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ON FIBRED SASAKIAN SPACES WITH VANISHING CONTACT BOCHNER CURVATURE TENSOR

BY

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1. Introduction. Recently, Y. Tashiro and B. H. Kim ([12]) studied fibred Riemannian spaces with almost complex, almost contact or contact structures. For fibred Sasakian spaces with conformal fibres, B. H. Kim ([4], [5]) studied total spaces of constant $\widetilde{\phi}$-holomorphic sectional curvature and total spaces with vanishing contact Bochner curvature tensor, and obtained the following theorems:

THEOREM A ([4]). *Let $\widetilde{M}$ be a fibred Sasakian space with conformal fibres. If $\widetilde{M}$ is a space of constant $\widetilde{\phi}$-holomorphic sectional curvature $\widetilde{c}$, then*

(1) *the total space is a Sasakian space of constant $\widetilde{\phi}$-holomorphic sectional curvature* -3,

(2) *the base space M is locally Euclidean*, *and*

(3) *each fibre F is a Sasakian space of constant $\bar{\phi}$-holomorphic sectional curvature* -3.

Conversely, *if the base space M is locally Euclidean and each fibre F is a Sasakian space of constant $\bar{\phi}$-holomorphic sectional curvature* -3, *then $\widetilde{M}$ is a Sasakian space of constant $\widetilde{\phi}$-holomorphic sectional curvature* -3.

THEOREM B ([5]). *Let $\widetilde{M}$ be a fibred Sasakian space with conformal fibres of dimension $s > 3$. If the contact Bochner curvature tensor of $\widetilde{M}$ vanishes*, *then the base space M is of constant holomorphic sectional curvature and each fibre F is of constant $\bar{\phi}$-holomorphic sectional curvature*

$$\frac{4(\overline{K} - s + 1) - (3s - 5)(s - 1)}{(s - 1)(s + 1)}\,.$$

We recall the definition of the Bochner curvature tensor in a Kählerian space and the contact Bochner curvature tensor in a Sasakian space in §2. In §3, we define fibred Sasakian spaces and prove certain equations valid in such spaces. We discuss fibred Sasakian spaces of constant $\widetilde{\phi}$-holomorphic sectional curvature in §4 and fibred Sasakian spaces with vanishing contact

Bochner curvature tensor in §5, without the assumption that the space in question has conformal fibres.

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2. Preliminaries. Let M be an n-dimensional Riemannian space. Throughout this paper, we assume that the spaces considered are connected and of class C^∞. Denote respectively by g_{ji}, $R_{kji}{}^h$, $R_{ji} = R_{hji}{}^h$ and R the metric tensor, the curvature tensor, the Ricci tensor and the scalar curvature of M in terms of local coordinates $\{x^i\}$, where the Latin indices run over the range $\{1, \ldots, n\}$.

An $n(= 2l)$-dimensional *Kählerian space* with metric g is a Riemannian space admitting a structure tensor $\phi_i{}^h$ such that

$$\phi_i{}^r \phi_r{}^j = -\delta_i^j\,, \qquad \phi_{ji} = -\phi_{ij}, \qquad \nabla_k \phi_{ji} = 0\,,$$

where we put $\phi_{ji} = \phi_j{}^r g_{ri}$ and ∇_k denotes the covariant derivative.

A Kählerian space is said to be *of constant holomorphic sectional curvature* c if the curvature tensor satisfies

$$R_{kji}{}^h = \frac{c}{4}(g_{ji}\delta_k^h - g_{ki}\delta_j^h + \phi_{ji}\phi_k{}^h - \phi_{ki}\phi_j{}^h - 2\phi_{kj}\phi_i{}^h)\,.$$

The *Bochner curvature tensor* $B_{kji}{}^h$ of a Kählerian space M^n is defined by

$$\begin{aligned} B_{kji}{}^h = R_{kji}{}^h & \\ &+ \frac{1}{n+4}(g_{ki}R_j{}^h - g_{ji}R_k{}^h + R_{ki}\delta_j^h - R_{ji}\delta_k^h + \phi_{ki}S_j{}^h - \phi_{ji}S_k{}^h \\ &\qquad + S_{ki}\phi_j{}^h - S_{ji}\phi_k{}^h + 2S_{kj}\phi_i{}^h + 2\phi_{kj}S_i{}^h) \\ &- \frac{R}{(n+2)(n+4)}(g_{ki}\delta_j^h - g_{ji}\delta_k^h + \phi_{ki}\phi_j{}^h - \phi_{ji}\phi_k{}^h + 2\phi_{kj}\phi_i{}^h)\,, \end{aligned}$$

where we put $S_{ji} = \phi_j{}^r R_{ri}$.

