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EXAMPLES OF NON-LOCAL TIME DEPENDENT OR PARABOLIC DIRICHLET SPACES

BY

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In [23] M. Pierre introduced parabolic Dirichlet spaces. Such spaces are obtained by considering certain families $(E^{(\tau)})_{\tau\in\mathbb{R}}$ of Dirichlet forms. He developed a rather far-reaching and general potential theory for these spaces. In particular, he introduced associated capacities and investigated the notion of related quasi-continuous functions. However, the only examples given by M. Pierre in [23] (see also [22]) are Dirichlet forms arising from strongly parabolic differential operators of second order.

To our knowledge, only very recently, when Y. Oshima in [20] was able to construct a Markov process associated with a time dependent or parabolic Dirichlet space, these parabolic Dirichlet spaces attracted the attention of probabilists. The proof of the existence of such a Markov process depends much on the potential theory developed by M. Pierre. Moreover, in [21] Y. Oshima proved that a lot of results valid for symmetric Dirichlet spaces (see [7] as a standard reference) also hold for time dependent Dirichlet spaces.

The purpose of this note is to give some concrete examples of time dependent Dirichlet spaces which are generated by pseudo-differential operators and therefore are non-local. In Section 1 we recall the basic definition of a time dependent Dirichlet space and in Section 2 we give some auxiliary results. Sections 3–5 are devoted to examples. In Section 3 we discuss some spatially translation invariant operators. We do not really give there any surprising examples, but we emphasize the relation to the theory of balayage spaces. In Section 4 we consider time dependent Dirichlet spaces constructed from a special class of symmetric pseudo-differential operators analogous to those handled in our joint paper [9] with W. Hoh. Finally, in Section 5 we construct time dependent generators of (symmetric) Feller semigroups following [15]. The estimates used in this construction already ensure that we get non-local time dependent Dirichlet spaces.

We would like to mention that non-local Dirichlet forms have recently been investigated by U. Mosco [19] in his study of composite media.

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1. Time dependent symmetric Dirichlet spaces on $\mathbb{R}^n$. Let $L^2(\mathbb{R}^n)$ be the usual space of real-valued measurable functions $u : \mathbb{R}^n \to \mathbb{R}$ which are square integrable. The norm in $L^2(\mathbb{R}^n)$ is denoted by $\|\cdot\|_0$ with the corresponding scalar product $(\cdot,\cdot)_0$. Let $V \subset L^2(\mathbb{R}^n)$ be a dense subspace on which a norm $\|\cdot\|_V$ is defined turning V into a Hilbert space in which the test functions $C_0^\infty(\mathbb{R}^n)$ are dense. We assume that for all $u \in V$ the estimate $\|u\|_0 \le c\|u\|_V$ holds. As usual we identify $L^2(\mathbb{R}^n)$ with its dual space and hence we have

$$V \to L^2(\mathbb{R}^n) \to V' \tag{1.1}$$

with dense and continuous embeddings. Moreover, we have

$$\|u\|_{V'} = \sup_{v\in V, v\neq 0} \frac{|(u,v)_0|}{\|v\|_V}\,. \tag{1.2}$$

We suppose further that $u \in V$ always implies that $v := (0 \vee u) \wedge 1 \in V$. Now, for each $\tau \in \mathbb{R}$ let a symmetric bilinear form $E^{(\tau)} : V \times V \to \mathbb{R}$ be given which satisfies the following conditions:

D.1. For all $u, v \in V$ the function $\tau \mapsto E^{(\tau)}(u,v)$ is measurable on $\mathbb{R}$.

D.2. For any $c \ge 0$ the bilinear form $E_c^{(\tau)}(u,v) := E^{(\tau)}(u,v) + c(u,v)_0$ is uniformly continuous on $V \times V$, i.e. there exists a constant $M = M(c)$ such that

$$|E_c^{(\tau)}(u,v)| \le M\|u\|_V\|v\|_V \tag{1.3}$$

for all $u, v \in V$ and all $\tau \in \mathbb{R}$.

D.3. There exist two constants $c_1 \ge 0$ and $c_0 > 0$ such that

$$E^{(\tau)}(u,u) \ge c_0\|u\|_V^2 - c_1\|u\|_0^2 \tag{1.4}$$

for all $u \in V$ and all $\tau \in \mathbb{R}$.

D.4. For any $v = (0 \vee u) \wedge 1$, $u \in V$,

$$E^{(\tau)}(v,v) \le E^{(\tau)}(u,u)\,. \tag{1.5}$$

Remark 1.1. A. In [20] instead of (1.4) the estimate

$$E^{(\tau)}(u,u) \ge c_0\|u\|_V^2 \tag{1.6}$$

is required, but it is also mentioned that the weaker condition (1.4) is sufficient for all considerations.

