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MULTIPLIER THEOREM ON GENERALIZED HEISENBERG GROUPS

BY

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Introduction. Let N be a homogeneous stratified Lie group (cf. [3]),

$$L = -(X_1^2 + \ldots + X_k^2)$$

a sublaplacean. Let

$$Lf = \int_0^\infty \lambda \, dE(\lambda) f$$

be its spectral resolution (on $L^2(N)$), and for $m \in L^\infty(\mathbb{R}_+)$,

$$m(L)f = \int_0^\infty m(\lambda) \, dE(\lambda) f \, .$$

Conditions on the function m which guarantee boundedness of $m(L)$ on $L^p(N)$, $1 < p < \infty$, have a long history. In 1960 L. Hörmander proved that if N is abelian and for a nonzero $\phi \in C_c^\infty(\mathbb{R}_+)$,

$$\sup_{t>0} \|\phi m(t\,\cdot)\|_{H(s)} < \infty$$

for an s greater than half of the (topological) dimension of N, then $m(L)$ is of weak type 1-1 and bounded on L^p, $1 < p < \infty$.

For general stratified groups M. Christ [1] and G. Mauceri and S. Meda [10] showed that the Hörmander theorem holds if the topological dimension is replaced by the homogeneous dimension. Recently D. Müller and E. M. Stein [12] showed that if N is a cartesian product of copies of Heisenberg groups and abelian groups then, in fact, in the Hörmander theorem s greater than half of the topological dimension suffices. A bit earlier J. Randall [13] obtained estimates for the heat kernel on generalized Heisenberg groups which imply the multiplier theorem with s greater than half of the euclidean dimension plus a constant, so if the dimension of the center is large this is less than half of the homogeneous dimension.

The aim of this note is to prove by a different method a somewhat more general theorem than the above mentioned theorem by D. Müller and E. M. Stein.

Preliminaries. We say that a two-step nilpotent Lie algebra N is a *generalized Heisenberg Lie algebra* if there is a scalar product on N and an orthogonal decomposition

$$N = W \oplus [N, N]$$

such that for each $x \in W$ of length 1 the mapping ad_x^* is an isometry from $[N, N]^*$ into W^*. We call W the *generating subspace* of N. We identify Lie algebras with Lie groups (using the exponential map), and we say that N is a *generalized Heisenberg group* if, as a Lie algebra, it is a generalized Heisenberg Lie algebra. With this identification 0 is the neutral element in our groups.

As a matter of fact we use only two properties of a generalized Heisenberg group: first, that the dimension of its center is at most half of the topological dimension of the group; second, that the euclidean Fourier transform of the heat kernel has a particularly simple form [13] which implies our formula replacing the convolution kernels associated with L by kernels associated with a Grushin operator.

In the sequel we assume that $N = \prod N_i$ with each N_i being a generalized Heisenberg group with generating subspace W_i. Let $|x|_i$ be the length of x in W_i (we fix a scalar product). We write $W = \bigoplus W_i$. We may consider N as a direct sum of W_i and $[N_i, N_i]$ so the projection $\pi_i : N \to W_i$ is well defined. Put

$$w_i(x) = |\pi_i(x)|_i \,.$$

N has also a natural structure of a homogeneous group: elements in W are of degree 1 and elements in $[N, N]$ are of degree 2. Assume that X_j are left invariant vector fields on N such that there exist j_i such that $j_0 = 1$ and $\{X_j : j = j_{i-1}, \ldots, j_i\}$ is an orthonormal basis of N_i. Let

$$L = -\sum_{j=1}^{\dim(W)} X_j^2 \,.$$

We say that a curve $\gamma \in C^\infty([0, 1], N)$ is *admissible* if

$$\gamma'(s) = \sum_{j=1}^{\dim(W)} a_j(s) X_j(\gamma(s))$$

and we write

$$|x|^2 = \inf \int_0^1 \sum_{j=1}^{\dim(W)} a_j(s)^2$$

where the inf is taken over all admissible curves γ such that $\gamma(0) = 0$ and $\gamma(1) = x$. $|x|$ is the optimal control distance to 0. Of course, on W_i we have $|x|_i = |x|$.

