C O L L O Q U I U M   M A T H E M A T I C U M

VOL. LXV 1993 FASC. 2

JACOBI OPERATOR FOR LEAF GEODESICS

BY

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Introduction. In [Wa2], while studying the geodesic flow of a foliation, we introduced the notion of Jacobi fields along geodesics on the leaves of a foliation $\mathcal{F}$ of a Riemannian manifold M. Jacobi fields occur as variation fields while varying a leaf geodesic c among leaf geodesics. They satisfy the equation

$$JY = 0\,,$$

where J is a second order differential operator acting in the space of vector fields along c (see (16) in Section 4). The Jacobi operator J depends on the curvature of M as well as on the second fundamental form B of $\mathcal{F}$. In the trivial case, $\mathcal{F} = \{M\}$, J reduces to the classical Jacobi operator studied in Riemannian geometry [Kl].

In this article, we show that J plays a role in the second variational formula for the arclength $\mathcal{L}$ and energy $\mathcal{E}$ of leaf curves (Section 4). Since leaf geodesics appear to be critical for $\mathcal{L}$ and $\mathcal{E}$ for some variations only (Section 3), we have to distinguish a suitable class of variations called admissible here (Section 4). We collect a number of properties of the operator J (Section 5) acting particularly on the tangent space $T_c\Omega$ of the space Ω of all the leaf curves. (The space $T_c\Omega$ is described in Section 2.) Some particular cases are considered in Section 6. The results lead to some consequences relating geometry and topology of $(M, \mathcal{F})$ (Propositions 2 and 9).

Further development of the variational theory is obstructed in general by the possibility of non-existence of admissible variations for some variation fields (see Proposition 4 and the Remark following it). The problem could be overcome by suitable assumptions on the exterior geometry of $\mathcal{F}$.

1. Notation. Throughout the paper ∇ is the Levi-Civita connection on an n-dimensional Riemannian manifold (M, g), R is its curvature tensor and K is the sectional curvature of M. $\mathcal{F}$ is a p-dimensional foliation of

1991 *Mathematics Subject Classification*: 57R30, 53C40.
Key words and phrases: foliation, leaf geodesic, variation, Jacobi operator.

M, $v = v^\top + v^\perp$ is the decomposition of a vector v into the parts tangent and orthogonal to $\mathcal{F}$. $\nabla^\top$ is the connection in $T\mathcal{F}$, the tangent bundle of $\mathcal{F}$, induced by ∇ and the orthogonal projection. $\nabla^\perp$ is the analogous connection in $T^\perp\mathcal{F}$, the orthogonal complement of $T\mathcal{F}$. All the connections in different tensor bundles induced by ∇, $\nabla^\top$ and $\nabla^\perp$ are denoted, maybe abusively, by ∇.

A (resp., $A^\perp$) is the Weingarten operator of $\mathcal{F}$ (resp., of the orthogonal distribution $T^\perp\mathcal{F}$), defined by $A^Y X = -(\nabla_X Y)^\top$ (resp., $A^{\perp X} Y = -(\nabla_Y X)^\perp$) for X tangent and Y orthogonal to $\mathcal{F}$. Similarly, B and $B^\perp$ are the second fundamental tensors of $\mathcal{F}$ and $T^\perp\mathcal{F}$: $\langle B(U,V),X\rangle = \langle A^X U, V\rangle$ and $\langle B^\perp(X,Y),U\rangle = \langle A^{\perp U} X, Y\rangle$ for U and V tangent to $\mathcal{F}$, and X and Y orthogonal to it. In other words, $B(U,V) = (\nabla_U V)^\perp$ and $B^\perp(X,Y) = (\nabla_X Y)^\top$. Note that the form B is symmetric while $B^\perp$ in general is not.

2. Space of curves. Let $\mathcal{F}$ be a foliation of a Riemannian manifold (M,g). Denote by Ω the space of piecewise smooth curves $c : [0,1] \to M$ tangent to the leaves of $\mathcal{F}$. We equip Ω with the uniform C^1-topology induced by g and the Sasaki metric g_S on $T\mathcal{F}$. In this way, Ω becomes a metric space with the distance function d_Ω given by

$$d_\Omega(c_1,c_2) = \sup_{0\le t\le 1} d(c_1(t),c_2(t)) + \sup_{0\le t\le 1} d_S(\dot c_1(t),\dot c_2(t)), \tag{1}$$

where d is the distance function on (M,g) and d_S the distance function on $(T\mathcal{F},g_S)$, and the supremum in the second term is taken over all the t's for which $\dot c_1(t)$ and $\dot c_2(t)$ do exist.

A *curve* in Ω is meant to be a continuous map $V : [0,1]\times(a,b)\to M$ such that $V(\cdot,s)\in\Omega$ for all s in (a,b) and there exist numbers $0 = t_0 < t_1 < \ldots < t_k = 1$ for which $V|[t_i,t_{i+1}]\times(a,b)$, $i = 1,\ldots,k-1$, are smooth. If $s_0\in(a,b)$ and $c = V(\cdot,s_0)$, then V is called an *$\mathcal{F}$-variation* of c.

The *tangent space* $T_c\Omega$ ($c\in\Omega$) is considered to be the space of all variation fields corresponding to all the $\mathcal{F}$-variations of c. $T_c\Omega$ consists of continuous piecewise smooth vector fields along c. Obviously, $T_c\Omega$ is a vector space containing all the fields tangent to $\mathcal{F}$.

