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THREE METHODS FOR THE STUDY OF SEMILINEAR EQUATIONS AT RESONANCE

BY

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Three methods for the study of the solvability of semilinear equations with noninvertible linear parts are compared: the alternative method, the continuation method of Mawhin and a new perturbation method [22]–[27]. Some extension of the last method and applications to differential equations in Banach spaces are presented.

1. Introduction. Most of nonlinear differential, integral or, more generally, functional equations have the form

$$Lx = N(x) \tag{1.1}$$

(called semilinear) where L is a linear and N a nonlinear operator, in appropriate function spaces. Usually, L is defined on a dense linear subspace Y of a Banach space X, takes values in a Banach space Z and is a closed operator, and $N : X \to Z$ is continuous. Moreover, if L has a trivial null space $\ker L$, then L is surjective; if $\ker L$ is nontrivial, it has a finite dimension equal to the codimension of the range space $L(Y) \subset Z$ (such operators are called Fredholm of index 0). We do not study the case $\ker L = \{0\}$ — it reduces to the fixed point problem for $L^{-1}N$. The mapping N is usually compact (or contractive or monotone or A-proper or $\ldots$) and a suitable topological degree theory works and gives a large number of results. The case $\ker L \neq \{0\}$ is more complicated; we then say that equation (1.1) is *at resonance*. In this survey, we are interested only in the existence of solutions to equation (1.1) at resonance. It should be noticed, however, that there are results concerning the number of solutions (see e.g. [2], [28]) or localization of at least one solution in a given set (it is usually the cone of nonnegative functions — see [25], [29]).

We present a few examples of resonance problems. The most typical are boundary value problems for second order ordinary differential equations:

$$x'' + m^2 x = f(t, x, x'), \quad x(0) = x(\pi) = 0\,, \tag{1.2}$$

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where $m = 1, 2, \ldots$ and $f : [0, \pi] \times \mathbb{R}^2 \to \mathbb{R}$ (see [7], [13], [21]) or

$$x' = f(t, x), \qquad x(0) = x(1)\,, \tag{1.3}$$

where $f : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$ ([4] and references therein). Ordinary differential equations can be replaced by partial differential ones ([17], [31], [15]):

$$Px - \lambda_0 x = f(t, x), \qquad x|\partial\Omega = 0, \tag{1.4}$$

where P is a linear strongly elliptic differential operator of the second order on Ω with smooth coefficients, λ_0 is its first eigenvalue and $f : \mathbb{R} \times \operatorname{cl}\Omega \to \mathbb{R}$, or (see [3], [6])

$$u_{tt} - \Delta u = f(t, z, u), \quad u|\partial\Omega = 0, \quad u\ 2\pi\text{-periodic in } t, \tag{1.5}$$

where Δ is the Laplace operator, $f : [0, 2\pi] \times \operatorname{cl}\Omega \times \mathbb{R} \to \mathbb{R}$ (here, $\dim\ker L = \infty$). Equations with retarded argument or integro-differential equations can also be considered within this framework; usually the retardation or the integral are included in the nonlinear term:

$$x' = f(t, x, x_h), \qquad x(0) = x(1), \tag{1.6}$$

where $x_h(t) = x(h(t))$, h is a measurable function with $h(t) \le t$ (see [1]), or

$$x'' + x = \int_0^{\pi} K(t, s) f(s, x(s))\, ds, \qquad x(0) = x(\pi) = 0, \tag{1.7}$$

where $K : [0, \pi]^2 \to \mathbb{R}$, $f : [0, \pi] \times \mathbb{R} \to \mathbb{R}$. We see that the nonlinear operator N is a superposition operator or its composition with linear ones, so we have two possible ways for the choice of suitable Banach spaces. If f is continuous, then Z is the space of continuous functions with the sup-norm

$$\|z\| = \sup_t |z(t)|\,,$$

$X = Z$ or X is the space of C^1-functions if f also depends on the first derivative. Then the solutions of all problems are classical in the sense that the derivatives exist everywhere. The second possibility: f is a Carathéodory function, i.e. $f(\cdot, x)$ is measurable for all x, $f(t, \cdot)$ is continuous for a.e. t, and f satisfies some growth condition

$$|f(t, x)| \le a|x|^{\varrho} + b(t)$$

with $a \ge 0$, $\varrho \in (0, 1]$ and $b \in L^r$. These assumptions ensure that

$$N(x)(t) = f(t, x(t))$$

maps $X = L^p$ into $Z = L^q$ with suitable p, q, r, ϱ (see [10]). If f depends on derivatives, these variables should behave similarly to x and the operator N maps an appropriate Sobolev space into L^q. The solutions obtained are called *strong*, i.e. the differential equation is satisfied a.e. The above examples of boundary value problems have a common feature: they have the form $Bx = 0$ where B is a linear operator. This enables us to restrict the subspace Y and X by the condition $Bx = 0$. For example, if f is continuous in (1.2), then $X = \{x \in C^1([0,\pi]) : x(0) = x(\pi) = 0\}$, $Z = C([0,\pi])$ and $Y = \{x \in X : x'' \in C([0,\pi])\}$; if f is a Carathéodory function then $X = H_0^1$, $Y = X \cap H^2$ and $Z = L^2$, where H^m is the Sobolev space of all functions whose derivatives up to order m are square integrable and H_0^m is the closure in H^m of the set of smooth functions with compact support contained in $(0,\pi)$.

One can also study the boundary conditions of the form $Bx = B_{\mathrm{n}}(x)$ where B_{n} is a nonlinear continuous operator taking values in the same space as B. Hence one can consider

$$x'' + m^2 x = f(t,x,x'), \quad x(0) = r_0, \quad x(\pi) = r_\pi\,, \tag{1.8}$$

instead of (1.2),

$$x' = f(t,x), \quad x(0) - x(1) = \int_0^1 x(s)\,ds\,,$$

instead of (1.3), and so on. This is possible since we can use product spaces for X, Y and Z. For instance, the boundary value problem (1.8) can be treated as equation (1.1) with $X = C^1([0,\pi])$, $Y = C^2([0,\pi])$, $Z = C([0,\pi]) \times \mathbb{R}^2$, and

$$Lx = (x'', x(0), x(\pi)), \quad N(x) = (f(\cdot, x(\cdot), x'(\cdot)), r_0, r_\pi)\,.$$

It is easy to introduce similar changes for all examples and both assumptions on f (continuity or the Carathéodory conditions).

We return to the first example (1.2)—it was most extensively studied—and notice that, even for a bounded function f, a solution need not exist:

$$x'' + m^2 x = f(t), \quad x(0) = x(\pi) = 0\,.$$

The problem is selfadjoint with one-dimensional kernel spanned by $w(t) = \sin mt$, so the solvability is equivalent to

$$\int_0^\pi w(t) f(t)\,dt = 0 \tag{1.9}$$

and, for $f = w$, this fails. In fact, equation (1.2) with $f(t,x) = g(x) + f(t)$, g continuous and $\lim_{x\to\pm\infty} g(x) = g_\pm$, has a solution if (1.9) holds and $g_+ g_- < 0$ (see below; this is a special case of the Landesman–Lazer condition). One can generalize this by taking $g_\pm$ not necessarily finite, the exact limits replaced by lim sup and lim inf and even (1.9) replaced by a weaker condition (see [9]), but the growth of g cannot be too fast. If we take $g(x) = -(2m+1)x$, $f(t) = \sin(m+1)t$, then (1.9) will be satisfied but the corresponding problem

$$x'' + (m+1)^2 x = \sin(m+1)t, \quad x(0) = x(\pi) = 0\,,$$

has no solution. It is easy to see that the upper bound on the growth of g is the distance between the eigenvalue m^2 which stands in (1.2) and the nearest eigenvalue $(m+1)^2$ or $(m-1)^2$. This situation is typical and we have a restriction on the constant

$$\gamma = \limsup_{\|x\|\to\infty} \|N(x)\|/\|x\|\,. \tag{1.10}$$

It is worth noticing, in connection with the above considerations, that one can generate many theorems by replacing equation (1.1) with $Lx + Ax = N(x) + Ax$ where A is a linear operator with the same (or larger) domain as L and $L + A$ is invertible, and by applying one of fixed point methods. However, the author does not refer to these theorems as resonance results.