For the Bochner curvature tensor M. Matsumoto and S. Tanno proved:

THEOREM C ([7]). *If a Kählerian space M with vanishing Bochner curvature tensor has constant scalar curvature, then either*

(1) *M is a space of constant holomorphic sectional curvature, or*

(2) *M is locally a product of two spaces of constant holomorphic sectional curvatures c (≥ 0) and $-c$.*

Next, an $n(= 2l + 1)$-dimensional Riemannian space M^n is called a *Sasakian space* if it admits a unit special Killing 1-form η with constant 1 such that

$$\nabla_k \phi_{ji} = \eta_j g_{ki} - \eta_i g_{kj}, \quad \phi_{kj} = \nabla_k \eta_j \quad \text{and} \quad \xi^i = \eta_j g^{ji}\,.$$

On a Sasakian space, the following identities are well-known:

$$R_{kji}{}^h\eta_h = \eta_k g_{ji} - \eta_j g_{ki}\,, \tag{2.1}$$

$$R_k{}^h\eta_h = (n-1)\eta_k\,, \tag{2.2}$$

$$\xi^s\nabla_s R_{kji}{}^h = 0\,. \tag{2.3}$$

The *contact Bochner curvature tensor* $B_{kji}{}^h$ of a Sasakian space is defined by

$$\begin{aligned}
B_{kji}{}^h = R_{kji}{}^h
&+ \frac{1}{n+3}(R_{ki}\delta_j{}^h - R_{ji}\delta_k{}^h + g_{ki}R_j{}^h - g_{ji}R_k{}^h + S_{ki}\phi_j{}^h - S_{ji}\phi_k{}^h \\
&\qquad + \phi_{ki}S_j{}^h - \phi_{ji}S_k{}^h + 2S_{kj}\phi_i{}^h + 2\phi_{kj}S_i{}^h \\
&\qquad - R_{ki}\eta_j\xi^h + R_{ji}\eta_k\xi^h - \eta_k\eta_i R_j{}^h + \eta_j\eta_i R_k{}^h) \\
&- \frac{k+n-1}{n+3}(\phi_{ki}\phi_j{}^h - \phi_{ji}\phi_k{}^h + 2\phi_{kj}\phi_i{}^h) - \frac{k-4}{n+3}(g_{ki}\delta_j{}^h - g_{ji}\delta_k{}^h) \\
&+ \frac{k}{n+3}(g_{ki}\eta_j\xi^h + \eta_k\eta_i\delta_j{}^h - g_{ji}\eta_k\xi^h - \eta_j\eta_i\delta_k{}^h)\,,
\end{aligned}$$

where $k = \frac{R+n-1}{n+1}$.

When the curvature tensor of a Sasakian space has the form

$$\begin{aligned}
R_{kji}{}^h = &\frac{c+3}{4}(g_{ji}\delta_k{}^h - g_{ki}\delta_j{}^h) \\
&+ \frac{c-1}{4}(g_{ki}\eta_j\xi^h - g_{ji}\eta_k\xi^h + \eta_k\eta_i\delta_j{}^h - \eta_j\eta_i\delta_k{}^h \\
&\qquad - \phi_{ki}\phi_j{}^h + \phi_{ji}\phi_k{}^h - 2\phi_{kj}\phi_i{}^h)\,,
\end{aligned}$$

then the Sasakian space is called a space *of constant ϕ-holomorphic sectional curvature c*. If the Ricci tensor R_{ji} of a Sasakian space M satisfies

$$R_{ji} = \left(\frac{R}{n-1} - 1\right)g_{ji} - \left(\frac{R}{n-1} - n\right)\eta_j\eta_i\,,$$

then M is called an *η-Einstein space*.

The following theorems were obtained by I. Hasegawa and T. Nakane:

THEOREM D ([2]). *Let M^n ($n \geq 7$) be a Sasakian space with constant scalar curvature R whose contact Bochner curvature tensor vanishes. If the square of the length of the Ricci tensor is less than*

$$\frac{n^3-5n^2+7n+29}{(n+1)^2(n-5)^2}R^2 - \frac{2(n^4-10n^3+58n+79)}{(n+1)^2(n-5)^2}R + \frac{(n-1)^2(n^4-7n^3+n^2+47n+54)}{(n+1)^2(n-5)^2}\,,$$

then M is a space of constant ϕ-holomorphic sectional curvature.

Theorem E ([2]). *Let M^5 be a Sasakian space with constant scalar curvature R whose contact Bochner curvature tensor vanishes. If the scalar curvature is not -4, then M is of constant ϕ-holomorphic sectional curvature.*

Finally, if a tensor $T_{i_1 \ldots i_p}{}^{j_1 \cdots j_q}$ on a Sasakian space satisfies

$$\phi_{i_1}{}^{a_1} \ldots \phi_{i_p}{}^{a_p} \phi_{b_1}{}^{j_1} \ldots \phi_{b_q}{}^{j_q} \phi_k{}^c \nabla_c T_{a_1 \ldots a_p}{}^{b_1 \ldots b_q} = 0 \,,$$

then the tensor $T_{i_1 \ldots i_p}{}^{j_1 \cdots j_q}$ is called *η-parallel*.