B. In this paper we only want to handle symmetric forms $E^{(\tau)}$. For this reason we stated the contraction property D.4 in a form different from, but in our case equivalent to, that stated in [20].

Next we will introduce certain function spaces on $\mathbb{R} \times \mathbb{R}^n$, namely the spaces

$$\mathbf{H} := L^2(\mathbb{R}, L^2(\mathbb{R}^n))\,, \tag{1.7}$$

$$\mathbf{V} := L^2(\mathbb{R}, V) \tag{1.8}$$

and

$$\mathbf{V}' := L^2(\mathbb{R}, V')\,, \tag{1.9}$$

which are equipped with the norms

$$\|u\|_{\mathbf{H}}^2 = \int_{\mathbb{R}} \|u(\tau,\cdot)\|_0^2\,d\tau\,, \tag{1.10}$$

$$\|u\|_{\mathbf{V}}^2 = \int_{\mathbb{R}} \|u(\tau,\cdot)\|_V^2\,d\tau \tag{1.11}$$

and

$$\|u\|_{\mathbf{V}'}^2 = \int_{\mathbb{R}} \|u(\tau,\cdot)\|_{V'}^2\,d\tau\,, \tag{1.12}$$

respectively. Basic properties of these spaces can be found in [17].

Now, following Y. Oshima [20] we define the *time dependent Dirichlet space* $(\mathbf{F},\mathbf{E})$ associated with the family $(E^{(\tau)})_{\tau\in\mathbb{R}}$ by

$$\mathbf{F} := \left\{ u \in \mathbf{V} : \frac{\partial u}{\partial \tau} \in \mathbf{V}' \right\}, \qquad \|u\|_{\mathbf{F}}^2 = \|u\|_{\mathbf{V}}^2 + \left\| \frac{\partial u}{\partial \tau} \right\|_{\mathbf{V}'}^2 \tag{1.13}$$

and

$$\mathbf{E}(u,v) = \int_{\mathbb{R}} E^{(\tau)}(u(\tau,\cdot), v(\tau,\cdot))\,d\tau - \int_{\mathbb{R}} \left(\frac{\partial u}{\partial \tau}(\tau,\cdot), v(\tau,\cdot) \right)_0 d\tau \tag{1.14}$$

for $u \in \mathbf{F}$ and $v \in \mathbf{V}$; for $u \in \mathbf{V}$ and $v \in \mathbf{F}$ we set

$$\mathbf{E}(u,v) = \int_{\mathbb{R}} E^{(\tau)}(u(\tau,\cdot), v(\tau,\cdot))\,d\tau + \int_{\mathbb{R}} \left(\frac{\partial v}{\partial \tau}(\tau,\cdot), u(\tau,\cdot) \right)_0 d\tau\,. \tag{1.15}$$

Clearly $(\mathbf{F}, \|\cdot\|_{\mathbf{F}})$ is a Hilbert space and by our assumptions $C_0(\mathbb{R}\times\mathbb{R}^n)\cap\mathbf{F}$ is dense in $(\mathbf{F}, \|\cdot\|_{\mathbf{F}})$ as well as in $(C_0(\mathbb{R}\times\mathbb{R}^n), \|\cdot\|_\infty)$.

2. Some auxiliary results. In the following the notion of a continuous negative definite function will be central.

DEFINITION 2.1. A function $\psi : \mathbb{R}^n \to \mathbb{C}$ is said to be *negative definite* if for all $m \in \mathbb{N}$ and all $\xi^j \in \mathbb{R}^n$, $1 \le j \le m$, the matrix $(\psi(\xi^i) + \overline{\psi(\xi^j)} - \psi(\xi^i - \xi^j))_{i,j=1,\ldots,m}$ is positive Hermitian.

The basic properties of negative definite functions can be found in [2]. We will only consider real-valued continuous negative definite functions. These functions are always non-negative. Without proof we state

LEMMA 2.1. *Let $a^2 : \mathbb{R}^n \to \mathbb{R}$ be a continuous negative definite function. Then*

$$0 \le a^2(\xi) \le c_a(1 + |\xi|^2)\,; \tag{2.1}$$

$$a^2(\xi) = c + Q(\xi) + \int_{\mathbb{R}^n} (1 - \cos(\xi,\eta)) \frac{1 + |\eta|^2}{|\eta|^2}\, d\sigma(\eta)\,, \tag{2.2}$$

where c is a non-negative constant, Q a non-negative quadratic form and σ is a positive measure on $\mathbb{R}^n$ not charging the origin and having finite total mass;

$$|a^2(\xi) - a^2(\eta)| \le 4a(\xi)a(\xi - \eta) + a^2(\xi - \eta)\,; \tag{2.3}$$

$$|a(\xi) - a(\eta)| \le a(\xi - \eta)\,. \tag{2.4}$$

Remark 2.1. The estimate (2.1) can be found in [2], as well as the Lévy–Khinchin formula (2.2). We have taken (2.3) from [11], and (2.4) is due to W. Hoh [8].