(1.1) THEOREM. *If N is a product of generalized Heisenberg groups, L a sublaplacean on N as described above, $n = \dim(N)$, $s > n/2$, $\phi \in C_c^\infty(\mathbb{R}_+)$, $\phi \neq 0$ and m satisfies*

$$\sup_{t>0} \|\phi m(t)\|_{H(s)} < \infty$$

then $m(L)$ is of weak type 1-1 and bounded on L^p, $1 < p < \infty$.

In the sequel we will identify functions of L (which are operators) with functions on N (their convolution kernels). Put $\varepsilon = (s - n/2)/3$. Let $\phi \in C_c^\infty(\mathbb{R}_+)$, $\operatorname{supp} \phi \subset (1/2, 2)$, $\sum \phi(2^k x) = 1$ for $x > 0$. Write $m_k(x) = \phi(2^k x) m(x)$. It is enough to show that:

(1.2) LEMMA. *There exists C such that for all m as in (1.1) and all integers k,*

$$\int |x|^\varepsilon |m_k(L)|(x)\, dx \le C 2^{k\varepsilon/2}$$

and

$$\int |x|^\varepsilon |X_j m_k(L)|(x)\, dx \le C 2^{k(\varepsilon-1)/2}, \quad j = 1, \ldots, \dim(W)\,.$$

We will prove the first estimate with $k = 0$. We pass to arbitrary k using dilations. The second estimate (for $k = 0$) follows from the first estimate applied to $\sqrt{L} m_0(L)$ and continuity of $X_j(\sqrt{L})^{-1}$ on weighted L^p spaces. In fact, the second estimate is not necessary to prove multiplier theorems (in [5] we show how to do this for L being a Schrödinger operator), but in the present setting it is easily available.

Before proving the theorem we need to study the structure of L. This part of our analysis remains valid on any two-step stratified group. First, let us note the following :

(1.3) LEMMA. *If $\{a_{i,k}\}_{i,k=1}^n$ is an orthogonal matrix and $Y_i = \sum a_{i,k} X_k$, then $\sum Y_i^2 = \sum X_k^2$.*

Next, in an appropriate basis of N we have

$$X_i = \partial_i + V_i$$

where V_i is a vector field with linear coefficients and containing only derivatives in $[N, N]$ directions. Then

$$L = -\sum X_i^2 = -\sum \partial_i^2 - \sum V_i^2 - 2 \sum \partial_i V_i\,.$$

The third term of this sum (we call it the *mixed terms*) complicates the structure of L, so we would like to get rid of it. To do this we are going to prove that the mixed terms commute with L. Note that the mixed terms are invariant under central translations. Hence it is enough to show that the Fourier transform of the mixed terms in $[N, N]$ variables commutes with the Fourier transform of L in $[N, N]$ variables. This is equivalent to looking at all quotients of N by codimension 1 subspaces of $[N, N]$. Such a quotient is (isomorphic to) the product of a euclidean space and the Heisenberg group. After a linear change of coordinates we may assume that the projection of L onto such a quotient has the following canonical form:

$$L = -\sum \lambda_i \left(X_i^2 + Y_i^2\right) - \sum R_j^2 ,$$

where the fields X_i, Y_i, R_j, Z form a basis, $[X_i, Y_i] = Z$ and the other brackets (of the base fields) are zero. Indeed, an orthogonal transformation (which by (1.3) leaves L invariant) allows us to reduce symplectic forms (i.e. the bracket) to the canonical form with $[X_i, Y_i] = 1/\lambda_i$. After rescaling X_i and Y_i we get our form.