3. Fibred Riemannian spaces. Let $\{\widetilde{M}, M, \widetilde{g}, \pi\}$ be a fibred space with Riemannian metric $\widetilde{g}$, that is, $(\widetilde{M}, \widetilde{g})$ is an m-dimensional total space with Riemannian metric $\widetilde{g}$, M an n-dimensional base space, and $\pi : \widetilde{M} \to M$ a projection with maximal rank n. The fibre through a point in $\widetilde{M}$ is denoted by F, and it is s-dimensional. Throughout this paper the ranges of indices are as follows:

$$\begin{aligned} A, B, C, D, \ldots &= 1, 2, \ldots, n, n+1, \ldots, m \,, \\ h, i, j, k, \ldots &= 1, 2, \ldots, n, \\ \alpha, \beta, \gamma, \delta, \ldots &= \qquad\qquad\quad n+1, \ldots, m \,. \end{aligned}$$

We take coordinate neighborhoods $\{\widetilde{U}, x^H\}$ on $\widetilde{M}$ and $\{U, v^h\}$ on M such that $\pi(\widetilde{U}) = U$, where x^H and v^h are coordinates in $\widetilde{U}$ and U, respectively. Then the projection π is expressed by

$$v^h = v^h(x^H) \,,$$

and the Jacobian $(\partial v^h / \partial x^H)$ has maximum rank n. Take a fibre F such that $F \cap \widetilde{U} \neq \emptyset$. Then there is a coordinate system (v^h, y^α) in $\widetilde{U}$ such that y^α are local coordinates in $F \cap \widetilde{U}$.

If we put

$$E_I{}^h = \frac{\partial v^h}{\partial x^I} \quad \text{and} \quad C^H{}_\alpha = \frac{\partial x^H}{\partial y^\alpha} \,,$$

then $E_I{}^h$ are components of a local covector field E^h in $\widetilde{U}$ for each fixed index h and $C^H{}_\alpha$ are those of a vector field C_α for each fixed index α. Denoting by $\widetilde{g}_{JI}$ the components of $\widetilde{g}$ in $\{\widetilde{U}, x^H\}$, we put

$$\overline{g}_{\gamma\beta} = \widetilde{g}_{JI} C^J{}_\gamma C^I{}_\beta \,.$$

Then the $\overline{g}_{\gamma\beta}$ are the components of the induced metric tensor $\overline{g}$ of F along $F \cap \widetilde{U}$. If we put

$$C_I{}^\alpha = \widetilde{g}_{IJ} \overline{g}^{\alpha\beta} C^J{}_\beta \,,$$

where $(\overline{g}^{\alpha\beta})$ is the inverse matrix of $(\overline{g}_{\alpha\beta})$, and denote by C^α the local covector field with components $C_I{}^\alpha$ in $\widetilde{U}$ for each index α, then (E^h, C^α) forms a coframe in $\widetilde{U}$. Denoting by $(E^H{}_h, C^H{}_\beta)$ the inverse matrix of $(E_I{}^i, C_I{}^\alpha)$, we have

$$E_I{}^h E^I{}_i = \delta_i{}^h, \quad E_I{}^h C^I{}_\beta = 0\,,$$
$$C_I{}^\alpha E^I{}_i = 0\,, \quad C_I{}^\alpha C^I{}_\beta = \delta_\beta{}^\alpha \tag{3.1}$$

and

$$E_I{}^h E^H{}_h + C_I^\alpha C^H{}_\alpha = \delta_I{}^H\,. \tag{3.2}$$

Denoting by $(\widetilde{g}^{JI})$ the inverse matrix of $(\widetilde{g}_{JI})$ and putting

$$g_{ji} = \widetilde{g}_{JI} E^J{}_j E^I{}_i\,,$$

we obtain

$$E^H{}_h = \widetilde{g}^{HI} g_{hi} E_I{}^i\,.$$

The $E^H{}_h$ are the components of a local vector field E_h defined in $\{\widetilde{U}, x^H\}$, for each fixed index h. Thus, we find that the set (E_i, C_β) forms in $\widetilde{U}$ a frame dual to the coframe (E^h, C^α). By analogy with the above notation, we often denote by $(B^I{}_B)$ (resp. $(B_J{}^A)$) the matrix $(E^I{}_i, C^I{}_\beta)$ (resp. the matrix $(E_J{}^j, C_J{}^\alpha)$). Then we can write (3.1) and (3.2) as

$$B_I{}^A B^I{}_B = \delta_B{}^A \quad \text{and} \quad B_I{}^A B^H{}_A = \delta_I{}^H\,,$$

respectively.