For any continuous negative definite function $a^2 : \mathbb{R}^n \to \mathbb{R}$ and any $s \ge 0$ we introduce the Hilbert space

$$H^{a^2,s}(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : \|u\|_{s,a^2} < \infty\}\,, \tag{2.5}$$

where

$$\|u\|^2_{s,a^2} = \int_{\mathbb{R}^n} (1 + a^2(\xi))^{2s} |\widehat{u}(\xi)|^2\, d\xi\,. \tag{2.6}$$

Here $\widehat{u}$ denotes the Fourier transform of u. Clearly $H^{a^2,0}(\mathbb{R}^n) = L^2(\mathbb{R}^n)$ and if we identify $[L^2(\mathbb{R}^n)]^*$ with $L^2(\mathbb{R}^n)$ we have (see [14])

$$[H^{a^2,s}(\mathbb{R}^n)]^* = H^{a^2,-s}(\mathbb{R}^n)\,, \tag{2.7}$$

where

$$H^{a^2,-s}(\mathbb{R}^n) = \{u \in S'(\mathbb{R}^n) : \|u\|_{-s,a^2} < \infty\}$$

and the "negative norm" is given on $L^2(\mathbb{R}^n)$ by

$$\|u\|^2_{-s,a^2} = \int_{\mathbb{R}^n} (1 + a^2(\xi))^{-2s} |\widehat{u}(\xi)|^2\, d\xi = \sup_{0 \ne v \in H^{a^2,s}(\mathbb{R}^n)} \frac{|(u,v)_0|}{\|v\|_{s,a^2}}\,.$$

Later we will often assume that a^2 also satisfies

$$a^2(\xi) \geq c_r|\xi|^r \tag{2.8}$$

for some $r > 0$ and all $\xi \in \mathbb{R}^n$, $|\xi| \geq \varrho \geq 0$. In this case $H^{a^2,s}(\mathbb{R}^n)$ is continuously embedded in the usual Sobolev space $H^{sr}(\mathbb{R}^n)$ and for $sr > n/2$ we find $H^{a^2,s}(\mathbb{R}^n) \subset C_\infty(\mathbb{R}^n)$ with a continuous embedding.

For later considerations it is useful to note that if $a^{-2} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then $H^{a^2,1/2}(\mathbb{R}^n)$ is a Dirichlet space with respect to the form

$$B^{a^2}(u,v) = \int_{\mathbb{R}^n} a^2(\xi)\widehat{u}(\xi)\overline{\widehat{v}(\xi)}\, d\xi\,. \tag{2.9}$$

In particular, the form B^{a^2} satisfies D.1–D.4 with $V = H^{a^2,1/2}(\mathbb{R}^n)$. Moreover, all translation invariant symmetric Dirichlet forms on $L^2(\mathbb{R}^n)$ are of this type (see [3] or [6]).

If $a^2 : \mathbb{R}^n \to \mathbb{R}$ is a continuous negative definite function we can define the pseudo-differential operator $a^2(D)$ on $H^{a^2,1}(\mathbb{R}^n)$ by

$$a^2(D)u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a^2(\xi)\widehat{u}(\xi)\, d\xi\,. \tag{2.10}$$

3. Spatially translation invariant time dependent Dirichlet forms. Let $a^2 : \mathbb{R}^n \to \mathbb{R}$ be a continuous negative definite function such that $a^{-2} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Further, let $p : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that for each $\tau \in \mathbb{R}$ the function $p(\tau,\cdot) : \mathbb{R}^n \to \mathbb{R}$ is negative definite. For p we assume that with two constants $0 < \gamma_0 \leq \gamma_1$,

$$\gamma_0(1 + a^2(\xi)) \leq 1 + p(\tau,\xi) \leq \gamma_1(1 + a^2(\xi)) \tag{3.1}$$

for all $\tau \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$. Now, for $\tau \in \mathbb{R}$ we define on $H^{a^2,1/2}(\mathbb{R}^n)$,

$$E^{(\tau)}(u,v) = \int_{\mathbb{R}^n} p(\tau,\xi)\widehat{u}(\xi)\overline{\widehat{v}(\xi)}\, d\xi\,. \tag{3.2}$$