Next, in exponential coordinates,

$$X_i = \frac{\partial}{\partial x_i} - \frac{1}{2} y_i \frac{\partial}{\partial z}, \qquad Y_i = \frac{\partial}{\partial y_i} + \frac{1}{2} x_i \frac{\partial}{\partial z} ,$$

so

$$\begin{aligned} L = & -\sum \lambda_i \left(\left(\frac{\partial}{\partial x_i}\right)^2 + \left(\frac{\partial}{\partial y_i}\right)^2 \right) - \sum R_j^2 \\ & - \frac{1}{4} \sum \lambda_i (x_i^2 + y_i^2) \left(\frac{\partial}{\partial z}\right)^2 - \sum \lambda_i \left(x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i} \right) \frac{\partial}{\partial z} . \end{aligned}$$

In this case the mixed terms reduce to the product of the derivative in the central direction and the vector field generating rotations in the (x_i, y_i) planes. Of course L commutes with such rotations, which gives our claim.

Another important property of the mixed terms is that when applied to δ_0 (the unit mass at 0), they give zero. This follows from the argument above (δ_0 is rotation invariant). Let A be L with the mixed terms omitted. Our previous considerations imply

$$e^{-tL}\delta_0 = e^{-tA}\delta_0 .$$

Indeed,

$$e^{-tL}\delta_0 = e^{-tA-t(L-A)}\delta_0 = e^{-tA}e^{t(A-L)}\delta_0 = e^{-tA}\delta_0 .$$

This formula is an abstract reformulation of the formulas given by A. Hulanicki [8], B. Gaveau [4] and J. Cygan [2].

We need the form of A on products of generalized Heisenberg groups. Let Δ_0 be the Laplace operator on W, and Δ_i be the Laplace operator on $[N_i, N_i]$. Then

$$A = -\Delta_0 - \frac{1}{4} \sum w_i^2 \Delta_i .$$

($[N, N]^*$ is the product of the $[N_i, N_i]^*$.) This holds since the quotient of a generalized Heisenberg group by a codimension one subspace of its center is the Heisenberg group, and the image of our sublaplacean is the standard sublaplacean on the Heisenberg group. In the case of generalized Heisenberg groups our formula for e^{-tL} and the expression for A are a restatement of the formula given by J. Randall [13] (where the reader can find a more detailed exposition).

The essential part of our argument is contained in the following lemma.

(1.4) LEMMA. *For every* $0 < \alpha_i < \dim([N_i, N_i])$ *there exists* C *such that for every* $f \in C_c^\infty(\mathbb{R}_+)$ *with* $\operatorname{supp} f \subset [1/2, 2]$,

$$\int \prod w_i^{\alpha_i} |f(L)|^2(x)\, dx \le C\|f\|_{L^2}^2 .$$

Remark. From [7] and [9] we know that $f(L)$ is a well defined rapidly decaying (Schwartz class) function, so all we do is to get the estimate.

We write s_i for the projection of $s \in [N, N]^*$ onto $[N_i, N_i]^*$. Put

$$A_s = -\Delta_0 + \frac{1}{4}\sum |s_i|^2 w_i^2 .$$

A_s is a (rescaled) Hermite operator on $L^2(W)$. Put $w = \prod w_i^{\alpha_i/2}$. By the Plancherel formula on $[N, N]$ we have

$$\int_N w^2 |f(L)|^2(x)\, dx = C \int_{[N,N]^*} \|wf(A_s)\delta_0\|^2\, ds .$$

(1.5) LEMMA. *There exists* C *such that for all* s *and all* $f \in C_c^\infty(\mathbb{R}_+)$ *with* $\operatorname{supp} f \subset [1/2, 2]$ *we have*

$$\int_1^2 \|f(A_{ts})\delta_0\|_{L^2}^2\, dt \le C\|f\|_{L^2}^2 .$$

Proof. We define D_t, for $t > 0$, by the formula

$$(D_t\phi)(x) = t^{\dim(W)}\phi(t^{-1}x) \quad \text{ for } \phi \in L^1(W)$$

and extend it by continuity to measures. One easily checks that

$$D_{t^{-1/2}}(f(A_{ts})\delta_0) = f(D_{t^{-1/2}}A_s D_{t^{1/2}})D_{t^{-1/2}}\delta_0 = f(tA_s)\delta_0$$

so

$$\int_1^2 \|f(A_{ts})\delta_0\|_{L^2}^2\, dt \le C\int_1^2 \|D_{t^{-1/2}}(f(A_{ts})\delta_0)\|_{L^2}^2\, dt = C\int_1^2 \|f(tA_s)\delta_0\|_{L^2}^2\, dt.$$