Any tensor field in $\widetilde{M}$, say $\widetilde{T}$ of type $(1,2)$, is represented in $\widetilde{U}$ in the form

$$\begin{aligned}\widetilde{T} = {} & T_{ji}{}^h E^j \otimes E^i \otimes E_h + T_{ji}{}^\alpha E^j \otimes E^i \otimes C_\alpha + \dots \\ & + T_{\gamma\beta}{}^h C^\gamma \otimes C^\beta \otimes E_h + T_{\gamma\beta}{}^\alpha C^\gamma \otimes C^\beta \otimes C_\alpha\,,\end{aligned} \tag{3.3}$$

where

$$T_{ji}{}^h = E^J{}_j E^I{}_i E_H{}^h \widetilde{T}_{JI}{}^H, \qquad T_{ji}{}^\alpha = E^J{}_j E^I{}_i C_H{}^\alpha \widetilde{T}_{JI}{}^H, \quad \dots,$$
$$T_{\gamma\beta}{}^h = C^J{}_\gamma C^I{}_\beta E_H{}^h \widetilde{T}_{JI}{}^H, \quad T_{\gamma\beta}{}^\alpha = C^J{}_\gamma C^I{}_\beta C_H{}^\alpha \widetilde{T}_{JI}{}^H\,.$$

The first term $T_{ji}{}^h E^j \otimes E^i \otimes E_h$ is called the *horizontal part* of $\widetilde{T}$ and denoted by $\widehat{T}$. The last term $T_{\gamma\beta}{}^\alpha C^\gamma \otimes C^\beta \otimes C_\alpha$ is called the *vertical part* of $\widetilde{T}$ and denoted by $\overline{T}$. For a function $\widetilde{f}$ on $\widetilde{M}$, we define its horizontal part $\widehat{f}$ and vertical part $\bar{f}$ by $\widetilde{f} = \widehat{f} = \bar{f}$.

A tensor field, say $\widetilde{T}$ of type $(1,2)$ with local expression (3.3) on $\widetilde{M}$, is projectable if and only if the $T_{ji}{}^h$ are projectable, or equivalently, if and only if

$$\frac{\partial}{\partial y^\alpha} T_{ji}{}^h = 0\,.$$

If the metric tensor $\widetilde{g}$ in a fibred space $\{\widetilde{M}, M, \widetilde{g}, \pi\}$ is projectable, then $\{\widetilde{M}, M, \widetilde{g}, \pi\}$ or simply $(\widetilde{M}, \widetilde{g})$ is called a *fibred Riemannian space.*

Let $\widetilde{\nabla}$ be the Riemannian connection of the Riemannian space $(\widetilde{M}, \widetilde{g})$ and denote by $\widetilde{\left\{ {H \atop JI} \right\}}$ the Christoffel symbols constructed from $\widetilde{g}_{JI}$ in $\{\widetilde{U}, x^H\}$. Let ∇ and $\overline{\nabla}$ be the Riemannian connections determined by the induced metrics g in M and $\overline{g}$ in F, respectively.

We denote by $\left\{ {h \atop ji} \right\}$ and $\overline{\left\{ {\alpha \atop \gamma\beta} \right\}}$ the Christoffel symbols constructed from g_{ji} in $\{U, v^h\}$ and $\overline{g}_{\gamma\beta}$ in $\{F \cap \widetilde{U}, y^\alpha\}$, respectively.

If we put

$$\widetilde{\nabla}_J B^H{}_B = \Gamma_C{}^A{}_B B_J{}^C B^H{}_A$$

in $\widetilde{U}$, where $\Gamma_C{}^A{}_B$ are local functions defined in $\widetilde{U}$, then the following hold [14]:

(a) $\Gamma_j{}^h{}_i = \left\{ {h \atop ji} \right\}$.

(b) $\Gamma_\gamma{}^\alpha{}_\beta = \overline{\left\{ {\alpha \atop \gamma\beta} \right\}}$.

(c) Writing $\Gamma_j{}^\alpha{}_i$ and $\Gamma_j{}^h{}_\beta$ $(= \Gamma_\beta{}^h{}_j)$ as $h_{ji}{}^\alpha$ and $h^h{}_{j\beta}$ respectively, we have

$$h_{ji}{}^\alpha + h_{ij}{}^\alpha = 0, \qquad h^h{}_{j\beta} = g^{hi} h_{ij}{}^\alpha \overline{g}_{\alpha\beta}.$$

Along each fibre F, the $h^h{}_{j\gamma}$ are the connection coefficients of the induced connection of the normal bundle of the submanifold F embedded in $(\widetilde{M}, \widetilde{g})$ with respect to the normals E_h.

(d) Writing $\Gamma_\gamma{}^h{}_\beta$ $(= \Gamma_\beta{}^h{}_\gamma)$ and $\Gamma_\gamma{}^\alpha{}_i$ as $L_{\gamma\beta}{}^h$ and $-L_\gamma{}^\alpha{}_i$ respectively, we have

$$L_\gamma{}^\alpha{}_i = L_{\gamma\beta}{}^h g_{hi} \overline{g}^{\beta\alpha}, \qquad \Gamma_j{}^\alpha{}_\beta = P_{j\beta}{}^\alpha - L_\beta{}^\alpha{}_j\,.$$