PROPOSITION 1. *$Z\in T_c\Omega$ if and only if $Z^{\perp\cdot} = -A^{\perp\dot c}Z^\perp$.*

Here and in the sequel, the upper dot denotes the covariant differentiation in the bundle $T^\perp\mathcal{F}$ in the direction of c.

Proof. Let $V : [0,1]\times(-\varepsilon,\varepsilon)\to M$ be a smooth $\mathcal{F}$-variation of $c = V(\cdot,0)$ and let $Z = V_*(d/ds)(\cdot,0)$ be the variation field. Assume that Z is orthogonal to $\mathcal{F}$. Let $X = V_*(d/dt)$ and $Y = V_*(d/ds)$ be fields along V so that $Z = Y(\cdot,0)$. Since the fields d/dt and d/ds commute, and the Levi-Civita connection ∇ on M is torsion free, we have $\nabla_{d/ds}X = \nabla_{d/dt}Y$

and therefore,

$$Z^{\cdot} = (\nabla_{d/dt}Y)^{\perp} = (\nabla_{d/dt}X)^{\perp} = -A^{\perp\dot{c}}Z\,. \tag{2}$$

Conversely, assume that Z is orthogonal to $\mathcal{F}$ and satisfies (2). Consider a chart x on M distinguished by $\mathcal{F}$ and such that $x(c(t)) = (t, 0, \ldots, 0)$ for any t. (This can be done for any short piece of any curve $c \in \Omega$ for which $\dot{c} \neq 0$, so it is sufficient to consider curves of this form.) Take an $(n-1)$-dimensional ($n = \dim M$) ball $B(\varepsilon)$ centered at the origin and extend Z along $\{0\} \times B(\varepsilon)$ keeping it orthogonal to $\mathcal{F}$. For any $u \in B(\varepsilon)$ there exists a unique solution Y_u along the curve $t \mapsto (t, u)$ of $Y^{\cdot} = -A^{\perp(d/dt)}Y$ satisfying the initial condition $Y_u(0, u) = Z(0, u)$. The field Y defined by all the fields Y_u satisfies

$$[d/dt, Y]^{\perp} = 0 \tag{3}$$

on $[0,1] \times B(\varepsilon)$. Let (φ_s) be a local flow of Y in a neighbourhood of $[0,1] \times \{0\}$. The map $V : [0,1] \times (-\varepsilon, \varepsilon) \ni (t, s) \mapsto \varphi_s(c(t))$ is a variation of c, $V_*(d/ds) = Z$ along c and $V(\cdot, s)$ is tangent to $\mathcal{F}$ for any s because of (3). ∎

Remark. For any leaf curve $c : [0,1] \to L$ the linear isomorphism

$$Z^{\perp}_{c(0)}\mathcal{F} \ni v \mapsto Z_v(1) \in T^{\perp}_{c(1)}\mathcal{F}\,,$$

where Z_v is the unique solution of (2) satisfying the initial condition $Z_v(0) = v$, represents the linear holonomy h_c of $\mathcal{F}$ along c. In particular, $Z_v(1)$ depends only on the homotopy class of c.

In fact, if $H : [0,1] \times [0,1] \to L$ is a homotopy satisfying $H(0, s) = x$ and $H(1, s) = y$ for all s and some x and y in L, Z is a vector field along H perpendicular to $\mathcal{F}$, $X = H_*(d/dt)$, $Y = H_*(d/ds)$,

$$\nabla^{\perp}_X Z = -A^{\perp X}Z\,, \tag{4}$$

$W = \nabla^{\perp}_Y Z$ and $f = \|W\|^2$, then for any $s \in [0,1]$ we have

$$\begin{aligned}\frac{1}{2}\frac{df}{dt} = \langle \nabla^{\perp}_X W, W\rangle &= \langle R(X,Y)Z, W\rangle + \langle \nabla^{\perp}_Y \nabla^{\perp}_X Z, W\rangle \\ &\quad - \langle B(A^Z X, Y), W\rangle + \langle B(X, A^Z Y), W\rangle\,.\end{aligned} \tag{5}$$

Ranjan's formula (*) ([Ra], p. 87) implies that

$$\begin{aligned}\langle R(X,Y)Z, W\rangle &= \langle (\nabla_Y B^{\perp})(Z,W), X\rangle - \langle (\nabla_X B^{\perp})(Z,W), Y\rangle \\ &\quad - \langle A^Z Y, A^W X\rangle + \langle A^Z X, A^W Y\rangle \\ &\quad - \langle A^{\perp X}A^{\perp Y}Z, W\rangle + \langle A^{\perp Y}A^{\perp X}Z, W\rangle\,.\end{aligned} \tag{6}$$

The formulae (4)–(6) together with the obvious relations between A and B ($A^{\perp}$ and $B^{\perp}$, resp.) and their covariant derivatives imply that

$$\frac{1}{2}\frac{df}{dt} = \frac{d}{dt}\langle A^{\perp Y}Z, W\rangle\,.$$

Therefore,

$$f(1,s)-f(0,s)=\langle A^{\perp Y}Z,W\rangle(1,s)-\langle A^{\perp Y}Z,W\rangle(0,s)=0$$

because $Y(0,s)=0$ and $Y(1,s)=0$ for all s. If $Z(0,s)=v$ for all s, then $f(0,s)=0$, $f(1,s)=0$ and $Z(1,s)$ is constant on the interval $[0,1]$. ■

3. First variational formula. The *arclength* $\mathcal{L}$ and the *energy* $\mathcal{E}$ are continuous functionals on Ω given, as usual, by

$$\mathcal{L}(c)=\int_0^1\|\dot c(t)\|\,dt\quad\text{and}\quad\mathcal{E}(c)=\int_0^1\|\dot c(t)\|^2\,dt\,. \tag{7}$$

They are differentiable in the sense that if V is a smooth variation, then the functions $s\mapsto\mathcal{E}(V(\cdot,s))$ and $s\mapsto\mathcal{L}(V(\cdot,s))$ are differentiable provided, in the second case, that the curves $V(\cdot,s)$ are regular.