2. Alternative method. This method seems to be the most general one for the study of equation (1.1). We have $L : X \supset Y \to Z$, $N : X \to Z$ and we do not assume anything about $\ker L$ and $L(Y)$. The unique assumption is: X and Z admit decompositions

$$X = X_0 \oplus X_1, \quad Z = Z_0 \oplus Z_1$$

(topological direct sums) such that $\ker L \subset X_0$ and $Z_1 \subset L(Y)$. Denote by $P : X \to X$ (resp. $Q : Z \to Z$) the projection onto X_0 (resp. Z_0) along X_1 (resp. Z_1). Suppose that there exists a linear operator $H : Z_1 \to X_1$ such that

$$H(I-Q)Lx = (I-P)x \quad \text{for } x \in Y\,, \tag{2.1}$$

$$L(Hz) = z \quad \text{for } z \in Z_1, \tag{2.2}$$

$$Q(Lx) = L(Px) \quad \text{for } x \in Y. \tag{2.3}$$

If $Z_1 = L(Y)$, then $\ker L = X_0 \cap Y$ by (2.3), and conditions (2.1), (2.2) mean that H is the inverse of $L|X_1 \cap Y : X_1 \cap Y \to Z_1$. However, this simplification fails in some examples.

We shall show that, under (2.1)–(2.3), equation (1.1) is equivalent to the system

$$x = Px + H(I-Q)N(x)\,, \tag{2.4}$$

$$Q(Lx - N(x)) = 0\,. \tag{2.5}$$

In fact, we get (2.4) (resp. (2.5)) by applying $H(I-Q)$ (resp. Q) to (1.1) and using (2.1). Conversely, applying L to (2.4) and using (2.2) and (2.3), we obtain

$$Lx = Q(Lx) + (I-Q)N(x)\,,$$

so $(I-Q)(Lx - N(x)) = 0$ and we have (1.1) by (2.5).

Equation (2.4) is usually called *auxiliary* and equation (2.5) the *bifurcation equation*. Generally, there are two possible ways to find a solution of the system (2.4)–(2.5). If we know that the auxiliary equation has a unique solution $S(\widehat{x})$ for any $\widehat{x} = Px \in X_0$:

$$S(\widehat{x}) = \widehat{x} + H(I-Q)N(S(\widehat{x}))\,,$$

then the bifurcation equation will have the form

$$Q(L-N)(S(\widehat{x})) = 0\,. \tag{2.6}$$

The unique solvability is obtained if $H(I-Q)N$ is contractive or monotone and the last equation is defined and takes values in X_0, which is usually a finite-dimensional space. Thus equation (2.6) can be studied with the use of finite-dimensional methods, the Brouwer degree for example. We shall show below a sample of such arguments.

The second approach to the system (2.4)–(2.5) lies in the study of the map $T : X \times X_0 \to X \times X_0$, $T(x,\widehat{x}) = (\widehat{x} + H(I-Q)N(x), \widehat{x} + Q(L-N)(x))$. Its fixed points (or rather their first components) are exactly solutions of the system examined. The typical method for finding such fixed points is the Leray–Schauder degree theory and its generalizations. We refer to the excellent survey by Cesari [4] where many examples of this and other types are given.

Example. Consider the typical problem

$$x'' + x = f(t,x), \qquad x(0) = x(\pi) = 0\,, \tag{2.7}$$

where $f : [0,\pi] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function which is bounded:

$$|f(t,x)| \le M, \qquad \text{for } t \in [0,\pi],\ x \in \mathbb{R}\,, \tag{2.8}$$

and satisfies the Lipschitz condition with respect to x:

$$|f(t,x_1) - f(t,x_2)| \le M|x_1 - x_2|, \qquad \text{for } t \in [0,\pi],\ x_1, x_2 \in \mathbb{R}\,. \tag{2.9}$$

Since the system of functions

$$w_k(t) = \sqrt{2/\pi}\sin kt, \qquad k = 1, 2, \ldots\,,$$

is orthonormal and complete in $X = Z = L^2(0,\pi)$, we can take the orthogonal projectors

$$Px = Qx = \sum_{k=1}^{m} (w_k, x) w_k$$

where $(\cdot,\cdot)$ stands for the L^2-scalar product and m will be chosen later. The null space of $Lx = x'' + x$ equals $\ker L = \mathrm{Lin}\{w_1\} \subset X_0 = \mathrm{Lin}\{w_1, \ldots, w_m\}$ and by the Hilbert–Schmidt theory, H should be defined by the formula

$$Hz = \sum_{k=2}^{\infty} (k^2-1)^{-1} (w_k, z) w_k\,;$$

$\{w_k\}$ is a system of eigenfunction for x'' and the corresponding sequence of eigenvalues is $\{k^2\}$. Conditions (2.1)–(2.3) are easily verified, hence we can study the auxiliary equation. By (2.9), the right-hand side of (2.4) is a contraction for sufficiently large m:

$$\begin{aligned}\|H(I-Q)(N(x)-N(y))\|^2 &= \Big\| \sum_{k=m+1}^{\infty} (k^2-1)^{-1}(N(x)-N(y),w_k)w_k\Big\|^2 \\ &= \sum_{k=m+1}^{\infty} (k^2-1)^{-1}|(N(x)-N(y),w_k)|^2 \\ &\le M^2 \sum_{k=m+1}^{\infty} (k^2-1)^{-1}\|x-y\|^2 .\end{aligned}$$

On the other hand, the functions $H(I-Q)N(x)$ are bounded:

$$|H(I-Q)N(x)(t)| \le \sum_{k=m+1}^{\infty} (k^2-1)^{-1}(2\pi)^{-1}M , \tag{2.10}$$

and the last constant tends to 0 as $m\to\infty$. Hence, for any $\widehat{x} = \sum_{k=1}^m a_k w_k$, there exists a unique $S\widehat{x}$ such that

$$S\widehat{x} = \widehat{x} + H(I-Q)N(S\widehat{x})$$

and we can pass to equation (2.6) which assumes the form

$$\int_0^\pi (S\widehat{x})''w_k + \int_0^\pi (S\widehat{x})w_k = \int_0^\pi f(t,S\widehat{x}(t))w_k(t)\,dt, \qquad k=1,\dots,m ,$$

or, after integration by parts,

$$(1-k^2)a_k = \int_0^\pi f(t,S\widehat{x}(t))w_k(t)\,dt, \qquad k=1,\dots,m ,$$

where $a_k = (w_k,\widehat{x})$. Let us introduce the mapping $F:\mathbb{R}^m\to\mathbb{R}^m$, $F=(F_1,\dots,F_m)$, where

$$\begin{aligned}F_k(a_1,\dots,a_m) &= (k^2-1)a_k \\ &\quad + \int_0^\pi f\Big(t,\sum_{k=1}^m a_k w_k(t) + H(I-Q)N(S\widehat{x})(t)\Big)w_k(t)\,dt .\end{aligned}$$

