kernel K is compact from $L^\infty(\Omega)$ into $L^\infty(\Omega)$ (see [9, Ch. XI, Sec. 3, Ths. 1, 3]). This, together with the continuity of $\mathcal{T}$, allows us to apply the Schauder fixed point theorem, and find a function $\Psi = \mathcal{T}(\Psi)$ solving our problem.

(ii) We can again apply the Schauder fixed point theorem to the nonlinear operator $\mathcal{T}$ in (13). In this situation the linear operator J with kernel K is compact from $L^\infty(\Omega)$ into $C^\alpha(\Omega)$ for every $\alpha \in (0,\beta)$ (cf. [9, Ch. XI, Sec. 3, Th. 4]). We skip the details of this standard reasoning. Note that further regularity of solutions Φ can be obtained under suitable assumptions on the derivatives of V and smoothness of the kernel K off the diagonal

$\{(x,x) : x \in \Omega\}$. Let us also remark that regularity properties of Φ near the boundary $\partial\Omega$ can be derived from the boundary behavior of the kernel K.

Our final remark concerns the equation (12) for the stationary density U. The existence of solutions to (12) can also be proved with the use of the Schauder theorem. The advantage of the approach with (12) instead of (11) lies in a simpler way to prove regularity of solutions (Φ is a priori more smooth than U). But establishing compactness properties of the right hand side of (12) requires additional assumptions on either translations or derivatives of K, i.e. on its average smoothness.

(iii) Concerning the uniqueness of solutions to (11) we begin with a general computation in the framework of (i). Consider two solutions Φ_1, Φ_2, and the corresponding Ψ_1, Ψ_2 with $R = \max(|\Psi_1|_\infty, |\Psi_2|_\infty)$. Since

$$\mu_1\mu_2(\Psi_1 - \Psi_2) = M(\mu_1 J(\exp\Psi_2) - \mu_2 J(\exp\Psi_1)),$$

where $\mu_i = \int_\Omega \exp\Psi_i \in [e^{-R}|\Omega|, e^R|\Omega|]$, $i = 1,2$, we can write

$$\begin{aligned}
\mu_1\mu_2 e^{-R}|\exp\Psi_1 - \exp\Psi_2|_\infty &\le \mu_1\mu_2|\Psi_1 - \Psi_2|_\infty \\
&\le M\mu_1|J(\exp\Psi_2 - \exp\Psi_1)|_\infty + M|\mu_1 - \mu_2||J(\exp\Psi_1)|_\infty \\
&\le M\mu_1 k|\Omega|^{1/r'}|\exp\Psi_1 - \exp\Psi_2|_\infty \\
&\quad + M|\Omega|\,|\exp\Psi_1 - \exp\Psi_2|_\infty k|\Omega|^{1/r'}e^R \\
&\le 2Mk|\Omega|^{1+1/r'}e^R|\exp\Psi_1 - \exp\Psi_2|_\infty .
\end{aligned}$$

Then the inequality

$$|\Omega|^{1/r}e^{-4R}|\exp\Psi_1 - \exp\Psi_2|_\infty \le 2Mk|\exp\Psi_1 - \exp\Psi_2|_\infty$$

shows that $\Psi_1 = \Psi_2$ in $L^\infty(\Omega)$ if we have an a priori bound on $|\Psi|_\infty$ and $M > 0$ is sufficiently small. So, under the assumptions (iii) we should prove a uniform bound valid for all the solutions of (11) when $M \in (0, M_3)$, for suitably small $M_3 > 0$.

Observe that an analogous computation shows that $\mathcal{T}$ is a contraction in the ball $\{|\Psi| \le R\} \subset L^\infty(\Omega)$ for sufficiently small $M > 0$ provided $\sup_{x\in\Omega}\int_\Omega |K(x,y)|\,dy < \infty$, but the reasoning in (i) has implied the existence for a larger range of M's.

First, let us estimate $\mu = \int_\Omega \exp(-(\Phi + V))$ from below. Evidently, the Cauchy–Schwarz inequality gives

$$|\Omega|^2 \le \mu \int_\Omega \exp(\Phi + V) \le \mu e^\ell \int_\Omega \exp|\Phi|,$$

so $\mu^{-1} \le C(\Omega, V)\int_\Omega \exp|\Phi|$.

Now we recall the Jensen inequality:

$$\exp\Big(|f|_1^{-1}\int_\Omega fg\Big) \le |f|_1^{-1}\int_\Omega f\exp g\,, \tag{14}$$

which we shall use with $f = M\mu^{-1}\exp(-(\Phi+V)) \ge 0$ (so $J(f) = \Phi$ from (11), $|f|_1 = M$) and

$$g = g(x-y) = \varepsilon k_1(|\log|x-y|| + 1) \le \varepsilon k_1(\log(|x-y|^{-1}) + 2\log d + 1)\,,$$

where $\varepsilon \in (0, 2/k_1)$, $d = \max(1, 2\operatorname{diam}(\Omega))$.