Let $V:[0,1]\times(-\varepsilon,\varepsilon)\to M$ be a smooth $\mathcal{F}$-variation of a leaf curve $c=V(\cdot,0)$ parametrized proportionally to arclength ($\|\dot c\|\equiv$ const.). Let $\mathcal{L}(s)=\mathcal{L}(V(\cdot,s))$, $X=V_*(d/dt)$ and $Y=V_*(d/ds)$. Then

$$\mathcal{L}'(s)=\int_0^1\frac{\langle\nabla_{d/ds}X,X\rangle(t,s)}{\|X(t,s)\|}\,dt=\int_0^1\frac{\langle\nabla_{d/dt}Y,X\rangle(t,s)}{\|X(t,s)\|}\,dt\,, \tag{8}$$

$$\mathcal{L}'(0)=\frac{1}{l}\int_0^1\langle Y',\dot c\rangle\,dt$$

and

$$\mathcal{L}'(0)=\frac{1}{l}\Big(\langle\dot c,Y\rangle|_0^1-\int_0^1\langle Y^\top,\dot c'^\top\rangle\,dt-\int_0^1\langle Y^\perp,B(\dot c,\dot c)\rangle\,dt\Big)\,, \tag{9}$$

where l is the length of c.

A similar formula holds for piecewise smooth curves and $\mathcal{F}$-variations. One has to consider the integrals over the intervals $[t_i,t_{i+1}]$, $0=t_0<t_1<\ldots$ $\ldots<t_k=1$, for which both c and V are differentiable.

In the same way,

$$\mathcal{E}'(s)=2\int_0^1\langle\nabla_{d/dt}Y,X\rangle(t,s)\,dt$$

and

$$\mathcal{E}'(0)=2l\cdot L'(0)\,, \tag{10}$$

where $\mathcal{E}(s)=\mathcal{E}(V(\cdot,s))$.

From (8) and (9) it follows that any leaf curve c which is to minimize either arclength or energy for $\mathcal{F}$-variations V satisfying

$$Y(0)\perp\dot c(0)\quad\text{and}\quad Y(1)\perp\dot c(1) \tag{11}$$

should be a leaf geodesic. In this case, the variation formula (9) reduces to

$$\mathcal{L}'(0) = -\frac{1}{l}\int_0^1 \langle Y^\perp, B(\dot c, \dot c)\rangle\, dt\,. \tag{12}$$

Therefore, a leaf geodesic c is a critical point of $\mathcal{L}$ (equivalently, of $\mathcal{E}$) for all the $\mathcal{F}$-variations V for which the variation field Y satisfies (11) and

$$\int_0^1 \langle Y^\perp, B(\dot c, \dot c)\rangle\, dt = 0\,. \tag{13}$$

The proposition below is a simple application of the above considerations.

PROPOSITION 2. *Let $\mathcal{F}$ be a transversely oriented codimension-one foliation of a manifold M. Let X be a non-vanishing vector field transverse to $\mathcal{F}$. Assume that there exists a Riemannian metric g on M for which $X \perp \mathcal{F}$ and the scalar fundamental form h of $\mathcal{F}$ is positive. Then any leaf of $\mathcal{F}$ admits at most one closed trajectory of X intersecting it.*

Proof. Assume that a leaf of $\mathcal{F}$ intersects two closed trajectories T_1 and T_2 of X. The subspace $\widehat{\Omega} \subset \Omega$ consisting of all the leaf curves joining T_1 to T_2 is non-void and there exists a leaf geodesic $c : [0,1] \to M$ for which $\mathcal{L}|\widehat{\Omega}$ attains its minimum. There exists a positive function f for which the field $Y = f \cdot X \circ c$ belongs to $T_c\Omega$, and an $\mathcal{F}$-variation V for which the variation field equals Y. For this variation

$$\int_0^1 f(t)\|X(c(t))\| h(\dot c(t), \dot c(t))\, dt = 0.$$

Since $h(v,v) > 0$ for $v \neq 0$, the last equality implies that $\dot c(t) = 0$ for any t. Therefore, $c(0) = c(1) \in T_1 \cap T_2$ and $T_1 = T_2$. ■

4. Admissible variations and second variational formula. Assume that $V : [0,b] \times (-\varepsilon, \varepsilon) \to M$ is a smooth $\mathcal{F}$-variation of a leaf geodesic $c : [0,b] \to M$ for which the variation field Y satisfies

$$Y(0,\cdot) \perp \mathcal{F}, \quad Y(b,\cdot) \perp \mathcal{F}, \quad \int_0^b \langle Y^\perp, B(X,X)\rangle(t,\cdot)\, dt \equiv 0\,, \tag{14}$$

where, as before, $X = V_*(d/dt)$. $\mathcal{F}$-variations satisfying (14) are said to be *admissible* here.