If $k\ne 1$, then the sign of F_k is determined by the sign of a_k for $|a_k| \ge (k^2-1)^{-1}M\sqrt{2/\pi}$. Set $r_k = 2(k^2-1)^{-1}M\sqrt{2/\pi}$ for $k\ge 2$. The situation for $k=1$ is different. Using (2.9), we can estimate

$$\Big|\sum_{k=2}^m a_k w_k(t) + H(I-Q)N(S\widehat{x})(t)\Big| \le C .$$

On the other hand, $a_1 w_1(t) \to \pm\infty$ as $a_1 \to \pm\infty$ uniformly on any set $[\delta, \pi - \delta]$. Hence if we assume that

$$x\, f(t,x) \geq 0 \tag{2.11}$$

for sufficiently large $|x|$, or

$$x\, f(t,x) \leq 0 \tag{2.12}$$

for large $|x|$, where strict inequalities hold on a set of a positive measure, then we can find $r_1 > 0$ such that F_1 has opposite signs on both sides $a_1 = \pm r_1$ of the cube $U = (-r_1, r_1) \times \ldots \times (-r_m, r_m)$. The same was proved above for F_k, $k \geq 2$, and $a_k = \pm r_k$. Therefore the Brouwer degree [18] $\deg(F, U, 0)$ is defined and F can be homotopically deformed to the antipodal map $\widehat{F}$, $\widehat{F}(-a) = -\widehat{F}(a)$. By the Borsuk Antipodensatz [18],

$$\deg(F, U, 0) \equiv 1 \pmod 2$$

and the equation $F(a) = 0$ has a solution in U. But this is a solution of the bifurcation equation. We have proved that problem (2.7) has a solution provided that f satisfies (2.8), (2.9) and (2.11) or (2.12).

Remark. Condition (2.12) is exactly as in the recent paper by Iannacci and Nkashama [13], however, they omit the strong assumption of the boundedness of f. Similarly, we can prove by the alternative method that problem (2.7) has a solution if, instead of (2.12), we have

$$\int_0^\pi f_+(t) \sin t\, dt < 0 < \int_0^\pi f_-(t) \sin t\, dt \tag{2.13}$$

where

$$f_+(t) = \limsup_{x \to +\infty} f(t,x), \qquad f_-(t) = \liminf_{x \to -\infty} f(t,x)\,.$$

This is exactly the Landesman–Lazer condition in the form of [7].

3. Coincidence degree. Let us go back to the alternative scheme and suppose that $X_0 = \ker L$, $Z_1 = L(Y)$ and both X_0 and Z_0 are linear subspaces of the same finite dimension. Then there exists an isomorphism (not unique) $J : Z_0 \to X_0$. Conditions (2.1), (2.3) are trivially satisfied and, by (2.2), H should be a right inverse of L. The bifurcation equation can be rewritten as

$$JQN(x) = 0\,. \tag{3.1}$$

Obviously, the system of equations (2.4), (3.1) is equivalent to one equation

$$x = Px + JQN(x) + H(I - Q)N(x)\,. \tag{3.2}$$

The nonlinear operator N is called *L-compact* if QN and $H(I-Q)N$ are compact operators. Usually, H is compact (maps bounded sets into compact ones) and N maps bounded sets into bounded ones, which implies the L-compactness of N. The main theorem for L-compact mappings was proved by J. Mawhin [19].

THEOREM 1. *Let Ω be an open bounded subset in X such that*

(i) *the equation $Lx = \lambda N(x)$ has no solution for $\lambda \in (0,1]$ and $x \in Y \cap \partial\Omega$,*

(ii) *the following Brouwer degree is defined and does not vanish*:

$$\deg(JQN|\ker L, \Omega \cap \ker L, 0) \neq 0\,.$$

Then equation (1.1) *has a solution in Ω.*

Proof. Let $h(x,\lambda) = (P + JQN)(x) + \lambda H(I - Q)N(x)$ for $\lambda \in [0,1]$, $x \in \operatorname{cl}\Omega$. Since N is L-compact, the homotopy h is compact. Moreover, fixed points of h are exactly solutions of $Lx = \lambda N(x)$, so h does not admit fixed points on $\partial\Omega$ (for $\lambda > 0$, this follows from (i); for $\lambda = 0$, it is a consequence of the fact that the Brouwer degree is defined). Hence the Leray–Schauder degree

$$\deg_{\mathrm{LS}}(I - h(\cdot,\lambda), \Omega, 0)$$

is independent of λ (comp. [18], [6]). But, for $\lambda = 0$, it reduces to the Brouwer degree (ii). Therefore, $\deg_{\mathrm{LS}}(I - h(\cdot,1), \Omega, 0) \neq 0$, hence $h(\cdot,1)$ has a fixed point in Ω which is a solution to (1.1).

The Brouwer degree (ii) is called the *coincidence degree* of N with respect to L. We refer to [11] for more information about this theory. Notice also that this theorem is closely related to Fučik's theorem [10] where only Hilbert spaces are considered.

We present one application of the theorem (comp. [20]). Consider the periodic boundary value problem

$$x' = f(t,x), \qquad x(0) = x(T)\,, \tag{3.3}$$

where $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function, T-periodic with respect to t. Here, $X = Z = C_T(\mathbb{R},\mathbb{R}^n)$, the space of continuous T-periodic functions, $Y = C^1_T(\mathbb{R},\mathbb{R}^n)$, $Lx = x'$, $N(x)(t) = f(t,x(t))$, $\ker L$ is the space of constant functions which is naturally isomorphic to $\mathbb{R}^n$, and $L(Y)$ is the subspace of continuous functions for which the integral over $[0,T]$ vanishes. Obviously, $\ker L \oplus L(Y) = X = Z$ and we can take $P = Q$ given by

$$Px = \int\limits_0^T x(t)\,dt \qquad \text{(constant function)}.$$

Moreover, $J = I$, and $H : Z \supset L(Y) \to L(Y) \subset X$ is defined by

$$Hz(t) = \int\limits_0^t z(s)\,ds\,.$$

All introductory assumptions are satisfied and we can pass to (i) and (ii). It is easy to see that (i) is guaranteed by

$$f_i(t,x) \neq 0 \quad \text{for } |x_i| = M_i,\ i = 1,\ldots,n\,, \tag{3.4}$$

where M_i are constants and f_i denotes the ith coordinate of f. The choice of Ω is simple:

$$x \in \Omega \subset C_T(\mathbb{R},\mathbb{R}^n) \Leftrightarrow |x_i(t)| < M_i,\ i = 1,\ldots,n,\ t \in \mathbb{R}\,,$$

and the proof of (i) is obvious.

The mapping $JQN|\ker L$ is given by the formula

$$F(a) = \int\limits_0^T f(t,a)\,dt, \qquad a \in \mathbb{R}^n\,,$$

and $\Omega \cap \ker L$ is the cube $(-M_1, M_1) \times \ldots \times (-M_n, M_n)$. Thus we have to strengthen condition (3.4):

$$f_i(t,x_1,\ldots,x_{i-1},-M_i,\ldots,x_n) f_i(t,x_1,\ldots,x_{i-1},M_i,\ldots,x_n) < 0 \tag{3.5}$$

for any $(x_1,\ldots,x_n) \in \mathbb{R}^n$ such that $|x_i| \leq M_i$ for all $i = 1,\ldots,n$. This implies that F_i takes opposite signs on both sides $a_i = \pm M_i$ of the cube. The arguments as in the example from Section 2 show that $\deg(F, \ker L \cap \Omega, 0)$ is odd and, therefore, does not vanish.