Clearly, for fixed $u, v \in H^{a^2,1/2}(\mathbb{R}^n)$ the function $\tau \mapsto E^{(\tau)}(u,v)$ is measurable and from (3.1) it follows immediately that

$$|E^{(\tau)}(u,v)| \leq c\|u\|_{1/2,a^2}\|v\|_{1/2,a^2} \tag{3.3}$$

and

$$E^{(\tau)}(u,u) \geq c_0\|u\|^2_{1/2,a^2} - c_1\|u\|^2_0\,. \tag{3.4}$$

In particular, each $E^{(\tau)}$ is a closed symmetric form. Moreover, by our assumption that $p(\tau,\cdot)$ is a continuous negative definite function, it follows that $E^{(\tau)}$ also has the contraction property D.4. Thus on

$$\mathbf{F} = \left\{ u \in L^2(\mathbb{R}, H^{a^2,1/2}(\mathbb{R}^n)) : \frac{\partial u}{\partial \tau} \in L^2(\mathbb{R}, H^{a^2,-1/2}(\mathbb{R}^n)) \right\}$$

we get the time dependent Dirichlet form

$$\begin{aligned} \mathbf{E}(u,v) &= \int_{\mathbb{R}} E^{(\tau)}(u(\tau,\cdot), v(\tau,\cdot))\, d\tau - \int_{\mathbb{R}} \left(\frac{\partial u}{\partial \tau}(\tau,\cdot), v(\tau,\cdot) \right)_0 d\tau \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^n} p(\tau,\xi)\widehat{u}(\tau,\xi)\overline{\widehat{v}(\tau,\xi)}\, d\xi\, d\tau \\ &\quad - \int_{\mathbb{R}} \int_{\mathbb{R}^n} \frac{\partial u}{\partial \tau}(\tau,x) v(\tau,x)\, dx\, d\tau\,, \end{aligned} \tag{3.5}$$

where $u \in \mathbf{F}$ and $v \in L^2(\mathbb{R}, H^{a^2,1/2}(\mathbb{R}^n))$. The case $u \in L^2(\mathbb{R}, H^{a^2,1/2}(\mathbb{R}^n))$ and $v \in \mathbf{F}$ gives the formula analogous to (1.15). Note that in (3.5), $\widehat{u}(\tau,\xi)$ denotes the Fourier transform with respect to x only, i.e.

$$\widehat{u}(\tau,\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(\tau,x)\, dx. \tag{3.6}$$

In particular, the example covers the case $p(\tau,\xi) = a^2(\xi)$. In [10] it was proved that if (2.8) holds, then $a^2(D)$ generates a balayage space in the sense of [4]. Moreover, in [5] M. Brzezina showed that under the same conditions on a^2 the operators $\partial/\partial t + a^2(D)$ and $\partial/\partial t - a^2(D)$ also generate a balayage space. But the time dependent Dirichlet form $\mathbf{E}$ associated with the family $E^{(\tau)}(u,v) = \int_{\mathbb{R}^n} a^2(\xi)\widehat{u}(\xi)\overline{\widehat{v}(\xi)}\, d\xi$ is generated by the operator $\partial/\partial t - a^2(D)$, namely

$$\mathbf{E}(u,v) = -\int_{\mathbb{R}} \int_{\mathbb{R}^n} \left[\frac{\partial u}{\partial \tau}(\tau,x) - a^2(D)u(\tau,x) \right] v(\tau,x)\, dx\, d\tau$$

for $u \in L^2(\mathbb{R}, H^{a^2,1}(\mathbb{R}^n)) \cap \mathbf{F}$ and $v \in L^2(\mathbb{R}, H^{a^2,1/2}(\mathbb{R}^n))$. Thus it seems reasonable to conjecture that also each of the operators $p(\tau,D) - \partial/\partial\tau$ generates a balayage space.

In [1] non-symmetric translation invariant Dirichlet spaces were characterized by complex-valued continuous negative definite functions ψ, which have the property that for all $\xi \in \mathbb{R}^n$ the estimate $|\operatorname{Im}\psi(\xi)| \le c \operatorname{Re}\psi(\xi)$ holds. It is an easy exercise to construct as above time dependent spatially translation invariant non-symmetric and non-local Dirichlet spaces by starting with this characterization.

4. Non-local time dependent Dirichlet forms with variable coefficients. I. We will closely follow our joint paper with W. Hoh [9]. For this reason let $a_j^2 : \mathbb{R} \to \mathbb{R}$, $1 \le j \le n$, be a continuous negative definite function which has the representation

$$a_j^2(\xi_j) = \int_{\mathbb{R}} (1 - \cos(\eta_j \xi_j))\, \widetilde{\mu}_j(d\eta_j)\,. \tag{4.1}$$

Further, let $b_j : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $(\tau, x) \mapsto b_j(\tau, x)$, $1 \le j \le n$, be a function satisfying the following conditions:

(4.2) b_j is independent of x_j;

(4.3) $b_j(\tau, \cdot)$ is bounded and measurable;

(4.4) $\tau \mapsto b_j(\tau, x)$ is a continuous function;

(4.5) $b_j(\tau, x) \ge d_0 > 0$ for all $(\tau, x) \in \mathbb{R} \times \mathbb{R}^n$ and $1 \le j \le n$.