PROPOSITION 3. *For any admissible variation V of a normal leaf geodesic c one has*

$$\mathcal{L}''(0) = \int_0^b (\langle JY, Y\rangle - \langle Y', \dot c\rangle^2)(t,0)\, dt\,, \tag{15}$$

where

$$JZ = -Z'' + R(\dot c, Z)\dot c + (\nabla_Z B)(\dot c, \dot c) + 2B(Z'^{\top}, \dot c) \tag{16}$$

for any vector field Z along c. Similarly,

$$\mathcal{E}''(0) = 2\int_0^b \langle JY, Y\rangle(t,0)\,dt\,. \tag{17}$$

The differential operator J defined by (16) is called the *Jacobi operator* here. It appeared in [Wa2], where the variations of leaf geodesics among leaf geodesics were considered. Some properties of J are studied in the next section.

Proof. From (8) we get

$$\mathcal{L}''(s) = \int_0^b \|X\|^{-3}\bigg(\frac{d}{ds}\langle\nabla_{d/dt}Y, X\rangle\|X\|^2 - \langle\nabla_{d/dt}Y, X\rangle^2\bigg)(t,s)\,dt$$

and

$$\mathcal{L}''(0) = \int_0^b (\langle\nabla_{d/ds}\nabla_{d/dt}Y, \dot c\rangle + \|Y'\|^2 - \langle Y', \dot c\rangle^2)(t)\,dt\,. \tag{18}$$

Since the fields d/ds and d/dt commute,

$$\langle\nabla_{d/ds}\nabla_{d/dt}Y, \dot c\rangle = \langle R(Y,\dot c)Y, \dot c\rangle + \langle\nabla_{d/dt}\nabla_{d/ds}Y, \dot c\rangle\,. \tag{19}$$

Also,

$$\begin{aligned}\langle\nabla_{d/dt}\nabla_{d/ds}Y, \dot c\rangle &= \frac{d}{dt}\langle\nabla_{d/ds}Y, \dot c\rangle - \langle\nabla_{d/ds}Y, B(\dot c,\dot c)\rangle\\ &= \frac{d}{dt}\langle\nabla_{d/ds}Y, \dot c\rangle - \frac{d}{ds}\langle Y, B(X,X)\rangle\\ &\quad + \langle Y, (\nabla_Y B)(\dot c,\dot c) + 2B(Y'^{\top}, \dot c)\rangle\,,\end{aligned}$$

$$\int_0^b \frac{d}{dt}\langle\nabla_{d/ds}Y, \dot c\rangle\,dt = \langle\nabla_{d/ds}Y, \dot c\rangle|_0^b$$

and

$$\int_0^b \frac{d}{ds}\langle Y, B(X,X)\rangle\,dt = \frac{d}{ds}\int_0^b \langle Y, B(X,X)\rangle\,dt = 0$$

because of (14). It follows that

$$\begin{aligned}\mathcal{L}''(0) = \int_0^b &(\langle R(\dot c, Y)\dot c + (\nabla_Y B)(\dot c,\dot c) + 2B(\dot c, Y'^{\top}), Y\rangle\\ &+ \|Y'\|^2 - \langle Y', \dot c\rangle^2)\,dt + \langle\nabla_{d/ds}Y, \dot c\rangle|_0^b\,.\end{aligned} \tag{20}$$

Finally,

$$\|Y'\|^2 = \frac{d}{dt}\langle Y, Y'\rangle - \langle Y'', Y\rangle\,, \tag{21}$$

$$\int_0^b \frac{d}{dt}\langle Y, Y'\rangle\, dt = \langle Y, Y'\rangle|_0^b \tag{22}$$

and

$$\langle \nabla_{d/dt} Y, Y\rangle + \langle \nabla_{d/ds} Y, X\rangle|_0^b = \frac{d}{ds}\langle X, Y\rangle|_0^b = 0\,. \tag{23}$$

The formulae (20)–(23) yield (15). ■

Corollary 1. *If an admissible variation V is geodesic, then*

$$\mathcal{L}''(0) = \mathcal{E}''(0) = 0\,.$$

Proof. If all the curves $V(\cdot, s)$ are leaf geodesics, then the variation field Y is Jacobi, i.e. it satisfies the Jacobi equation $JY = 0$. For a Jacobi field Y along a leaf geodesic c one has $\langle Y', \dot c\rangle \equiv \mathrm{const}$ ([Wa2], Lemma 1). Also, $\langle Y, \dot c\rangle' = \langle Y', \dot c\rangle + \langle Y, B(\dot c, \dot c)\rangle$ and if $Y(t) \perp \mathcal{F}$ for $t = 0$ and $t = b$, then

$$\int_0^b \langle Y', \dot c\rangle\, dt = -\int_0^b \langle B(\dot c, \dot c)\rangle\, dt\,.$$

If Y comes from an admissible variation, then

$$\int_0^b \langle Y', \dot c\rangle^2\, dt = \Big(\int_0^b \langle Y', \dot c\rangle\, dt\Big)^2 = \Big(\int_0^b \langle B(\dot c, \dot c)\rangle\, dt\Big)^2 = 0\,. \quad ■$$

Now, we shall show the existence of admissible variations with prescribed variation fields. To this end we need the following elementary fact.

Lemma 1. *If $f : [0, b] \times (-\varepsilon, \varepsilon) \to \mathbb{R}$ is a smooth function such that $\int_0^b f(t, 0)\, dt = 0$ and $f(t, 0) \neq 0$ for some t, then there exists a smooth function $\lambda : [0, b] \times (-\eta, \eta) \to \mathbb{R}$ $(0 < \eta < \varepsilon)$ for which $\lambda(t, 0) = t$, $\lambda(0, s) = 0$, $\lambda(b, s) = b$, $\partial\lambda/\partial t > 0$ and*

$$\int_0^b \frac{\partial \lambda}{\partial t}(\lambda(\cdot, s)^{-1}(u), s) f(u, s)\, du = 0 \tag{24}$$

for all s and t.