If we denote by $\widetilde{\mathcal{L}}_{C_\beta}$ the Lie derivation with respect to C_β on $\widetilde{M}$, then the $P_{j\beta}{}^\alpha$ appear in

$$\widetilde{\mathcal{L}}_{C_\beta} E^h = 0, \quad \widetilde{\mathcal{L}}_{C_\beta} E_j = -P_{j\beta}{}^\alpha C_\alpha, \quad \widetilde{\mathcal{L}}_{C_\beta} C_\gamma = 0, \quad \widetilde{\mathcal{L}}_{C_\beta} C^\alpha = P_{j\beta}{}^\alpha E^j\,.$$

Along each fibre F, the $L_{\gamma\beta}{}^h$ are the components of the second fundamental tensor of the submanifold F embedded in $(\widetilde{M}, \widetilde{g})$ with respect to the normals E_h. If $L_{\gamma\beta}{}^h = 0$, then $\{\widetilde{M}, M, \widetilde{g}, \pi\}$ is called a *fibred Riemannian space with isometric fibres.* If $L_{\gamma\beta}{}^h = \overline{g}_{\gamma\beta} A^h$, where $A = A^h E_h$ is the mean curvature vector along each fibre and a horizontal vector field in $\widetilde{M}$, then $\{\widetilde{M}, M, \widetilde{g}, \pi\}$ is called a *fibred Riemannian space with conformal fibres.*

Summing up the results mentioned above, we have

$$\Gamma_j{}^h{}_i = \left\{ {h \atop ji} \right\}, \qquad \Gamma_j{}^h{}_\beta = \Gamma_\beta{}^h{}_j = h^h{}_{j\beta}\,,$$

$$(3.4)\qquad \Gamma_\gamma{}^h{}_\beta = L_{\gamma\beta}{}^h, \quad \Gamma_j{}^\alpha{}_i = h_{ji}{}^\alpha, \quad \Gamma_\gamma{}^\alpha{}_i = -L_\gamma{}^\alpha{}_i\,,$$
$$\Gamma_j{}^\alpha{}_\beta = P_{j\beta}{}^\alpha - L_\beta{}^\alpha{}_j, \quad \Gamma_\gamma{}^\alpha{}_\beta = \overline{\left\{ \begin{matrix} \alpha \\ \gamma\beta \end{matrix} \right\}}\,.$$

Let $\mathcal{F}^p_q(\widetilde{M})$ be the space of all tensor fields of type (p,q) on $\widetilde{M}$. Let $\mathcal{F}^r_s(h\widetilde{M})$ (resp. $\mathcal{F}^t_u(v\widetilde{M})$) be the space of all horizontal (resp. vertical) tensor fields of type (r,s) (resp. (t,u)) on $\widetilde{M}$. We consider formal tensor products on $\widetilde{M}$ such as $\mathcal{F}^p_q(\widetilde{M}) \otimes \mathcal{F}^r_s(h\widetilde{M}) \otimes \mathcal{F}^t_u(v\widetilde{M})$. We call an element $\widetilde{T}$ of this space a $\begin{pmatrix} prt \\ qsu \end{pmatrix}$-*partial tensor* on $\widetilde{M}$. We may identify $\mathcal{F}^{p00}_{q00}(\widetilde{M})$, $\mathcal{F}^{0r0}_{0s0}(\widetilde{M})$ and $\mathcal{F}^{00t}_{00u}(\widetilde{M})$ with $\mathcal{F}^p_q(\widetilde{M})$, $\mathcal{F}^r_s(h\widetilde{M})$ and $\mathcal{F}^t_u(v\widetilde{M})$, respectively. For any element of $\mathcal{F}^{prt}_{qsu}(\widetilde{M})$, say $\widetilde{T}$ in $\mathcal{F}^{111}_{111}(\widetilde{M})$ with components $T_J{}^I{}_i{}^h{}_\beta{}^\alpha$, we define the $(*)$-*covariant derivative* $\nabla^*\widetilde{T}$ of $\widetilde{T}$ as a partial tensor with components

$$(3.5)\qquad \nabla^*_K T_J{}^I{}_i{}^h{}_\beta{}^\alpha = \frac{\partial}{\partial x^K} T^{\cdots}_{\cdots} + \widetilde{\left\{ \begin{matrix} I \\ KH \end{matrix} \right\}} T^{H\cdot\cdot}_{\cdots} - T^{\cdots}_{H\cdot\cdot} \widetilde{\left\{ \begin{matrix} H \\ KJ \end{matrix} \right\}}$$
$$+ (\Gamma_C{}^h{}_e T^{\cdot e \cdot}_{\cdots} + \Gamma_C{}^\alpha{}_\varepsilon T^{\cdot\cdot\varepsilon}_{\cdots} - T^{\cdots}_{\cdot e \cdot} \Gamma_C{}^e{}_i - T^{\cdots}_{\cdot\cdot\varepsilon} \Gamma_C{}^\varepsilon{}_\beta) B_K{}^C$$

in $\widetilde{U}$, where Γ's are given by (3.4) (see [14]). For any element $\widetilde{T}$ of $\mathcal{F}^{prt}_{qsu}(\widetilde{M})$, we have $\nabla^*\widetilde{T} = \widetilde{\nabla}\widetilde{T}$.