R e m a r k. One can study the same problem in another way:

$$\frac{d}{dt}\|x\|^2 = 2(x',x) = 2(f(t,x),x)\,.$$

If $(f(t,a),a) \neq 0$ for $\|a\| = R$, then $\Omega = B(0,R)$ satisfies (i). Moreover, the Krasnosel'skiĭ condition

$$(F(a),a) \leq 0 \quad \text{for } \|a\| = R$$

implies (ii) (comp. [18]). Thus the following condition is sufficient for the solvability of (3.3):

$$(f(t,a),a) < 0 \quad \text{for } \|a\| = R,\ t \in [0,T]\,.$$

4. The perturbation method. The third method for the study of resonance problems is based on the observation that a linear noninvertible operator L after an arbitrarily small perturbation, $L + \lambda I$ for instance, becomes invertible. One can solve perturbed equations and find conditions that ensure the existence of a convergent sequence of solutions. Its limit is the solution to (1.1) we look for. This rather natural observation was first made by P. Hess in his elegant proof of the Landesman–Lazer theorem [12]. The abstract result was given by de Figueiredo [5] and, then, more specific methods attracted the attention of mathematicians. Recently, the present author has applied this approach to resonance problems independently of the above mentioned papers. The use of inverse operators of perturbed linear maps enables us to pass to unbounded nonlinearities and even to nonlinearities with linear growth. The method is developed in a series of papers [22]–[27] where also applications to boundary value problems are given.

Let $L(\lambda)$ be a continuous family of linear operators $Y \to Z$ for λ from a nbhd of $\lambda_0 \in \mathbb{R}$. We assume that Y and Z are Banach spaces such that all $L(\lambda)$ are bounded and the continuity of $L(\cdot)$ is meant to be the norm continuity. Let $L(\lambda)$ be invertible for $\lambda \neq \lambda_0$, and $L(\lambda_0)$ be a Fredholm linear operator (of index zero). Let the inclusion map $J : Y \to X$ be compact. Denote by $w_1, \ldots, w_n$ a basis of $\ker L$ and suppose that the limits

$$\lim_{\lambda \to \lambda_0} \|L(\lambda)w_j\|^{-1} L(\lambda)w_j, \quad j = 1, \ldots, n,$$

exist, spanning a topological complement of $L(\lambda_0)(Y)$ (these limits always exist for $L(\lambda) = L(\lambda_0) + (\lambda - \lambda_0)I$). Under this assumption, the inverse operators $G(\lambda) = L(\lambda)^{-1}$ have the form

$$G(\lambda) = G_0(\lambda) + \sum_{j=1}^{n} c_j(\lambda)\langle u_j(\lambda), \cdot\rangle w_j$$

where $G_0(\lambda) : Z \to Y$, $c_j(\lambda) \in \mathbb{R}$, $u_j(\lambda) \in Z^*$, while G_0 and u_j have continuous extensions to λ_0 and

$$\lim_{\lambda \to \lambda_0} |c_j(\lambda)| = \infty, \quad j = 1, \ldots, n,$$

(comp. [26]). Moreover, $G_0(\lambda)$ takes values in a fixed space which is a topological complement of $\ker L$. Notice that

$$L(\lambda_0)(Y) = \bigcap_{j=1}^{n} \ker u_j(\lambda_0).$$

The examined equation

$$L(\lambda_0)y = N(Jy) \tag{4.1}$$

is equivalent to the system

$$x = JG_0(\lambda_0)N(x) + \sum d_j Jw_j, \tag{4.2}$$

$$\langle u_j(\lambda_0), N(x)\rangle = 0, \quad j = 1, \ldots, n, \tag{4.3}$$

where $x = Jy$ and the first summand in (4.2) is an element of a topological complement $\widetilde{X}$ of $\overline{X} = \mathrm{Lin}\{Jw_1, \ldots, Jw_n\}$. Fix λ_1 close to λ_0.

Theorem 2. *Suppose there exist open bounded subsets $\widetilde{U}$ of $\widetilde{X}$ and U of $\mathbb{R}^n$ such that*

(∗) *there is no solution $(\widetilde{x}, (d_1, \ldots, d_n))$ to the system*

$$\widetilde{x} = (\lambda - \lambda_1)(\lambda_0 - \lambda_1)^{-1} JG_0(\lambda)N(x),$$

$$d_j = c_j(\lambda)\langle u_j(\lambda), N(x)\rangle, \qquad j = 1, \ldots, n\,,$$

on the boundary of $\widetilde{U} \times U$ *where* $x = \widetilde{x} + \sum d_j Jw_j$.

If we denote $g = (g_1, \ldots, g_n) : \mathrm{cl}\, U \to \mathbb{R}^n$ *by the formula*

$$g_j(d_1, \ldots, d_n) = d_j - c_j(\lambda_1)\Big\langle u_j(\lambda_1), N\Big(\sum d_i Jw_i\Big)\Big\rangle$$

and if

$$\deg(g, U, 0) \neq 0\,,$$

then equation (4.1) *has a solution* y *such that* $Jy = \widetilde{x} + \sum d_j Jw_j$ *and* $\widetilde{x} \in \mathrm{cl}\,\widetilde{U}$, $(d_1, \ldots, d_n) \in \mathrm{cl}\, U$.

Proof. Define an open bounded set $V \subset X$ by $V = \{\widetilde{x} + \sum d_j Jw_j : \widetilde{x} \in \widetilde{U},\ (d_1, \ldots, d_n) \in U\}$. Then condition $(*)$ means that the homotopy

$$H(x, \lambda) = \frac{\lambda - \lambda_1}{\lambda_0 - \lambda_1} JG_0(\lambda)N(x) + \sum_{j=1}^{n} c_j(\lambda)\langle u_j(\lambda), N(x)\rangle Jw_j\,,$$

for $x \in X$ and $\lambda \in (\lambda_0, \lambda_1]$, is fixed point free on the boundary ∂V. It follows that the Leray–Schauder degree

$$\deg_{\mathrm{LS}}(I - H(\cdot, \lambda), V, 0)$$

is defined and independent of $\lambda \in (\lambda_0, \lambda_1]$. On the other hand, $H(\cdot, \lambda_1)$ has a finite-dimensional range and, therefore,

$$\deg_{\mathrm{LS}}(I - H(\cdot, \lambda_1), V, 0) = \deg(g, U, 0) \neq 0\,.$$

Thus, for any $\lambda > \lambda_0$, there exists a solution to the equation $x = H(x, \lambda)$ in V.

Take $\lambda_k \to \lambda_0$ and $x_k = \widetilde{x}_k + \sum_j d_j^k Jw_j = H(x_k, \lambda_k)$ such that $\widetilde{x}_k \in \widetilde{U}$, $(d_1^k, \ldots, d_n^k) \in U$. Since J is compact and (x_k) is bounded, we can choose subsequences (still denoted by (λ_k) and (x_k) for simplicity) such that

$$\frac{\lambda_k - \lambda_1}{\lambda_0 - \lambda_1} JG_0(\lambda_k)N(x_k) \to x_0\,,$$
$$c_j(\lambda_k)\langle u_j(\lambda_k), N(x_k)\rangle \to d_j, \qquad j = 1, \ldots, n\,.$$

Hence $x_k \to x_0 + \sum d_j Jw_j =: x$. Obviously, x satisfies (4.2)–(4.3).