Proof. We shall find a piecewise linear function λ satisfying all the conditions. It could be made smooth by a procedure analogous to that of the proof of Lemma 2 of [Wa1], for example.

First, we can find $d\in(0,b)$ and $\eta\in(0,\varepsilon)$ such that $\int_0^d f(t,s)\,dt\neq 0$, for example

$$\int_0^d f(t,s)\,dt>0 \quad\text{and}\quad \int_d^b f(t,s)\,dt<0\,,$$

for all $s\in(-\eta,\eta)$. Let

$$\lambda_c(t)=\begin{cases}\dfrac{d}{c}t & \text{if } 0\le t\le c\,,\\ \dfrac{b-d}{b-c}(t-c) & \text{if } c\le t\le d\,,\end{cases}$$

and

$$I(s,c)=\int_0^b \lambda_c'(\lambda_c^{-1}(u))f(u,s)\,du=\frac{d}{c}\int_0^d f(u,s)\,du+\frac{b-d}{b-c}\int_d^b f(u,s)\,du\,.$$

Then

$$\frac{\partial I}{\partial c}<0,\qquad \lim_{c\to 0^+} I(s,c)=+\infty,\qquad \lim_{c\to b^-} I(s,c)=-\infty\,,$$

so for any s there exists a unique c_s such that $I(s,c_s)=0$. Obviously, $c_0=d$. The function λ given by $\lambda(t,s)=\lambda_{c_s}(t)$ satisfies all the conditions of the lemma. ■

PROPOSITION 4. *Assume that $Z\in T_c\Omega$ is a vector field orthogonal to $\mathcal{F}$ and such that*

$$\int_0^b \langle Z,B(\dot c,\dot c)\rangle\,dt=0 \quad\textit{and}\quad \langle Z,B(\dot c,\dot c)\rangle(t)\neq 0$$

for some t. There exists an admissible $\mathcal{F}$-variation $V:[0,b]\times(-\eta,\eta)\to M$ for which Z is the normal component of the variation field.

Proof. Take any $\mathcal{F}$-variation $W:[0,b]\times(-\varepsilon,\varepsilon)\to M$ for which $Z(t)=W_*(d/ds)(t,0)$ $(0\le t\le b)$. Apply Lemma 1 to the function

$$f=\langle W_*(d/ds),B(W_*(d/dt),W_*(d/dt))\rangle\,.$$

Let

$$V(t,s)=W(\lambda(t,s),s),\qquad 0\le t\le b,\ -\eta<s<\eta\,,$$

where λ is any function satisfying the conditions of Lemma 1. Then

$$V_*\left(\frac{\partial}{\partial s}\right)=\frac{\partial\lambda}{\partial s}W_*\left(\frac{\partial}{\partial s}\right)+W_*\left(\frac{\partial}{\partial s}\right) \tag{25}$$

and

$$V_*\left(\frac{\partial}{\partial t}\right)=\frac{\partial\lambda}{\partial t}W_*\left(\frac{\partial}{\partial t}\right). \tag{26}$$

Formula (25) shows that the normal component of $V_*(\partial/\partial s)$ equals Z along c. Formulae (25) and (26) together with (24) show that the variation V is admissible. ■

Remark. (i) Note that the tangent component of the variation field constructed in the course of the proof above is of the form $f \cdot \dot{c}$, where $f : [0,b] \to \mathbb{R}$ satisfies $f(0) = f(b) = 0$.

(ii) The assumption $\langle Z, B(\dot{c},\dot{c})\rangle(t) \neq 0$ is essential here. If, for example, $\operatorname{codim}\mathcal{F} = 1$, $\mathcal{F}$ is transversely oriented and totally umbilical, $B = \lambda g \otimes N$ for a unit field N orthogonal to $\mathcal{F}$ and a function $\lambda : M \to \mathbb{R}$, L is an isolated totally geodesic leaf, λ is strictly positive in $U \setminus L$ for some neighbourhood U of L and $c : [0,b] \to L$ is a geodesic, then there are no non-trivial transverse to $\mathcal{F}$ admissible variations of c in spite of the identity $B(\dot{c},\dot{c}) \equiv 0$.

5. Properties of the Jacobi operator. Consider the operator J defined by (16) for a normal leaf geodesic $c : [0,b] \to L$. Clearly, J is $\mathbb{R}$-linear and maps the space of vector fields along c into itself. Its kernel is of dimension $2n$ while the intersection $T_c\Omega \cap \ker J$ of dimension $n+p$. It consists of Jacobi fields (in the sense of [Wa2]) obtained by varying c among leaf geodesics.

Proposition 5. *Let $X = Y + Z$ satisfy $JX = 0$, $Y^\perp = 0$ and $Z^\top = 0$. Then $X \in T_c\Omega$ if and only if*

$$Z^{\cdot}(0) = -A^{\perp \dot{c}(0)} Z(0)\,.$$

Proof. The "only if" part of the statement follows immediately from Proposition 1. To prove the "if" part put

$$\zeta = Z^{\cdot} - A^{\perp \dot{c}} Z\,.$$

From Proposition 1 again it follows that it is sufficient to show that ζ satisfies an ODE of the form

$$\zeta^{\cdot} = \Lambda\zeta\,,$$

Λ being a linear operator on the space of vector fields along c orthogonal to $\mathcal{F}$.