For $E_s(\lambda)$ being the spectral measure of A_s we write $d\mu(\lambda) = d(E_s(\lambda)e^{-A_s}\delta_0, e^{-A_s}\delta_0)$. By the Feynman–Kac formula (see for example [11]), $0 \le e^{-A_s}\delta_0(x) \le p_1(x)$, $p_1(x) = (4\pi)^{-\dim(W)/2}e^{-|x|^2/4}$ being the euclidean heat kernel. Hence

$$\int d\mu = \|e^{-A_s}\delta_0\|_{L^2}^2 \le C_{\dim(W)} .$$

We have

$$\int_1^2 \|f(tA_s)\delta_0\|_{L^2}^2\,dt \le \int_1^2\int |f(t\lambda)|^2 e^{2\lambda}\,d\mu(\lambda)\,dt \le C\|f\|_{L^2}^2 \int d\mu \le C\|f\|_{L^2}^2,$$

which gives (1.5).

Put

$$\|s\| = \max|s_i|, \qquad s^{(\alpha)} = \prod |s_i|^{\alpha_i}.$$

Note that

$$s^{(\alpha)}\|wf(A_s)\delta_0\|_{L^2}^2 \le C\|A_s^{|\alpha|/2} f(A_s)\delta_0\|_{L^2}^2 \le C'\|f(A_s)\delta_0\|_{L^2}^2 .$$

Also, if $s_i \ge 2$ then A_s is greater than the two-dimensional Hermite operator so $A_s \ge 2$. Therefore if $\|s\| \ge 2$ then $f(A_s) = 0$. We need a version of polar coordinates: there exist measures η_k such that for all positive Borel measurable ϕ we have

$$\int_{[N,N]^*} \phi = C\sum_k 2^k \int_{\|s\|=2^k} \int_1^2 t^{\dim([N,N])-1}\phi(ts)\,dt\,d\eta_k(s).$$

Using those observations and (1.5) we have

$$\begin{aligned} C\int_{[N,N]^*} \|wf(A_s)\delta_0\|^2\,ds &\le C\int_{\|s\|\le 2} s^{(-\alpha)}\|f(A_s)\delta_0\|^2\,ds \\ &\le C\sum_{k=0}^{\infty} 2^{-k}\int_{\|s\|=2^{-k}}\int_1^2 s^{(-\alpha)}\|f(A_{ts})\delta_0\|^2\,dt\,d\eta_k(s) \\ &\le C\|f\|_{L^2}^2 \sum_{k=0}^{\infty} 2^{-k}\int_{\|s\|=2^{-k}}\int_1^2 s^{(-\alpha)}\,dt\,d\eta_k(s) \\ &\le C\|f\|_{L^2}^2 \int_{\|s\|\le 2} s^{(-\alpha)} \le C\|f\|_{L^2}^2 , \end{aligned}$$

which ends the proof of (1.4).

From (1.4) by simple application of the Schwarz inequality, and since $\dim[N_i, N_i] < \dim(W_i)$, we obtain:

(1.6) LEMMA. *For every $\varepsilon > 0$ there exists C such that for every real $k > 0$ and every $f \in C_c^\infty(\mathbb{R}_+)$ with $\operatorname{supp} f \subset [1/2, 2]$ we have*

$$\int_{|x|<k} |f(L)|(x)\,dx \le Ck^{n/2+\varepsilon}\|f\|_{L^2} .$$

We also need:

(1.7) LEMMA. *Let I and J be closed intervals such that $I \subset \operatorname{Int} J \subset J \subset (-\pi,\pi)$, $f \in H(s)$, $\operatorname{supp} f \subset I$, $l > 0$. There exist functions f_j, $j = 0,1,\ldots,$ satisfying the following conditions*:

$$f = \sum f_j, \qquad \operatorname{supp} f_j \subset J\,,$$
$$|\widehat{f}_j(k)| \le C(s,I,J,l)2^{-sj}(1+\max(0,|k|-2^j))^{-l-3}\,,$$
$$\|f_j\|_{L^2} \le C(s,I,J,l)2^{-sj}\|f\|_{H(s)}\,,$$

where $C(s,I,J,l)$ depend only on s, I, J, l.