Denote two covariant derivations $'\nabla$ and $''\nabla$ acting on elements of $\mathcal{F}^{prt}_{qsu}(\widetilde{M})$ by

$$'\nabla_k = E^K{}_k \nabla^*_K, \qquad ''\nabla_\gamma = C^K{}_\gamma \nabla^*_K\,.$$

For any element of $\mathcal{F}^{prt}_{qsu}(\widetilde{M})$, say $\widetilde{T}$ in $\mathcal{F}^{111}_{111}(\widetilde{M})$ with components $T_J{}^I{}_i{}^h{}_\beta{}^\alpha$, $'\nabla\widetilde{T}$ and $''\nabla\widetilde{T}$ are elements of $\mathcal{F}^{111}_{121}(\widetilde{M})$ and $\mathcal{F}^{111}_{112}(\widetilde{M})$ respectively, with components

$$'\nabla_k T_J{}^I{}_i{}^h{}_\beta{}^\alpha = \frac{\partial}{\partial v^k} T^{\cdots}_{\cdots} + \left(\widetilde{\left\{ \begin{matrix} I \\ KH \end{matrix} \right\}} T^{H\cdot\cdot}_{\cdots} - T^{\cdots}_{H\cdot\cdot} \widetilde{\left\{ \begin{matrix} H \\ KJ \end{matrix} \right\}} \right) E^K{}_k$$
$$+ \Gamma_k{}^h{}_e T^{\cdot e \cdot}_{\cdots} + \Gamma_k{}^\alpha{}_\varepsilon T^{\cdot\cdot\varepsilon}_{\cdots} - T^{\cdots}_{\cdot e \cdot} \Gamma_k{}^e{}_i - T^{\cdots}_{\cdot\cdot\varepsilon} \Gamma_k{}^\varepsilon{}_\beta\,,$$
$$''\nabla_\gamma T_J{}^I{}_i{}^h{}_\beta{}^\alpha = \frac{\partial}{\partial y^\gamma} T^{\cdots}_{\cdots} + \left(\widetilde{\left\{ \begin{matrix} I \\ KH \end{matrix} \right\}} T^{H\cdot\cdot}_{\cdots} - T^{\cdots}_{H\cdot\cdot} \widetilde{\left\{ \begin{matrix} H \\ KJ \end{matrix} \right\}} \right) C^K{}_\gamma$$
$$+ \Gamma_\gamma{}^h{}_e T^{\cdot e \cdot}_{\cdots} + \Gamma_\gamma{}^\alpha{}_\varepsilon T^{\cdot\cdot\varepsilon}_{\cdots} - T^{\cdots}_{\cdot e \cdot} \Gamma_\gamma{}^e{}_i - T^{\cdots}_{\cdot\cdot\varepsilon} \Gamma_\gamma{}^\varepsilon{}_\beta\,.$$

Proposition F ([14]). *On* $\widetilde{M}$ *we have*

$$\nabla^*_K \widetilde{g}_{JI} = 0, \quad \nabla^*_K g_{ji} = 0, \quad \nabla^*_K \overline{g}_{\gamma\beta} = 0\,,$$
$$'\nabla_k \widetilde{g}_{JI} = 0, \quad '\nabla_k g_{ji} = 0, \quad '\nabla_k \overline{g}_{\gamma\beta} = 0\,,$$
$$''\nabla_\alpha \widetilde{g}_{JI} = 0, \quad ''\nabla_\alpha g_{ji} = 0, \quad ''\nabla_\alpha \overline{g}_{\gamma\beta} = 0\,.$$

We denote by $\widetilde{K}_{KJI}{}^H$, $K_{kji}{}^h$ and $\overline{K}_{\delta\gamma\beta}{}^\alpha$ the components of the curvature tensor of $(\widetilde{M}, \widetilde{g})$ in $\{\widetilde{U}, x^H\}$, of the base space (M, g) in $\{U, v^h\}$, and of each fibre $(F, \overline{g})$ in $\{F \cap \widetilde{U}, y^\alpha\}$, respectively.

If we put $P_{DCB}{}^A = B^K_D B^J_C B^I_B B_H{}^A \widetilde{K}_{KJI}{}^H$, then

$$P_{DCB}{}^A + P_{CDB}{}^A = 0, \qquad P_{DCB}{}^A + P_{CBD}{}^A + P_{BDC}{}^A = 0$$

and the following equations hold [14]:

$$P_{kji}{}^h = K_{kji}{}^h - 2h_{kj}{}^\varepsilon h^h{}_{i\varepsilon} + h_{ji}{}^\varepsilon h^h{}_{k\varepsilon} - h_{ki}{}^\varepsilon h^h{}_{j\varepsilon}\,, \tag{3.6}$$

$$P_{kj\beta}{}^h = {}'\nabla_k h^h{}_{j\beta} - {}'\nabla_j h^h{}_{k\beta} - 2h_{kj}{}^\varepsilon L_{\varepsilon\beta}{}^h\,, \tag{3.7}$$