Take any vector field $N = N^\perp$ along c. From the definitions of ζ, A and $A^\perp$ it follows easily that

$$\begin{aligned}(27)\qquad \langle \zeta^{\cdot}, N\rangle = \langle X'', N\rangle - \langle Y'', N\rangle &+ \langle B(A^Z \dot{c}, \dot{c}), N\rangle \\ &- \langle (\nabla_{\dot{c}} B^\perp)(Z,N), \dot{c}\rangle - \langle B^\perp(Z^{\cdot}, N), \dot{c}\rangle\,.\end{aligned}$$

Ranjan's structure equation ([Ra], p. 87) in our notation reads

$$\begin{aligned}(28)\qquad \langle R(\dot{c}, Z)\dot{c}, N\rangle = \langle B(A^Z \dot{c}, \dot{c}), N\rangle &+ \langle B^\perp(A^{\perp\dot{c}} Z, N), \dot{c}\rangle \\ &- \langle (\nabla_Z B)(\dot{c},\dot{c}), N\rangle - \langle (\nabla_{\dot{c}} B^\perp)(Z,N), \dot{c}\rangle\,.\end{aligned}$$

We also have the Codazzi equation

$$\langle R(\dot c, Y)\dot c, N\rangle = \langle(\nabla_{\dot c}B)(Y,\dot c), N\rangle - \langle(\nabla_Y B)(\dot c,\dot c), N\rangle \tag{29}$$

and the equality

$$\begin{aligned}\langle Y'', N\rangle &= \langle B(Y'^{\top},\dot c), N\rangle + \langle B(Y,\dot c)^{\cdot}, N\rangle\\ &= \langle(\nabla_{\dot c}B)(Y,\dot c), N\rangle + 2\langle B(Y'^{\top},\dot c), N\rangle\,.\end{aligned} \tag{30}$$

Now, $JX = 0$ together with (27)–(30) yield

$$\langle \zeta^{\cdot}, N\rangle = -\langle B^{\perp}(\zeta, N), \dot c\rangle\,.$$

This shows that ζ satisfies the required ODE with $\Lambda = -\langle B^{\perp}(\cdot, N), \dot c\rangle$. ∎

Proposition 6. *If $Y \in T_c\Omega$, then*

(i) $(JY)^{\perp} = 0$,

(ii) $JY = J_L Y$ *if* $Y^{\perp} = 0$,

(iii) $\langle JY, X\rangle = \langle R(\dot c, X)\dot c, Y\rangle + \langle B(\dot c,\dot c), A^{\perp X}Y\rangle - \langle A^{\perp\dot c}Y, B(\dot c, X)\rangle - \langle Y', X\rangle'$ *if* $Y^{\top} = 0$, $X^{\perp} = 0$ *and X is $\nabla^{\top}$-parallel along c.*

Here, J_L denotes the standard Jacobi operator on the leaf L [Kl]: If $Z^{\perp} = 0$, then $J_L Z = -\nabla^{\top}_{\dot c}\nabla^{\top}_{\dot c}Z + R_L(\dot c, Z)\dot c$ with R_L being the curvature tensor on L.

Proof. (i) Assume first that Y is orthogonal to $\mathcal F$ and take a $\nabla^{\perp}$-parallel section X of $T^{\perp}\mathcal F$ along c. Then

$$\langle B(Y'^{\top},\dot c), X\rangle = -\langle B(A^Y\dot c,\dot c), X\rangle \tag{31}$$

and

$$Y'' = (Y^{\cdot} - A^Y\dot c)' = -(A^{\perp\dot c}Y + A^Y\dot c)'. \tag{32}$$

The last formula implies

$$\langle Y'', X\rangle = -\langle(\nabla_{\dot c}B^{\perp})(Y,X),\dot c\rangle + \langle B^{\perp}(A^{\perp\dot c}Y, X),\dot c\rangle + \langle A^Y\dot c, A^X\dot c\rangle\,. \tag{33}$$

Substitution of (31), (33) and (28) (where one has to replace Z by Y and N by X) to (16) yields

$$\langle JY, X\rangle = 0\,. \tag{34}$$

If Y is tangent to $\mathcal F$ and X is, as before, orthogonal to $\mathcal F$ and satisfies $X^{\cdot} = 0$, then (34) follows immediately from (16) and the Codazzi equation

$$\langle R(\dot c, Y)\dot c, X\rangle = \langle(\nabla_{\dot c}B)(Y,\dot c), X\rangle - \langle(\nabla_Y B)(\dot c,\dot c), X\rangle\,.$$

(ii) The Gauss equation

$$\langle R(\dot c, Y)\dot c, X\rangle = \langle R_L(\dot c, Y)\dot c, X\rangle + \langle B(\dot c,\dot c), B(X,Y)\rangle - \langle B(\dot c, X), B(\dot c, Y)\rangle$$

implies that if $X^{\perp} = 0$ and X is $\nabla^{\perp}$-parallel along c, then

$$\begin{aligned}\langle JY, X\rangle &= \langle R_L(\dot c, Y)\dot c, X\rangle + \langle B(\dot c,\dot c), B(X,Y)\rangle - \langle B(\dot c, X), B(\dot c, Y)\rangle\\ &\quad + \langle(\nabla_Y B)(\dot c,\dot c), X\rangle - \langle Y'', X\rangle\,.\end{aligned}$$