This theorem is closely related to Theorem 1. Since λ_1 is near λ_0, the real coefficients $c_j(\lambda_1)$ are arbitrarily large, so the degree of g equals (up to sign) the degree of the map $(c_j(\lambda_1)^{-1} g_j)_{j \le n}$, and the latter equals $\deg(h, U, 0)$ with

$$h_j(d_1, \ldots, d_n) = \Big\langle u_j(\lambda_0), N\Big(\sum d_i Jw_i\Big)\Big\rangle\,.$$

The last mapping is exactly $JQN|\ker L$ from Theorem 1 but here it is described better for applications. It should also be noticed that (4.2) is the auxiliary equation and (4.3) the bifurcation equation from the alternative method.

Theorem 2 produces results concerning nonlinearities with restricted growth. If N is sublinear:

$$\lim_{\|x\|\to\infty} \|N(x)\|/\|x\| = 0\,,$$

then the condition guaranteeing the solvability of equation (4.1) has the form: for any $(x_\nu) \subset X$ with $\|x_\nu\| \to \infty$ and $\|x_\nu\|^{-1}x_\nu \to \sum d_j J w_j$, there exists $j_0 \in \{1,\ldots,n\}$ such that

$$\limsup_{\nu\to\infty} d_{j_0}\langle u_{j_0}(\lambda_0), N(x_\nu)\rangle < 0\,.$$

We refer to it as L-L. The same is true if we reverse the inequality and replace lim sup by lim inf. The proof via Theorem 2 can be found in [24], and the direct one in [23]. The case of nonlinearities with linear growth is more difficult. If

$$\gamma = \limsup_{\|x\|\to\infty} \|N(x)\|/\|x\| \in (0,\infty)\,,$$

then we have a natural restriction on γ:

$$\gamma\|JG_0(\lambda_0)\| < 1\,,$$

and the L-L condition has to be replaced by a stronger one, L-L$_1$: there exist $R > 0$ and $\sigma > \gamma\|JG_0(\lambda_0)\|(1-\gamma\|JG_0(\lambda_0)\|)^{-1}$ such that, for any $j = 1,\ldots,n$ and $|d_j| > R$,

$$\sup\Big\{d_j\Big\langle u_j(\lambda_0), N\Big(\widetilde{x} + \sum d_i J w_i\Big)\Big\rangle : \\ |d_i| \le |d_j|,\ \widetilde{x} \in \widetilde{X},\ \|\widetilde{x}\| \le \sigma\Big\|\sum d_i J w_i\Big\|\Big\} < 0\,.$$

We can replace the last inequality by inf > 0, but for all j simultaneously. Here, the proof can be carried out only via homotopy arguments, as in Theorem 2 of [24]. That paper also contains a result on superlinear nonlinearities (N is called *superlinear* if $\|N(x)\| \le a + b\|x\|^\varrho$ with $\varrho > 1$).

The abstract theorems have numerous applications to boundary value problems. The first application found was the existence result for partial differential equations, obtained via the alternative method by Landesman and Lazer [17]. We present a slight generalization of it but refer to [23] where the full discussion is given.

Consider the Dirichlet boundary value problem

$$Pu - \lambda_0 u = f(x,u), \qquad u \in H_0^m(\Omega)\,, \tag{4.4}$$

where P is a selfadjoint strongly elliptic operator of order $2m$ on an open bounded domain $\Omega \subset \mathbb{R}^k$ with Lipschitzian boundary, λ_0 is an eigenvalue for P, i.e. $Pu - \lambda_0 u = 0$ has nontrivial solutions belonging to $H_0^m(\Omega)$,

$f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (measurable with respect to x and continuous with respect to u) with a sublinearly restricted growth:

$$\lim_{|u|\to\infty} \operatorname*{ess\,sup}_{x} |f(x,u)|/|u| = 0\,. \tag{4.5}$$

Since f need not be continuous, we should look for strong solutions of the problem. Put $X = Z = L^2(\Omega)$ and $Y = H^{2m}(\Omega) \cap H_0^m(\Omega)$ where $H^m(\Omega)$ is the Sobolev space of all functions whose distributional derivatives up to order m are in $L^2(\Omega)$ and $H_0^m(\Omega)$ is the closure in this space of the set of all smooth functions with supports contained in Ω. The condition $u \in H_0^m(\Omega)$ corresponds to the null Dirichlet data $D^\alpha u|\partial\Omega = 0$ for $|\alpha| < m$.

Let $L(\lambda) = P - \lambda I$ and $N(u)(x) = f(x, u(x))$. By the Hilbert–Schmidt theory,

$$G(\lambda) = \sum_{j=n+1}^{\infty} (\lambda_j - \lambda)^{-1}(w_j, \cdot)w_j + \sum_{j=1}^{n}(\lambda_j - \lambda)^{-1}(w_j, \cdot)w_j$$

where $\{\lambda_j : \ j = 1, 2, \ldots\}$ is the sequence of all eigenvalues of P, $\{w_j : j = 1, 2, \ldots\}$ is the sequence of the corresponding eigenfunctions which form an orthonormal basis in $L^2(\Omega)$, $(\cdot,\cdot)$ is the L^2-scalar product. We choose $\lambda_1 = \ldots = \lambda_n \neq \lambda_j$ for $j > n$ and denote this eigenvalue by λ_0. The first summand (denote it by $G_0(\lambda)$) has a continuous extension to $\lambda = \lambda_0$. Moreover,

$$c_j(\lambda) = (\lambda_0 - \lambda)^{-1}, \qquad \langle u_j(\lambda), z\rangle = (w_j, z)\,,$$

in our notation.The inclusion map $J : H^{2m} \cap H_0^m \to L^2$ is obviously compact. By (4.5), N is sublinear and we can use the L-L condition. It now has the form: for any $(u_\nu) \subset L^2$ such that $\|u_\nu\| \to \infty$ and $\|u_\nu\|^{-1}u_\nu \to \sum d_j w_j$, there exists j_0 such that

$$\limsup_{\nu\to\infty} d_{j_0} \int_{\Omega} w_{j_0}(x) f(x, u_\nu(x))\, dx \ < 0$$

or $\liminf \ldots > 0$. If we add all inequalities, we shall get a stronger assumption but having some advantages. Suppose that the limits

$$\lim_{u\to\pm\infty} f(x,u) = f_\pm(x) \tag{4.6}$$

exist. Then the following condition is sufficient for the solvability of (4.4): for any $d \in \mathbb{R}^n$ such that $\|d\| = 1$, the numbers

$$\int_{A_+} f_+(x) \sum d_j w_j(x)\, dx + \int_{A_-} f_-(x) \sum d_j w_j(x)\, dx\,,$$

$$\int_{A_+} f_-(x) \sum d_j w_j(x)\, dx + \int_{A_-} f_+(x) \sum d_j w_j(x)\, dx$$

have opposite signs, where $A_+ = \{x : \sum d_j w_j(x) > 0\}$ and $A_- = \{x : \sum d_j w_j(x) < 0\}$ (cf. [31]). The one-dimensional resonance ($n = 1$) was studied by Landesman and Lazer [17]: the sufficient condition is then simpler since we can take only $d_1 = +1$ (for $d_1 = -1$, we get reversed signs).

The limits in (4.6) can be replaced by lim sup and lim inf and the nonlinearity f can also depend also on derivatives (comp. [23]). Moreover, the method can be applied to all equations with selfadjoint operators having inverses compact in L^2. This enables us to study periodic problems via the perturbation method [25].