On $C_0^\infty(\mathbb{R}^n)$ we consider the family of pseudo-differential operators

$$L^{(\tau)}(x, D)u(x) = \sum_{j=1}^{n} b_j(\tau, x) a_j^2(D_j)\,. \tag{4.6}$$

We can associate with $L^{(\tau)}(x, D)$ the bilinear form

$$\begin{aligned} E^{(\tau)}(u, v) &= \int_{\mathbb{R}^n} L^{(\tau)}(x, D)u(x) \cdot v(x)\, dx \\ &= \sum_{j=1}^{n} (b_j(\tau, \cdot) a_j(D_j)u, a_j(D_j)v)_0\,. \end{aligned} \tag{4.7}$$

Now let

$$a^2(\xi) = \sum_{j=1}^{n} a_j^2(\xi_j)\,. \tag{4.8}$$

Then a^2 is a continuous negative definite function on $\mathbb{R}^n$, and the space $H^{a^2,1/2}(\mathbb{R}^n)$ is well defined. Using (4.2)–(4.5) we get, as in [9],

PROPOSITION 4.1. *For all $u, v \in H^{a^2,1/2}(\mathbb{R}^n)$,*

$$|E^{(\tau)}(u, v)| \le c\|u\|_{1/2,a^2} \|v\|_{1/2,a^2} \tag{4.9}$$

and

$$E^{(\tau)}(u, u) \ge d_0 \|u\|^2_{1/2,a^2} - d_0 \|u\|_0^2\,. \tag{4.10}$$

Clearly for all $u, v \in H^{a^2,1/2}(\mathbb{R}^n)$ the function $\tau \mapsto E^{(\tau)}(u, v)$ is measurable. As in [9], Proposition 3.1, we find, for $u, v \in H^{a^2,1/2}(\mathbb{R}^n)$,

$$\begin{aligned} &E^{(\tau)}(u, v) \\ &\quad = \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (u(x + y) - u(x))(v(x + y) - v(x)) \sum_{j=1}^{n} b_j(\tau, x)\, \mu_j(dy)\, dx\,, \end{aligned} \tag{4.11}$$

where μ_j is the image of $\widetilde{\mu}_j$ under the mapping $T_j : \mathbb{R} \to \mathbb{R}^n$, $\xi_j \mapsto (0,\ldots,0,\xi_j,0,\ldots,0)$, i.e. ξ_j is in the jth position. But (4.11) implies D.4. Thus

$$\mathbf{E}(u,v) = \frac{1}{2}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} (u(\tau,x+y)-u(\tau,x))(v(\tau,x+y)-v(\tau,x)) \times \sum_{j=1}^{n} b_j(\tau,x)\,\mu_j(dy)\,dx\,d\tau - \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^n} \frac{\partial u}{\partial \tau}(\tau,x)\cdot v(\tau,x)\,dx\,d\tau \tag{4.12}$$

defined for $u \in \{w \in L^2(\mathbb{R}, H^{a^2,1/2}(\mathbb{R}^n)) : \partial w/\partial\tau \in L^2(\mathbb{R}, H^{a^2,-1/2}(\mathbb{R}^n))\}$ and $v \in L^2(\mathbb{R}, H^{a^2,1/2}(\mathbb{R}^n))$ gives a time dependent Dirichlet form. (We have to make the usual modification in the case that $u \in L^2(\mathbb{R}, H^{a^2,1/2}(\mathbb{R}^n))$ and $v \in \{w \in L^2(\mathbb{R}, H^{a^2,1/2}(\mathbb{R}^n)) : \partial w/\partial\tau \in L^2(\mathbb{R}, H^{a^2,-1/2}(\mathbb{R}^n))\}$.)

5. Non-local time dependent Dirichlet forms with variable coefficients. II. In a series of papers [12]–[15] we constructed Feller semigroups starting with certain pseudo-differential operators. The most general case was treated in [15]. If these pseudo-differential operators are symmetric on $L^2(\mathbb{R}^n)$, we also get certain non-local Dirichlet spaces. It turns out that we can make the whole construction also time dependent. The result will be handled in this section while the proofs, which are essentially based on the proofs in [15], are sketched in the appendix.

Let $p : \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ be a continuous function such that for fixed $\tau \in \mathbb{R}$ and fixed $x \in \mathbb{R}^n$ the function $p(\tau,x,\cdot) : \mathbb{R}^n \to \mathbb{R}$ is negative definite. Further, suppose that for some $x_0 \in \mathbb{R}^n$ we have

$$p(\tau,x,\xi) = p(\tau,x_0,\xi)+(p(\tau,x,\xi)-p(\tau,x_0,\xi)) = p_1(\tau,\xi)+p_2(\tau,x,\xi)\,. \tag{5.1}$$

For p_1 and p_2 we impose the following assumptions:

P.1. There exists a continuous negative definite function $a^2 : \mathbb{R}^n \to \mathbb{R}$ satisfying (2.8) such that

$$|p_1(\tau,\xi)| \le \gamma_1(1+a^2(\xi)) \tag{5.2}$$

for all $\xi \in \mathbb{R}^n$ and all $\tau \in \mathbb{R}$.