Since

$$\langle Y'', X\rangle = \langle {Y'^{\top}}', X\rangle + \langle {Y'^{\perp}}', X\rangle = \langle {Y'^{\top}}'^{\top}, X\rangle + \langle B(\dot c, Y)', X\rangle$$

and $\langle B(\dot c, Y)', X\rangle = -\langle B(\dot c, Y), B(\dot c, X)\rangle$, we get

$$\langle JY, X\rangle = \langle J_L Y, X\rangle + \langle (\nabla_Y B)(\dot c, \dot c), X\rangle - \langle B(\dot c, \dot c), B(X, Y)\rangle = \langle J_L Y, X\rangle$$

because for any vector fields U, V and W tangent to $\mathcal{F}$ we have

$$\begin{aligned}\langle (\nabla_U B)(V, V), W\rangle &= \langle \nabla_U B(V, V), W\rangle = -\langle B(V, V), \nabla_U W\rangle \\ &= -\langle B(V, V), B(U, W)\rangle\,. \end{aligned} \tag{35}$$

(iii) The desired formula follows easily from (16) and (32). ∎

Corollary 2. *If $X = Z + f\cdot \dot c$ ($Z^{\top} = 0$, $f(0) = f(b) = 0$) is the variation field of an admissible variation V of a normal leaf geodesic $c : [0, b] \to L$, then the variational formula* (15) *reduces to*

$$\mathcal{L}''(0) = \int_0^b \{f'\langle B(\dot c, \dot c), Z\rangle - \langle B(\dot c, \dot c), Z\rangle^2\}\, dt\,. \tag{36}$$

P r o o f. The last proposition implies that

$$\langle JX, X\rangle = f\langle B(\dot c, \dot c), Z\rangle' - ff''\,. \tag{37}$$

Also,

$$\langle X', \dot c\rangle = f' - \langle B(\dot c, \dot c), Z\rangle\,. \tag{38}$$

Substituting (37) and (38) into (15) and integrating by parts we get (36). ∎

Corollary 3. *Assume that c is a leaf geodesic minimizing arclength for all the admissible variations. If Z is the variation field of an admissible variation and Z is the orthogonal to $\mathcal{F}$, then*

$$\langle B(\dot c, \dot c), Z\rangle \equiv 0\,.$$

If c admits $q = \operatorname{codim}\mathcal{F}$ admissible variations with variation fields $Z_1, \ldots, Z_q$ orthogonal to $\mathcal{F}$ and linearly independent at a point, then c is an M-geodesic contained in a leaf.

P r o o f. If c minimizes arclength, then $\mathcal{L}'' \ge 0$ for all the admissible variations of c. The formula (36) with $f \equiv 0$ implies that

$$\int_0^b \langle B(\dot c, \dot c), Z\rangle^2\, dt \le 0\,.$$

This holds if and only if $\langle B(\dot c, \dot c), Z\rangle \equiv 0$.

The second part of the statement follows from the first one and Proposition 1 which implies that the fields $Z_1, \ldots, Z_q$ are linearly independent everywhere. ∎

6. Some particular cases

6.1. *Totally geodesic foliations.* If $\mathcal{F}$ is totally geodesic ($B \equiv 0$), then any variation of a leaf geodesic for which the variation field is perpendicular to $\mathcal{F}$ at the ends of the geodesic is admissible. Take any geodesic $c : [0,b] \to L$ and any field $Y \in T_c\Omega$ such that $Y^\top(0) = 0$ and $Y^\top(b) = 0$. From Proposition 6 it follows that

$$\begin{aligned}\langle JY, Y\rangle - \langle Y', \dot c\rangle^2 &= \langle JY^\top, Y^\top\rangle + \langle JY^\perp, Y^\top\rangle - \langle Y^{\top\prime}, \dot c\rangle^2 \\ &= \langle R(\dot c, Y^\top)\dot c, Y^\top\rangle - \langle Y^{\top\prime\prime}, Y^\top\rangle \\ &\quad + \langle R(\dot c, Y^\perp)\dot c, Y^\perp\rangle - \langle Y^{\perp\prime\prime}, Y^\top\rangle - \langle Y^{\top\prime}, \dot c\rangle^2 \\ &= \langle R(\dot c, Y^\top)\dot c, Y^\top\rangle + \|Y^{\top\prime}\|^2 - \langle Y^{\top\prime}, \dot c\rangle^2 - \langle Y^\top, Y^{\top\prime}\rangle'\,.\end{aligned}$$

Integrating over $[0,b]$ we get, from (15),

$$\mathcal{L}''(0) = \int_0^b (\langle R(\dot c, Z)\dot c, Z\rangle + \|Z'_\perp\|^2)\,dt\,,$$

where $Z = Y^\top$ and $Z'_\perp$ is the component of Z' orthogonal to c. The last formula coincides with that for the second variation of arclength on L. Therefore, the classical results of Riemannian geometry imply the following.

PROPOSITION 7. *If $\mathcal{F}$ is totally geodesic, then a geodesic $c : [0,b] \to L$ minimizes arclength for all admissible variations if and only if there are no Jacobi fields Z along c tangent to L and satisfying $Z(0) = 0$ and $Z(t) = 0$ for some $t \in (0,b)$.* ∎

6.2. *Riemannian foliations.* Assume that $\mathcal{F}$ is a Riemannian foliation for which the Riemannian structure of M is bundle-like [Re]. In this case, $\mathcal{F}$ is given locally by a Riemannian submersion of an open subset of M onto another Riemannian manifold. The following fact is a direct consequence of Lemma 1.3 of [Es].