If the nonlinearity f in problem (4.4) has a linear growth:

$$\limsup_{|u|\to\infty} \operatorname*{ess\,sup}_{x} |f(x,u)|/|u| = \gamma \in (0,\infty)\,,$$

we have no integral conditions of the above type that ensure the existence of solutions to (4.4). We have only found the following sufficient condition if $n = 1$ [23]:

$$\check{a} \le \frac{f(x,u) - b(x)}{u} \le \hat{a} \quad \text{for } x \in \Omega,\ |u| > M\,, \tag{4.7}$$

where $\check{a}$, $\hat{a}$, M are positive constants, $b \in L^2(\Omega)$ and

$$\gamma(\min_{j\neq 0} |\lambda_0 - \lambda_j|)^{-1} < \frac{\check{a}}{\sqrt{\check{a}^2 + (\hat{a} - \check{a})^2}}\,.$$

Notice that condition (4.7) is satisfied by jumping nonlinearities:

$$f(x,u) = \hat{a}u^+ - \check{a}u^- + g(x,u)$$

with g sublinear.

Our method is useful not only for selfadjoint problems considered in Hilbert spaces: L^2 or Sobolev spaces. The boundary value problems for ordinary differential equations admit Green operators which are described sufficiently well. This enables us to divide this operator into two parts: G_0 and the other summand, and to consider the problem in different spaces. Sometimes, a perturbation of $L(\lambda_0)$ is not expressed by the identity operator: see [22] and [23] where the problems of the form

$$\sum_{j=0}^{m} a_j(\lambda)x^{(j)} = f(t,x)\,,$$
$$\sum_{j=0}^{m-1} (b_{ij}x^{(j)}(0) + c_{ij}x^{(j)}(1)) = 0, \qquad i = 1,\ldots,m\,,$$

are studied. The functions a_j are analytic in a nbhd of λ_0 and the corresponding linear homogeneous problem has nontrivial solutions.

The perturbation method can be applied to differential equations in Banach spaces. The following problem was studied in [27]:

$$x' = A(t)x + f(t,x), \qquad B_1x(0) + B_2x(1) = 0\,, \tag{4.8}$$

where $A : [0,1] \to L(E)$, $f : [0,1] \times E \to E$ were continuous functions, E a Banach space, $f(t,\cdot)$ compact, $f(\cdot,x)$ uniformly continuous with respect to x on bounded subsets, and B_1, $B_2 \in L(E)$ the space of linear bounded operators on E. Suppose that the operator $B_1 + B_2U(1)$, where $U(t)$, $t \in [0,1]$, is the resolvent for $x' = A(t)x$, is a Fredholm operator of index zero in E, $B_1 + e^{\lambda}B_2U(1)$ is invertible for small $|\lambda|$ and the limits

$$h_j = \lim_{\lambda\to 0} \frac{(B_1 + e^{\lambda}B_2U(1))x_j}{\|(B_1 + e^{\lambda}B_2U(1))x_j\|}$$

exist for $j = 1,\dots,n$, where $x_1,\dots,x_n$ form a basis of $\ker(B_1 + B_2U(1))$. Let f be sublinear. If we denote by v_j, $j = 1,\dots,n$, the linear continuous functional on E defined by

$$\langle v_j, h_i\rangle = \delta_{ij}, \qquad v_j|(B_1 + B_2U(1))(E) = 0\,,$$

then the L-L condition will have the form: for any $(x_\nu) \subset C([0,1],E)$ with $\|x_\nu\| \to \infty$ and $\|x_\nu\|^{-1}x_\nu \to \sum d_jU(\cdot)x_j$, there exists j_0 such that

$$\liminf_{\nu\to\infty} d_{j_0}\Big\langle v_{j_0}, B_2U(1)\int_0^1 U^{-1}(s)f(s,x_\nu(s))\,ds\Big\rangle > 0$$

(or $\limsup\ldots < 0$). The boundary condition can also involve nonlinear operators and both nonlinearities can have a linear growth. The suitable assumptions are much complicated (see [27]).

5. The case of the infinite-dimensional kernel. The perturbation method admits generalizations in many directions. Below, we present one of them and its applications to boundary value problems for infinite-dimensional ordinary differential equations. If one examines the boundary value problems such as

$$x' = f(t,x), \qquad x(0) = x(1)\,,$$
$$x'' + m^2x = f(t,x), \qquad x(0) = x(\pi) = 0\,,$$

where $x \in E$, an infinite-dimensional Banach space, then the spaces of solutions of the homogeneous linear problems are isomorphic to E, so we cannot apply the above theory. Although the corresponding linear operator is not Fredholm, its behaviour is similar to the finite-dimensional case.

Let X, Y, Z be Banach spaces, $J : Y \to X$ an injective bounded linear operator, and $L(\lambda) : Y \to Z$, $\lambda \in \mathbb{R}$, a continuous family of bounded linear operators which are invertible except $L(\lambda_0)$. Assume that

$$\ker L(\lambda_0) = J^{-1}(E_1) \oplus \ldots \oplus J^{-1}(E_n)\,,$$

where $E_1, \ldots, E_n$ are reflexive subspaces of X. Moreover, let

$$JL(\lambda)^{-1} =: G(\lambda) = G_0(\lambda) + \sum_{j=1}^{n} c_j(\lambda) u_j(\lambda)\,,$$

where $G_0(\lambda) : Z \to X$ and $u_j(\lambda) : Z \to E_j$, $j = 1, \ldots, n$, are bounded linear operators having continuous extensions to $\lambda = \lambda_0$, and $|c_j(\lambda)| \to \infty$ as $\lambda \to \lambda_0$. Assume that there exists a closed subspace $\widetilde{X} \subset X$ such that $G_0(\lambda)(Z) \subset \widetilde{X}$ and

$$\widetilde{X} \oplus E_1 \oplus \ldots \oplus E_n = X\,,$$
$$L(\lambda_0)(Y) = \bigcap_{j=1}^{n} \ker u_j(\lambda_0)\,,$$
$$L(\lambda_0) G_0(\lambda_0) z = z \quad \text{for } z \in L(\lambda_0)(Y)\,.$$

We look for a solution of (4.1) or, equivalently,

$$x = G_0(\lambda_0) N(x) + \sum_j d_j, \quad u_j(\lambda_0) N(x) = 0, \quad j = 1, \ldots, n\,, \tag{5.1}$$

where $d_j \in E_j$, $j = 1, \ldots, n$, are arbitrary. We suppose that the nonlinear operator $N : X \to Z$ is uniformly continuous on bounded sets and, for some sequences of finite rank bounded linear projectors $(P_{j,k})_{k \in \mathbb{N}}$ in E_j, $j = 1, \ldots, n$, such that $P_{j,k} d_j \to d_j$ as $k \to \infty$, the following condition is satisfied:

(W) for each bounded set $K \subset X$ and $\varepsilon > 0$, there exists k_0 such that

$$\Big\| N\Big(\widetilde{x} + \sum_j P_{j,k} d_j\Big) - N\Big(\widetilde{x} + \sum_j d_j\Big)\Big\| \le \varepsilon$$

for $k \ge k_0$, $\widetilde{x} \in \widetilde{X}$, $d_j \in E_j$, $j = 1, \ldots, n$, such that $\widetilde{x} + \sum d_j \in K$.

Moreover, suppose the operators $G_0(\lambda) N$, $\lambda \ne \lambda_0$, are compact.