P.2. Let a^2 be as in P.1. For some $q \in \mathbb{N}$ the function $x \mapsto p_2(\tau,x,\xi)$ is q times differentiable and for any $\alpha \in \mathbb{N}_0^n$, $|\alpha| \le q$, there exists a function $\varphi_\alpha \in L^1(\mathbb{R}^n)$ such that

$$|\partial_x^\alpha p_2(\tau,x,\xi)| \le \varphi_\alpha(x)(1+a^2(\xi))\,. \tag{5.3}$$

P.3. For all $\xi \in \mathbb{R}^n$, $|\xi| \ge \varrho \ge 0$,

$$p_1(\tau,\xi) \ge \gamma_0 a^2(\xi) \tag{5.4}$$

for all $\tau \in \mathbb{R}$.

P.4. Define

$$\gamma_b = c_a \widetilde{\gamma}_q \sum_{|\alpha|\le q} \|\varphi_\alpha\|_{L^1} \int\limits_{\mathbb{R}^n} (1+|y|^2)^{(1-q)/2}\, dy\,, \qquad q > n+1\,, \tag{5.5}$$

where $\widetilde{\gamma}_q$ is a constant such that

$$(1+|\xi|^2)^{q/2} \le \widetilde{\gamma}_q \sum_{|\alpha|\le q} |\xi^\alpha|\,.$$

Then for some ε, $0<\varepsilon<1$, we require the estimate

$$\gamma_b \le (1-\varepsilon)\gamma_0\,. \tag{5.6}$$

P.5. Let γ_0 be as in P.3 and set

$$\gamma_c = \widetilde{\gamma}_q \sum_{|\alpha|\le q} \|\varphi_\alpha\|_{L^1} \int\limits_{\mathbb{R}^n} (1+|y|^2)^{-q/2}\, dy\,, \qquad q > n\,. \tag{5.7}$$

Then we assume

$$\gamma_c \le \gamma_0/\sqrt{24}\,. \tag{5.8}$$

P.6. The operator $p(\tau,x,D)$ is symmetric, i.e. for all $u,v \in C_0^\infty(\mathbb{R}^n)$ and any fixed $\tau\in\mathbb{R}$ we have

$$\int\limits_{\mathbb{R}^n} p(\tau,x,D)u(x)\cdot v(x)\, dx = \int\limits_{\mathbb{R}^n} u(x)\cdot p(\tau,x,D)v(x)\, dx\,. \tag{5.9}$$

In [16], Lemma 2.1, we gave an easy sufficient condition for $p(\tau,x,D)$ to be symmetric if τ is fixed.

THEOREM 5.1. *Suppose that* P.1–P.6 *hold for some sufficiently large q. Then for each fixed $\tau\in\mathbb{R}$ the operator $-p(\tau,x,D)$ extends to a generator of a symmetric Feller semigroup. Further, for each $\tau\in\mathbb{R}$ it is possible to associate with the operator $-p(\tau,x,D)$ a Dirichlet form $E^{(\tau)}$ with domain $H^{a^2,1/2}(\mathbb{R}^n)$. The family $(E^{(\tau)})_{\tau\in\mathbb{R}}$ gives a time dependent Dirichlet space in the sense of Y. Oshima and M. Pierre.*

As mentioned before, the proof of the theorem will be sketched in the appendix.

Finally, let us indicate how to get further time dependent Dirichlet spaces from the examples already given. Let $\Omega\subset\mathbb{R}^n$ be an open set. Then $C_0^\infty(\Omega)\subset H^{a^2,1/2}(\mathbb{R}^n)$ for any a^2 under consideration. Clearly all estimates also hold for this subspace. Since the contraction property needs only to be checked on a dense subspace (see [18]) we conclude that if $E^{(\tau)}$ is a symmetric Dirichlet form on $H^{a^2,1/2}(\mathbb{R}^n)$, then it is also a symmetric Dirichlet form on $H_0^{a^2,1/2}(\Omega)$, the closure of $C_0^\infty(\Omega)$ in $H^{a^2,1/2}(\mathbb{R}^n)$. Thus we can apply our previous results. However, the spaces $H^{a^2,1/2}(\mathbb{R}^n)$ and $H^{a^2,-1/2}(\mathbb{R}^n)$ in the definition of **F** must be substituted by $H_0^{a^2,1/2}(\Omega)$ and its dual space $[H_0^{a^2,1/2}(\Omega)]'$.

Appendix. In order to prove Theorem 5.1 we have to inspect the time dependence in the key steps of the proof of Theorem 5.2 in [15].