$$P_{\delta ji}{}^h = -{}'\nabla_j h^h{}_{i\delta} + h^h{}_{i\varepsilon} L_\delta{}^\varepsilon{}_j + L_{\delta\varepsilon}{}^h h_{ji}{}^\varepsilon + h^h{}_{j\varepsilon} L_\delta{}^\varepsilon{}_i, \tag{3.8}$$

$$P_{\delta j\beta}{}^h = {}''\nabla_\delta h^h{}_{j\beta} - {}'\nabla_j L_{\delta\beta}{}^h + L_\delta{}^\varepsilon{}_j L_{\varepsilon\beta}{}^h + h^e{}_{j\delta} h^h{}_{e\beta}\,, \tag{3.9}$$

$$P_{\delta\gamma i}{}^h = {}''\nabla_\delta h^h{}_{i\gamma} - {}''\nabla_\gamma h^h{}_{i\delta} + h^h{}_{e\gamma} h^e{}_{i\delta} - h^h{}_{e\delta} h^e{}_{i\gamma} - L_{\delta\varepsilon}{}^h L_\gamma{}^\varepsilon{}_i + L_{\gamma\varepsilon}{}^h L_\delta{}^\varepsilon{}_i\,, \tag{3.10}$$

$$P_{\delta\gamma\beta}{}^h = {}''\nabla_\delta L_{\gamma\beta}{}^h - {}''\nabla_\gamma L_{\delta\beta}{}^h\,, \tag{3.11}$$

$$P_{\delta\gamma\beta}{}^\alpha = \overline{K}_{\delta\gamma\beta}{}^\alpha - L_\delta{}^\alpha{}_e L_{\gamma\beta}{}^e + L_\gamma{}^\alpha{}_e L_{\delta\beta}{}^e\,, \tag{3.12}$$

$$P_{\delta\gamma i}{}^\alpha = -{}''\nabla_\delta L_\gamma{}^\alpha{}_i + {}''\nabla_\gamma L_\delta{}^\alpha{}_i\,, \tag{3.13}$$

$$P_{\delta j\beta}{}^\alpha = {}''\nabla_\beta L_\delta{}^\alpha{}_j - \overline{g}^{\alpha\varepsilon} g_{je}{}''\nabla_\varepsilon L_{\delta\beta}{}^e\,, \tag{3.14}$$

$$P_{kj\beta}{}^\alpha = -{}'\nabla_k L_\beta{}^\alpha{}_j + {}'\nabla_j L_\beta{}^\alpha{}_k - 2{}''\nabla_\beta h_{kj}{}^\alpha - h_{ke}{}^\alpha h^e{}_{j\beta} + h_{je}{}^\alpha h^e{}_{k\beta} - L_\varepsilon{}^\alpha{}_k L_\beta{}^\varepsilon{}_j + L_\varepsilon{}^\alpha{}_j L_\beta{}^\varepsilon{}_k, \tag{3.15}$$

$$P_{\delta ji}{}^\alpha = {}''\nabla_\delta h_{ji}{}^\alpha + {}'\nabla_j L_\delta{}^\alpha{}_i - L_\delta{}^\varepsilon{}_j L_\varepsilon{}^\alpha{}_i + h^e{}_{j\delta} h_{ei}{}^\alpha\,, \tag{3.16}$$

$$P_{kji}{}^\alpha = {}'\nabla_k h_{ji}{}^\alpha - {}'\nabla_j h_{ki}{}^\alpha + 2h_{kj}{}^\varepsilon L_\varepsilon{}^\alpha{}_i\,. \tag{3.17}$$

Also, we denote by $\widetilde{K}_{JI}$, K_{ji} and $\overline{K}_{\beta\alpha}$ the components of the Ricci tensors of $\widetilde{M}$, M and F, respectively. Then from (3.6), (3.7), (3.9), (3.12), (3.14) and (3.16) we have

$$E^J{}_j E^I{}_i \widetilde{K}_{JI} = K_{ji} - 2h_{ej}{}^\varepsilon h^e{}_{i\varepsilon} + {}''\nabla_\varepsilon h_{ji}{}^\varepsilon + {}'\nabla_j L_\varepsilon{}^\varepsilon{}_i - N_{ji}\,, \tag{3.18}$$

$$E^J{}_j C^I{}_\alpha \widetilde{K}_{JI} = {}'\nabla_e h^e{}_{j\alpha} - 2h_{ej}{}^\varepsilon L_{\varepsilon\alpha}{}^e + {}''\nabla_\alpha L_\varepsilon{}^\varepsilon{}_j + Q_{\alpha j}\,, \tag{3.19}$$

$$C^J{}_\beta C^I{}_\alpha \widetilde{K}_{JI} = \overline{K}_{\beta\alpha} - h^f{}_{e\beta} h^e{}_{f\alpha} + {}'\nabla_e L_{\beta\alpha}{}^e - L_\varepsilon{}^\varepsilon{}_e L_{\beta\alpha}{}^e\,, \tag{3.20}$$

where we put $N_{ji} = L_\varepsilon{}^\tau{}_j L_\tau{}^\varepsilon{}_i$ and $Q_{\alpha j} = -{}''\nabla_\varepsilon L_\alpha{}^\varepsilon{}_j$.