We restrict ourselves to the sublinear case. For $\lambda \ne \lambda_0$ the equations $x = G(\lambda) N(X)$ have solutions due to the Rothe Fixed Point Theorem. If $\lambda_m \to \lambda_0$ and x_m is such a solution for $\lambda = \lambda_m$, then we prove that a bounded sequence (x_m) contains a subsequence convergent to a solution of our equation and we find a condition which excludes the case of unbounded (x_m).

Let (x_m) be bounded and $x_m = \widetilde{x}_m + \sum_j d_j^m$ where $\widetilde{x}_m = G_0(\lambda_m) N(x_m)$ and $d_j^m = c_j(\lambda_m) u_j(\lambda_m) N(x_m)$, $j = 1, \ldots, n$. All sequences $(\widetilde{x}_m)$, (d_j^m), $j = 1, \ldots, n$, are bounded, thus, by the compactness of $G_0(\lambda_m) N$ and the reflexivity of E_j, we can assume

$$\widetilde{x}_m \to \widetilde{x} \in \widetilde{X}, \quad d_j^m \rightharpoonup d_j, \quad j = 1, \ldots, n,$$

where $\rightharpoonup$ denotes weak convergence. Applying condition (W) and the uniform continuity of N, it is easy to show that

$$N\Big(\widetilde{x}_m + \sum_j d_j^m\Big) \to N\Big(\widetilde{x} + \sum_j d_j\Big).$$

Hence $x = \widetilde{x} + \sum d_j$ satisfies (5.1), by the boundedness of $(d_j^m)_m$ and the fact that $|c_j(\lambda_m)| \to \infty$.

THEOREM 3. *Under the above assumptions, if, for any sequence* $(x_m) \subset X$ *such that* $\|x_m\| \to \infty$ *and* $\|x_m\|^{-1} x_m \to \sum d_j$, *there exists* j_0 *and a functional* $r : E_{j_0} \times E_{j_0} \to \mathbb{R}$ *weakly continuous and homogeneous in the second variable such that* $r(d_{j_0}, d_{j_0}) > 0$ *and*

$$\limsup_{m\to\infty} \alpha_{j_0} r(d_{j_0}, u_{j_0}(\lambda_0) N(x_m)) < 0 \tag{5.2}$$

where $\alpha_j = \operatorname{sgn} c_j(\lambda)$ *for* λ *close to* λ_0, *then equation* (4.1) *has a solution.*

Proof. We shall see that the sequence of fixed points (x_m) cannot be unbounded. If it is, we can assume that $\|x_m\| \to \infty$. Then

$$\frac{x_m}{\|x_m\|} = G_0(\lambda_m) \frac{N(x_m)}{\|x_m\|} + \sum_j \frac{c_j(\lambda_m)}{\|x_m\|} u_j(\lambda_m) N(x_m).$$

The first summand tends to 0 since N is sublinear, hence the remaining summands are bounded and contain weakly convergent subsequences

$$\frac{c_j(\lambda_m)}{\|x_m\|} u_j(\lambda_m) N(x_m) \rightharpoonup d_j \in E_j, \quad j = 1, \ldots, n.$$

For $j = j_0$, this contradicts (5.2).

Remarks. If the nonlinear part N is bounded, then the condition (5.2) should be satisfied only for sequences $x_m = \widetilde{x}_m + \sum d_j^m$ such that $\|\widetilde{x}_m\| \le M$ where $M > \|G_0(\lambda_0)\| \sup \|N(x)\|$, $\|\sum d_j^m\| \to \infty$, $\|\sum d_j^m\|^{-1} d_j^m \to d_j \in E_j$, $j = 1, \ldots, n$. When, moreover, the limits of $N(x_m)$ exist and depend only on $\sum d_j$ (denote them by $N(d_1, \ldots, d_n)$) then the sufficient condition for the solvability of (4.1) becomes simpler: for any $(d_1, \ldots, d_n)$ on the unit sphere, there exists j such that

$$\alpha_j r(d_j, u_j(\lambda_0) N(d_1, \ldots, d_n)) < 0$$

where $r : E_j \times E_j \to \mathbb{R}$ has the properties from Theorem 3. In the previous section, we have $E_j = \mathbb{R}$ for any j, with r being the multiplication of real numbers.

Now, we shall show how this theorem works for two boundary value problems mentioned at the beginning of this section. Consider

$$x' = f(t,x), \qquad x(0) = x(T), \tag{5.3}$$

where $f : [0,T] \times H \to H$ is a continuous function, H is a Hilbert space, $f(0,\cdot) = f(T,\cdot)$ and the following condition is satisfied: for each bounded set $K \subset H$ and $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$\|f(t,x) - f(t,P_k x)\| \le \varepsilon \tag{5.4}$$

for $k \ge k_0$, $x \in K$ and $t \in [0,T]$, where (P_k) is a fixed sequence of finite rank orthogonal projectors in H strongly convergent to the identity map.

Put $X = Z = C([0,T],H)$, $Y = \{y \in C^1([0,T],H) : y(0) = y(T)\}$, let $J : Y \to X$ be the inclusion map, $L(\lambda)y = y' - \lambda y$, $\lambda_0 = 0$, and $N(x)(t) = f(t,x(t))$. Obviously $\ker L(0) \cong H$, $\widetilde{X} = \{x \in X : \int_0^T x(t)\,dt = 0\}$. It is easy to verify that N is uniformly continuous on bounded sets and satisfies condition (W). Moreover, one can calculate the Green operator:

$$G_0(\lambda)z(t) = e^{\lambda t}\int_0^t e^{-\lambda s}z(s)\,ds + \frac{e^{\lambda(t+T)}}{1-e^{\lambda T}}\int_0^T e^{-\lambda s}z(s)\,ds + \frac{1}{\lambda T}\int_0^T z(s)\,ds\,,$$

$$c_1(\lambda) = -\lambda^{-1}, \qquad u_1(\lambda)z = T^{-1}\int_0^T z(s)\,ds \in H$$

where we identify the space of constant functions with H. The operators $G_0(\lambda)N$ are compact by the General Ascoli–Arzelà Theorem. N will be sublinear if we assume that

$$\lim_{\|x\|\to\infty} \sup_t \|f(t,x)\|/\|x\| = 0\,.$$

Take r to be the scalar product in H. Then problem (5.4) has a solution if, for any sequence $(x_m) \subset X$ such that $\|x_m\| \to \infty$ and $\|x_m\|^{-1}x_m \to d \in H$ uniformly, we have

$$\limsup_{m\to\infty} \int_0^T (d, f(t,x_m(t)))\,dt < 0$$

or

$$\limsup_{m\to\infty} \int_0^T (x_m(t), f(t,x_m(t)))\,dt < 0\,.$$

The dual assumptions are obvious.