Proof. We choose smooth functions φ, ψ such that $\operatorname{supp}\varphi \subset J$, $\varphi|_I = 1$, $\psi = 1$ on $[-1/2,1/2]$, $\operatorname{supp}\psi \subset [-1,1]$ and we put

$$h_j(x) = \begin{cases} \sum \psi(k)\widehat{f}(k)e^{ikx} & \text{for } j = 0\,,\\ \sum[\psi(2^{-j}k) - \psi(2^{-j+1}k)]\widehat{f}(k)e^{ikx} & \text{otherwise,}\end{cases}$$
$$f_j = \varphi h_j\,.$$

The third condition holds because

$$\widehat{f}_j(k) = \sum_r \widehat{\varphi}(r)\widehat{h}_j(k-r)$$

and $|\widehat{\varphi}(k)| \le C(1+|k|)^{-l-4}$.

To handle error terms we will use the following (cf. [6], [7], [9]):

(1.8) LEMMA. *There exist l, c and M such that*

$$\int |x|^{\varepsilon}|e^{ike^{-L}}e^{-L}|(x)\,dx \le C(1+|k|^l), \qquad \int e^{c|x|}|e^{-L}|(x)\,dx \le M\,,$$
$$\int e^{c|x|}|e^{ike^{-L}}e^{-L}|(x)\,dx \le Me^{kM}\,.$$

Proof of (1.2). Put $f(\lambda) = m_0(-\log\lambda)\lambda$. We decompose f using (1.7) with $J = [e^{-2}, e^{-1/2}]$. Then

$$m_0(L) = \sum f_j(e^{-L})e^{-L}\,.$$

Also $f_j(e^{-L})e^{-L} = g_j(L)$ with $g_j(x) = f_j(e^{-x})e^{-x}$ so $\operatorname{supp} g_j \subset [1/2,2]$. Moreover,

$$\|g_j\|_{L^2} \le C2^{-js}\|f\|_{H(s)}\,.$$

Choose R so that $cR/2 > 4M$. We have

$$\int |x|^{\varepsilon}|f_j(e^{-L})e^{-L}|(x)\,dx$$
$$\le R^{\varepsilon}2^{j\varepsilon}\int\limits_{|x|<R2^j}|g_j(L)|(x)\,dx + \int\limits_{|x|>R2^j}|x|^{\varepsilon}|f_j(e^{-L})e^{-L}|(x)\,dx\,.$$

By (1.6) the first term is $\le C2^{-j\varepsilon}$. Expanding f_j in a Fourier series and using (1.8) and (1.7), we get

$$\begin{aligned}
&\int\limits_{|x|>R2^j} |x|^\varepsilon |f_j(e^{-L})e^{-L}|(x)\,dx \\
&\quad\le \sum_{|k|\le 2^{j+2}} |\widehat{f}_j(k)| \int\limits_{|x|>R2^j} |x|^\varepsilon |e^{ike^{-L}} e^{-L}|(x)\,dx \\
&\qquad\qquad + \sum_{|k|>2^{j+2}} |\widehat{f}_j(k)| \int |x|^\varepsilon |e^{ike^{-L}} e^{-L}|(x)\,dx \\
&\quad\le \sum_{|k|\le 2^{j+2}} |\widehat{f}_j(k)| e^{-cR2^j/2} \int e^{c|x|} |e^{ike^{-L}} e^{-L}|(x)\,dx + \sum_{|k|>2^{j+2}} |\widehat{f}_j(k)| k^l \\
&\quad\le \sum_{k\le 2^{j+2}} |\widehat{f}_j(k)| e^{-cR2^j/2} e^{kM} M + \sum_{|k|>2^{j+2}} C|k|^{-l-2}|k|^l \\
&\quad\le C2^{-j}\|f\|_{H(s)}\,.
\end{aligned}$$

This means that we can add the estimates of $g_j(L)$ to get an estimate of $m_0(L)$, which ends the proof of (1.2).

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