LEMMA 2. *If $\mathcal{F}$ is the foliation by the fibres of a Riemannian submersion $f : M \to N$, $c : [0,b] \to M$ is a curve tangent to $\mathcal{F}$ and Z is a vector field along c orthogonal to $\mathcal{F}$, then $Z \in T_c\Omega$ if and only if $f_* \circ Z \equiv$ const.* ∎

Now, let $c : [0,b] \in L$ be a leaf curve and $Z \in T_c\Omega$ a vector field orthogonal to $\mathcal{F}$. Put

(39) $$V(s,t) = \exp^M(sZ(t)) \quad \text{for } s \in (-\varepsilon, \varepsilon) \text{ and } t \in [0,b]\,.$$

LEMMA 3. *For any s, $V(s,\cdot)$ is a leaf curve.*

Proof. It suffices to consider $\mathcal{F}$ given by the fibres of a Riemannian submersion $f : M \to N$.

Let $v \in TN$ be a vector such that $f_*(Z(t)) = v$ for any t (Lemma 2). Let $\gamma : (-\varepsilon, \varepsilon) \to N$ be a geodesic satisfying $\dot\gamma(0) = v$. Since horizontal (i.e. orthogonal to the fibres) lifts of N-geodesics are M-geodesics, we have $f(V(s,t)) = \gamma(s)$ for all s and t. In particular, the maps $t \mapsto f(V(\cdot,t))$ are constant. ∎

For the variation given by (39), the variational formula (15) is much simpler. Also, since $\nabla_{d/ds} Y \equiv 0$ (we keep the notation of the proof of Proposition 3) we do not need the assumption of V being admissible. (Actually, in general it is not: the derivative

$$\frac{d}{ds}\int_0^b \langle B(X,X),Y\rangle\,dt = \int_0^b \langle \nabla_{d/ds}\nabla_{d/dt}X,Y\rangle\,dt = \int_0^b (\langle R(Y,X)X,Y\rangle - \|\nabla_{d/dt}Y\|^2)\,dt$$

need not vanish.)

Proposition 8. *For the variation V given by* (39) *one has*

$$\mathcal{L}''(0) = \int_0^b (\langle R(\dot c,Y)\dot c,Y\rangle + \|Y'\|^2 - \langle Y',\dot c\rangle^2)\,dt \tag{40}$$

and

$$\mathcal{E}''(0) = 2\int_0^b (\langle R(\dot c,Y)\dot c,Y\rangle + \|Y'\|^2)\,dt\,. \tag{41}$$

Proof. The first formula follows immediately from (18) and (19) because $\nabla_{d/ds} Y \equiv 0$ in our case. The second formula could be obtained in a similar way. ∎

Remark. Since Y is orthogonal to $\mathcal{F}$, the formulae (40) and (41) could be written in the form

$$\mathcal{L}''(0) = \int_0^b (\langle R(\dot c,Y)\dot c,Y\rangle + \|A^{\perp\dot c}Y\|^2 + \|A^Y(\dot c)\|^2 - \langle A^Y\dot c,\dot c\rangle^2)\,dt \tag{42}$$

and

$$\mathcal{E}''(0) = \int_0^b (\langle R(\dot c,Y)\dot c,Y\rangle + \|A^{\perp\dot c}Y\|^2 + \|A^Y(\dot c)\|^2)\,dt\,. \tag{43}$$

The following result gives an application of the last formula. We use the following notation:

$$\|A\|(x) = \sup\{\|A^v w\| \mid v\in T_x^\perp\mathcal{F},\ w\in T_x\mathcal{F},\ \|v\| = \|w\| = 1\}$$

and

$$K_{\min}(x) = \min\{K_M(v\wedge w) \mid v\in T_x^\perp\mathcal{F},\ w\in T_x\mathcal{F}\}\,.$$

The norm $\|A^{\perp}\|$ is defined similarly to that of A. The argument in the proof is analogous to that of Proposition 2.

PROPOSITION 9. *Assume that the inequality*

$$\|A\|^2 + \|A^{\perp}\|^2 < K_{\min}$$

holds along a leaf L of a Riemannian foliation $\mathcal{F}$. Then the bundle $T^{\perp}\mathcal{F}$ admits at most one closed integral manifold of dimension $q = \operatorname{codim}\mathcal{F}$ intersecting L.

Proof. Assume that T_1 and T_2 are two closed integral manifolds of $T^{\perp}\mathcal{F}$ such that $L \cap T_1$, $L \cap T_2 \neq \emptyset$. The space Ω_0 of leaf curves $\gamma : [0, b] \to M$ with $\gamma(0) \in T_1$ and $\gamma(b) \in T_2$ is non-empty and the functional $\mathcal{E}|\Omega_0$ (as well as $\mathcal{L}|\Omega_0$) attains its minimum for some curve c. From (9) it follows that c is a leaf geodesic. Let V be an $\mathcal{F}$-variation of c of the form (39). From (43) it follows that

$$\begin{aligned} 0 \le \mathcal{E}''(0) &= \int_0^b (-K_M(\dot c \wedge Z)\|Z\|^2 + \|A^{\perp \dot c} Z\|^2 + \|A^Z \dot c\|^2)\, dt \\ &\le \int_0^b (\|A^{\perp}\|^2(c(t)) + \|A\|^2(c(t)) - K_{\min}(c(t)))\, dt < 0\,. \end{aligned}$$

Contradiction. ■

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Reçu par la Rédaction le 23.10.1992