We shall study the infinite-dimensional Dirichlet problem:

$$x_k'' + m_k^2 x_k = f_k(t,x), \quad x_k(0) = x_k(\pi) = 0, \quad k \in \mathbb{N}\,, \tag{5.5}$$

where $x = (x_k) \in \mathbf{l}^p$ with $p > 1$ and $(m_k) \subset \mathbb{N}$. Assume that $f_k : [0, \pi] \times \mathbf{l}^p \to \mathbb{R}$ are continuous and, for all t and x,

$$\sum_{k=1}^{\infty} |f_k(t, x)|^p < \infty\,.$$

Consider the projectors $P_k(x_1, \ldots, x_k, x_{k+1}, \ldots) = (x_1, \ldots, x_k, 0, \ldots)$, $k \in \mathbb{N}$, in $\mathbf{l}^p$. Suppose that, for any $M > 0$ and $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} |f_n(t, x) - f_n(t, P_k x)|^p \le \varepsilon \tag{5.6}$$

for $k \ge k_0$, $t \in [0, \pi]$ and $\|x\| \le M$. Put $X = Z = C([0, \pi], \mathbf{l}^p)$, $Y = \{y \in C^2([0, \pi], \mathbf{l}^p) :\ y(0) = y(\pi) = 0\}$, $J : Y \to X$ the inclusion map, $L(\lambda)(y_k) = (y_k'' + m_k^2 y_k + \lambda y_k)$, $\lambda_0 = 0$ and $N(x)(t) = (f_k(t, x(t)))_k$. Then

$$\ker L(0) = \{(d_k \sin m_k\, \cdot) :\ (d_k) \in \mathbf{l}^p\} \cong \mathbf{l}^p.$$

It is easy to find the Green operator and its decomposition into a regular part $G_0(\lambda)$ and an irregular term. In particular,

$$u_1(0)z = \Big(\int_0^{\pi} z_k(s) \sin m_k s\, ds \cdot \sin m_k\ \cdot \Big)_{k \in \mathbb{N}}\,.$$

Condition (5.6) guarantees that N is uniformly continuous on bounded sets and satisfies condition (W), and the operators $G_0(\lambda)N$ are compact. Assume that $f = (f_k)$ is sublinear. The functional $r : \mathbf{l}^p \times \mathbf{l}^p \to \mathbb{R}$ which is the multiplication of the jth coordinates of both vectors has the required properties. Therefore, problem (5.5) has a solution if, for any sequence $(x_m) \subset X$ such that $\|x_m\| \to \infty$ and $\|x_m\|^{-1} x_m \to (d_k \sin m_k \cdot)$ uniformly on $[0, \pi]$, there exists $j \in \mathbb{N}$ such that

$$\limsup_{m\to\infty} d_j \int_0^{\pi} \sin m_j s f_j(s, x_m(s))\, ds < 0$$

or

$$\limsup_{m\to\infty} \int_0^{\pi} x_{m,j}(s) f_j(s, x_m(s))\, ds < 0$$

(or the dual conditions $\liminf \ldots > 0$). For $p = 2$, we can use a weaker condition

$$\limsup_{m\to\infty} \int_0^{\pi} (x_m(s), f(s, x_m(s)))\, ds < 0$$

where $(\cdot, \cdot)$ is the scalar product.

REFERENCES

[1] A. R. Abdullaev and A. B. Burmistrova, *On the solvability of boundary value problems at resonance*, Differentsial'nye Uravneniya 25 (1989), 2044–2048.

[2] A. Ambrosetti and G. Prodi, *On the inversion of some differentiable mappings with singularities between Banach spaces*, Ann. Math. Pura Appl. 93 (1973), 231–247.

[3] H. Brézis and L. Nirenberg, *Characterizations of the ranges of some nonlinear operators and applications to boundary value problems*, Ann. Scuola Norm. Sup. Pisa 5 (1978), 225–326.

[4] L. Cesari, *Functional analysis*, *nonlinear differential equations*, *and the alternative method*, in: Nonlinear Functional Analysis and Differential Equations, L. Cesari, R. Kannan and J. D. Schuur (eds.), Dekker, New York, 1976, 1–197.

[5] D. G. de Figueiredo, *On the range of nonlinear operators with linear asymptotes which are not invertible*, Comment. Math. Univ. Carolinae 15 (1974), 415–428.

[6] K. Deimling, *Nonlinear Functional Analysis*, Springer, 1985.

[7] P. Drábek, *Landesman–Lazer type condition and nonlinearities with linear growth*, Czechoslovak Math. J. 40 (1990), 70–87.

[8] —, *Landesman–Lazer condition for nonlinear problems with jumping nonlinearities*, J. Differential Equations, to appear.

[9] S. Fučik, *Nonlinear equations with noninvertible linear part*, Czechoslovak Math. J. 24 (1974), 259–271.

[10] —, *Solvability of Nonlinear Equations and Boundary Value Problems*, Reidel, Dordrecht, 1980.

[11] R. E. Gaines and J. L. Mawhin, *Coincidence Degree and Nonlinear Differential Equations*, Lecture Notes in Math. 568, Springer, 1977.

[12] P. Hess, *On a theorem by Landesman and Lazer*, Indiana Univ. Math. J. 23 (1974), 827–829.

[13] R. Iannacci and M. N. Nkashama, *Nonlinear two point boundary value problem at resonance without Landesman–Lazer condition*, Proc. Amer. Math. Soc. 106 (1989), 943–952.

[14] —, —, *Unbounded perturbations of forced second order ordinary differential equations at resonance*, J. Differential Equations 69 (1987), 289–309.

[15] R. Iannacci, M. N. Nkashama and J. R. Ward, *Nonlinear second order elliptic partial differential equations at resonance*, Trans. Amer. Math. Soc. 311 (1989), 710–727.

[16] R. Kannan, *Perturbation methods for nonlinear problems at resonance*, in: Nonlinear Functional Analysis and Differential Equations, L. Cesari, R. Kannan and J. D. Schuur (eds.), Dekker, New York, 1976, 209–225.

[17] E. M. Landesman and A. C. Lazer, *Nonlinear perturbations of linear elliptic boundary value problems at resonance*, J. Math. Mech. 19 (1970), 609–623.

[18] N. G. Lloyd, *Degree Theory*, Cambridge Univ. Press, Cambridge, 1978.

[19] J. Mawhin, *Topological degree methods in nonlinear boundary value problems*, Regional Conf. Series in Math. 40, Amer. Math. Soc., Providence, R.I., 1979.

[20] —, *Boundary value problems for vector second order nonlinear ordinary differential equations*, in: Lecture Notes in Math. 703, Springer, 1979, 241–249.

[21] L. C. Piccinini, G. Stampacchia and G. Vidossich, *Ordinary Differential Equations in* $\mathbb{R}^n$, Appl. Math. Sci. 39, Springer, 1984.

[22] B. Przeradzki, *An abstract version of the resonance theorem*, Ann. Polon. Math. 53 (1991), 35–43.

[23] B. Przeradzki, *Operator equations at resonance with unbounded nonlinearities*, preprint.
[24] —, *A new continuation method for the study of nonlinear equations at resonance*, J. Math. Anal. Appl., to appear.
[25] —, *A note on solutions of semilinear equations at resonance in a cone*, Ann. Polon. Math. 58 (1993), 95–103.
[26] —, *The solvability of nonlinear equations with noninvertible linear part*, Acta Univ. Lodzensis, habilitation thesis.
[27] —, *Nonlinear boundary value problems at resonance for differential equations in Banach spaces*, preprint.
[28] B. Ruf, *Multiplicity results for nonlinear elliptic equations*, in: Nonlinear Analysis, Function Spaces and Applications, Vol. 3 (Litomyšl 1986), Teubner-Texte zur Math. 93, Teubner, Leipzig, 1986, 109–138.
[29] J. Santanilla, *Existence of nonnegative solutions of a semilinear equations at resonance with linear growth*, Proc. Amer. Math. Soc. 105 (1989), 963–971.
[30] M. Schechter, J. Shapiro and M. Snow, *Solution of the nonlinear problem* $Au = Nu$ *in a Banach space*, Trans. Amer. Math. Soc. 241 (1978), 69–78.
[31] S. A. Williams, *A sharp sufficient condition for solution of a nonlinear elliptic boundary value problem*, J. Differential Equations 8 (1970), 580–586.
[32] —, *A connection between the Cesari and Leray–Schauder methods*, Michigan Math. J. 15 (1968), 441–448.

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