Integrating (14) over Ω we obtain

$$\begin{aligned}\int_\Omega \exp(\varepsilon M^{-1}|\Phi|) &\le \int_\Omega \exp\Big(|f|_1^{-1}\int_\Omega fg\Big)\\ &\le \int_\Omega |f|_1^{-1}\Big(\int_\Omega f(y)(d^2e)^{\varepsilon k_1}|x-y|^{-\varepsilon k_1}\,dy\Big)\,dx\\ &= C\int_\Omega |f|_1^{-1}f(y)\Big(\int_\Omega |x-y|^{-\varepsilon k_1}\,dx\Big)\,dy\\ &\le C\int_\Omega |f|_1^{-1}f(y)\Big(\int_{B(0,d)} |x|^{-\varepsilon k_1}\,dx\Big)\,dy\\ &= C\,2\pi(2-\varepsilon k_1)^{-1}d^{2-\varepsilon k_1} = C(\Omega, V, \varepsilon) < \infty\,.\end{aligned}$$

Choosing $M_* = \varepsilon$ we have $\varepsilon M^{-1} = s > 1$ for each $M \in (0, M_*)$, hence the above inequality gives $|\exp|\Phi||_s^s = \int_\Omega(\varepsilon M^{-1}|\Phi|) \le C < \infty$.

Finally, this better integrability of $\exp(-\Phi)$ leads to $(1/s + 1/s' = 1)$

$$\begin{aligned}|\Phi|_\infty &\le M\mu^{-1}|J(\exp(-(\Phi+V)))|_\infty\\ &\le C(\Omega, V)|\exp|\Phi||_1\Big(\sup_{x\in\Omega}\int_\Omega |K(x,y)|^{s'}\,dy\Big)^{1/s'}|\exp|\Phi||_s\\ &\le C(\Omega, V, k_1, \varepsilon, s) < \infty\,.\end{aligned}$$

The a priori estimate for Φ in $L^\infty(\Omega)$, hence for $|\Psi|_\infty \le |\Phi|_\infty + \ell$ (guaranteeing the uniqueness), is proved.

Note that the above proof is essentially that of the Moser–Trudinger inequality (e.g. [8, Secs. 7.8, 7.9]) in two dimensions.

(iv) In this situation the equation (11) implies

$$\Delta\Phi = M\mu^{-1}\exp(-(\Phi+V))\,, \qquad \mu = \int_\Omega \exp(-(\Phi+V))\,. \tag{15}$$

For a radially symmetric potential $\Phi = \text{const}$ on $\partial B = \{|x| = R\}$. The equation (15) is invariant under translations in Φ, hence we may assume

$\Phi_{|\partial B} = 0$. Applying the Pokhozhaev identity (cf. [10] or [12], where nonexistence of solutions has also been proved in some particular cases) we obtain

$$R\int_{\partial B}\left(\frac{\partial\Phi}{\partial\nu}\right)^2 = M\mu^{-1}\int_B e^{-V}((e^{-\Phi}-1)(-2\nabla V\cdot x+2n)+(n-2)\Phi e^{-\Phi}). \tag{16}$$

Since $\Phi\le 0$, the right hand side of (16) can be estimated by a linear function of M: CM with $C=C(n,V)$. Due to $\int_{\partial B}\partial\Phi/\partial\nu = M$, we have

$$M^2\le CR^{n-2}\int_{\partial B}\left(\frac{\partial\Phi}{\partial\nu}\right)^2 R\,,$$

which implies that (16) cannot be satisfied for sufficiently large M: $M\in(M_4,\infty)$.

Similarly, if K is the Green function and $V\in C^1(\overline{B})$ (not necessarily radial), then the above arguments prove that solutions to (15) cannot exist for large M.

An inspection of the proofs in (iii), (iv) shows that if $K=E_2$ then $M_*=4\pi$ (for any $V\in L^\infty(\Omega)$) and $M_4=8\pi$ (when $V=0$) do work.

Appendix. For completeness of exposition we present a concise proof of the uniqueness of solutions to (11) for arbitrary $M>0$ when the symmetric kernel K is positive definite.

Let Φ_i, $i=1,2$, be two solutions of (11), and $\Psi_i=-\Phi_i-V$, $\nu_i=\log(\int_\Omega\exp\Psi_i)$, $i=1,2$. We multiply the difference of the equations (13), i.e. (11) written for Ψ_i,

$$\Psi_2-\Psi_1 = MJ(\exp(\Psi_1-\nu_1)-\exp(\Psi_2-\nu_2))$$

by $w=\exp(\Psi_1-\nu_1)-\exp(\Psi_2-\nu_2)$ and integrate over Ω:

$$\int_\Omega(\Psi_2-\Psi_1)(\exp(\Psi_1-\nu_1)-\exp(\Psi_2-\nu_2)) = M\iint_{\Omega\times\Omega}K(x,y)w(x)w(y)\,dx\,dy\ge 0\,.$$

Since $\int_\Omega w(x)\,dx=0$ we also have

$$\int_\Omega((\Psi_2-\nu_2)-(\Psi_1-\nu_1))(\exp(\Psi_1-\nu_1)-\exp(\Psi_2-\nu_2))\ge 0\,.$$

From the monotonicity of the exponential function $\Psi\mapsto\exp\Psi$, the integrand is not positive for all $x\in\Omega$, so $w\equiv 0$, and consequently $\Psi_1=\Psi_2$.

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