LEMMA I. *Suppose that* P.1–P.5 *hold for* q *sufficiently large.*

A. *Define* $\widehat{p}_2(\tau,\eta,\xi)$ *by*

$$\widehat{p}_2(\tau,\eta,\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\eta} p_2(\tau,x,\xi)\,dx\,.$$

Then

$$|\widehat{p}_2(\tau,\eta,\xi)| \le \widetilde{\gamma}_q \sum_{|\alpha|\le q} \|\varphi_\alpha\|_{L^1}(1+|\eta|^2)^{-q/2}(1+a^2(\xi))\,. \tag{A.1}$$

B. *For all* $u \in H^{a^2,s+t+1/2}(\mathbb{R}^n)$,

$$\|[(1+a^2(D))^s, p_2(\tau,x,D)]u\|_{t,a^2} \le c\|u\|_{s+t+1/2,a^2} \tag{A.2}$$

holds if $q > n+2s+2|t|+4N+4$, *where* $N\in\mathbb{N}$ *satisfies* $0 < s-N \le 1$.

C. *For all* $t>0$ *and all* $u\in H^{a^2,t+1}(\mathbb{R}^n)$,

$$\|p(\tau,x,D)u\|_{t,a^2} \le c\|u\|_{t+1,a^2}\,. \tag{A.3}$$

D. *Let* $E^{(\tau)}(u,v) = (p(\tau,x,D)u,v)_0$. *Then for all* $u,v\in H^{a^2,1/2}(\mathbb{R}^n)$,

$$|E^{(\tau)}(u,v)| \le c\|u\|_{1/2,a^2}\|v\|_{1/2,a^2} \tag{A.4}$$

and

$$E^{(\tau)}(u,u) \ge \varepsilon\gamma_0\|u\|^2_{1/2,a^2} - c_0\|u\|_0^2\,. \tag{A.5}$$

Proof. A. For $\alpha\in\mathbb{N}_0^n$, $|\alpha|\le q$, we have

$$\Big|\eta^\alpha \int_{\mathbb{R}^n} e^{-ix\cdot\eta}p_2(\tau,x,\xi)\,dx\Big| = \Big|\int_{\mathbb{R}^n} e^{-ix\cdot\eta}\partial_x^\alpha p_2(\tau,x,\xi)\,dx\Big| \le \int_{\mathbb{R}^n}\varphi_\alpha(x)(1+a^2(\xi))\,dx\,,$$

which gives (A.1).

B. We have

$$|([(1+a^2(D))^s, p_2(\tau,x,D)]u,v)_0| \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\widehat{p}_2(\tau,\xi-\eta,\eta)|\cdot|(1+a^2(\xi))^s-(1+a^2(\eta))^s|\cdot|\widehat{u}(\eta)|\cdot|\widehat{v}(\xi)|\,d\eta\,d\xi$$

and (A.1) immediately implies that the proof of Theorem 2.1 of [15] applies.

C. Since for $p_1(\tau, D)$ the estimate is trivial and by part B we have

$$\|p_2(\tau,x,D)u\|_{t,a^2} \le \|p_2(\tau,x,D)(1+a^2(D))^t u\|_0 + \|[(1+a^2(D))^t, p_2(\tau,x,D)]u\|_0\,,$$

we only have to prove $\|p_2(\tau,x,D)u\|_0 \le c\|u\|_{1,a^2}$. For $u \in H^{a^2,1}(\mathbb{R}^n)$ and $v \in L^2(\mathbb{R}^n)$ we find

$$\text{(A.6)} \quad |(p_2(\tau,x,D)u,v)_0| = \Big|\int_{\mathbb{R}^n} \widehat{p}_2(\tau,\xi-\eta,\eta)\widehat{u}(\eta)\overline{\widehat{v}(\xi)}\,d\eta\,d\xi\Big|$$

$$\le \widetilde{\gamma}_q \sum_{|\alpha|\le q} \|\varphi_\alpha\|_{L^1} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (1+|\xi-\eta|^2)^{-q/2}(1+a^2(\eta))|\widehat{u}(\eta)|\cdot|\widehat{v}(\xi)|\,d\eta\,d\xi\,,$$

which gives (A.3).

D. Clearly it suffices to prove (A.4) and (A.5) for $u, v \in C_0^\infty(\mathbb{R}^n)$. First we prove (A.4). For this note that

$$\begin{aligned} |E^{(\tau)}(u,v)| &= |(p(\tau,x,D)u,v)_0| \\ &\le \Big|\int_{\mathbb{R}^n} \widehat{p}_1(\tau,\xi)\widehat{u}(\xi)\overline{\widehat{v}(\xi)}\,d\xi\Big| \\ &\quad + \Big|\int_{\mathbb{R}^n} \widehat{p}_2(\tau,\xi-\eta,\eta)\widehat{u}(\eta)\overline{\widehat{v}(\xi)}\,d\eta\,d\xi\Big|\,, \end{aligned}$$

which implies the estimate by our previous considerations. In order to get (A.5) note first that

$$\begin{aligned} E^{(\tau)}(u,u) &\ge \int_{\mathbb{R}^n} \widehat{p}_1(\tau,\xi)|\widehat{u}(\xi)|^2\,d\xi - |(p_2(\tau,x,D)u,u)_0| \\ &\ge \gamma_0\|u\|^2_{1/2,a^2} - c\|u\|_0^2 - |(p_2(\tau,x,D)u,u)_0| \end{aligned}$$

and the definition of γ_b implies by (A.6) that

$$|(p_2(\tau,x,D)u,u)_0| \le \gamma_b\|u\|^2_{1/2,a^2}\,,$$

which together with (5.6) gives the required estimate.

Now fix $\tau \in \mathbb{R}$. Then we find that for each $\lambda \ge c_0$, c_0 given as in (A.5), the representation problem:

Find $u \in H^{a^2,1/2}(\mathbb{R}^n)$ such that for a given $f \in L^2(\mathbb{R}^n)$,

$$\text{(A.7)} \qquad E^{(\tau)}(u,v) + \lambda(u,v)_0 = (f,v)_0$$

holds for all $v \in H^{a^2,1/2}(\mathbb{R}^n)$,

has a unique solution.

We need further regularity properties for this solution.

Lemma II. *Suppose that* P.1–P.5 *hold. Further, let* $t \ge 0$ *and* $q > n + 2|t| + 4N + 4$, $0 < t - N \le 1$.

A. *For all* $u \in H^{a^2,1/2}(\mathbb{R}^n)$ *we have the estimate*

$$\|u\|_{t+1,a^2} \le \gamma(\|p(\tau,x,D)u\|_{t,a^2} + \|u\|_0)\,. \tag{A.8}$$

B. *If* $u \in H^{a^2,1/2}(\mathbb{R}^n)$ *is a solution of the representation problem* (A.7), *then* $f \in H^{a^2,t}(\mathbb{R}^n)$, $t \ge 0$, *implies* $u \in H^{a^2,t+1}(\mathbb{R}^n)$.

P r o o f. A. By P.3 we have

$$\|p_1(\tau,D)u\|^2_{t,a^2} \ge \tfrac{1}{2}\gamma_0\|u\|^2_{t+1,a^2} - c(\varrho,t,a^2)\|u\|_0^2$$

and from (A.3) we get for any $\varepsilon \in (0,1)$,

$$\|p_2(\tau,x,D)u\|^2_{t,a^2} \le 2(1+\varepsilon)^2\gamma_c^2\|u\|^2_{t+1,a^2} + \widetilde{c}\,\|u\|_0^2\,.$$

Both estimates suffice in order to use the proof of Theorem 4.2 of [15] for the new situation.

B. Once we have proved that

$$\|[J_\varepsilon, p_2(\tau,x,D)]u\|_{t,a^2} \le c\|u\|_{t+1/2,a^2}\,,$$

where J_ε denotes the Friedrichs mollifier, we can use the proof of Theorem 4.5 of [15]. But for $u,v \in C_0^\infty(\mathbb{R}^n)$ we find

$$|([J_\varepsilon, p_2(\tau,x,D)]u,v)_0| = \Big|\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \widehat{p}_2(\tau,\eta-\xi,\xi)r_1(\xi,\eta,\varepsilon)\widehat{u}(\xi)\overline{\widehat{v}(\eta)}\,d\xi\,d\eta\Big|\,,$$

where $r_1(\xi,\eta,\varepsilon)$ denotes the remainder in the Taylor expansion of the function $\widehat{j}(\varepsilon\cdot)$, when j defines the mollifier, i.e. we have

$$\widehat{j}(\varepsilon\xi) = \widehat{j}(\varepsilon\eta) + r_1(\xi,\eta,\varepsilon)\,.$$

But now, by (A.1), it is clear that our estimates will be independent of τ.

Now, since $p(\tau,x,D)$ is symmetric, we can conclude from Lemmas I and II that for each fixed τ a Dirichlet form $E^{(\tau)}$ is defined on $H^{a^2,1/2}(\mathbb{R}^n)$ by

$$E^{(\tau)}(u,v) = \int\limits_{\mathbb{R}^n} p_1(\tau,\xi)\widehat{u}(\xi)\overline{\widehat{v}(\xi)}\,d\xi + \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \widehat{p}_2(\tau,\xi-\eta,\eta)\widehat{u}(\eta)\overline{\widehat{v}(\xi)}\,d\eta\,d\xi\,.$$

Indeed, from Lemmas I and II it follows that for each fixed $\tau \in \mathbb{R}$ the operator $-p(\tau,x,D)$ extends to the generator of a symmetric Feller semigroup, which gives the contraction property for $E^{(\tau)}$.

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