Let $\widetilde{M}$ be a Sasakian space with Sasakian structure $(\widetilde{\phi}, \widetilde{\xi}, \widetilde{\eta}, \widetilde{g})$ such that $\widetilde{\phi}$ is projectable and each fibre is $\widetilde{\phi}$-invariant and tangent to the vector $\widetilde{\xi}$. Then $\{\widetilde{M}, M, \widetilde{g}, \pi\}$ is called a *fibred Sasakian space*. In [4] and [5], the following is shown:

PROPOSITION G. *Let the induced almost contact metric structure* $(\widetilde{\phi},\widetilde{\xi},\widetilde{\eta},\widetilde{g})$ *on* $\widetilde{M}$ *be Sasakian. Then the base space* M *is Kählerian with Kählerian structure* (ϕ, g) *and each fibre* F *is Sasakian with Sasakian structure* $(\bar{\phi},\overline{\xi},\overline{\eta},\overline{g})$.

Also, we have the following equations:

$$h_{ji}{}^{\alpha} = -\phi_{ji}\overline{\xi}^{\alpha}\,, \tag{3.21}$$

$$\overline{\xi}^{\varepsilon} L_{\varepsilon}{}^{\alpha}{}_{i} = 0\,, \tag{3.22}$$

$$'\nabla_i \overline{\xi}^{\varepsilon} = 0\,, \tag{3.23}$$

$$'\nabla_i \bar{\phi}_{\beta}{}^{\alpha} = 0\,, \tag{3.24}$$

$$L_{\beta}{}^{\varepsilon}{}_{i}\bar{\phi}_{\varepsilon}{}^{\alpha} - L_{\beta}{}^{\alpha}{}_{e}\phi_i{}^{e} = 0\,, \tag{3.25}$$

$$\overline{g}^{\varepsilon\tau} L_{\varepsilon\tau}{}^{h} = 0\,. \tag{3.26}$$

From (3.26), each fibre F is minimal in $\widetilde{M}$. Moreover, if $\widetilde{M}$ is a fibred Sasakian space with conformal fibres, then $\widetilde{M}$ has isometric fibres.

We define skew-symmetric tensors $\widetilde{S}_{JI}$, S_{ji} and $\overline{S}_{\beta\alpha}$ by

$$\widetilde{S}_{JI} = \widetilde{\phi}_J{}^R \widetilde{K}_{RI}, \quad S_{ji} = \phi_j{}^r K_{ri} \quad \text{and} \quad \overline{S}_{\beta\alpha} = \bar{\phi}_{\beta}{}^{\tau}\overline{K}_{\tau\alpha}$$

on $\widetilde{M}$, M and F respectively. Since $\widetilde{S}_{JI}$ and $\overline{S}_{\beta\alpha}$ are skew-symmetric, from (3.9), (3.12), (3.21)–(3.24) and (3.25) we find

$$'\nabla_e L_{\beta\alpha}{}^{e} = 0\,. \tag{3.27}$$

From (3.6), (3.7), (3.9), (3.12), (3.14), (3.16) and (3.27), it is clear that

$$E^J{}_j E^I{}_i \widetilde{S}_{JI} = S_{ji} - 2\phi_{ji} - \phi_j{}^e N_{ei}\,, \tag{3.28}$$

$$E^J{}_j C^I{}_{\alpha} \widetilde{S}_{JI} = \phi_j{}^e Q_{\alpha e}\,, \tag{3.29}$$

$$C^J{}_{\beta} C^I{}_{\alpha} \widetilde{S}_{JI} = \overline{S}_{\beta\alpha}\,. \tag{3.30}$$

Moreover, by (3.21), (3.22), (3.26) and (3.27), equations (3.18)–(3.20) can be rewritten as follows:

$$E^J{}_j E^I{}_i \widetilde{K}_{JI} = K_{ji} - 2g_{ji} - N_{ji}, \tag{3.31}$$

$$E^J{}_j C^I{}_{\alpha} \widetilde{K}_{JI} = Q_{\alpha j}\,, \tag{3.32}$$

$$C^J{}_{\beta} C^I{}_{\alpha} \widetilde{K}_{JI} = \overline{K}_{\beta\alpha} + n\overline{\eta}_{\beta}\overline{\eta}_{\alpha}\,. \tag{3.33}$$

Denote by $\widetilde{K}$, K and $\overline{K}$ the scalar curvatures of $\widetilde{M}$, M and F, respectively. Then from (3.6), (3.9), (3.12) and (3.16) we find

$$\widetilde{K} = K^L + \overline{K} - n - N\,, \tag{3.34}$$

where K^L is the horizontal lift of K and $N = \overline{g}^{\varepsilon\beta}\overline{g}^{\tau\alpha} g_{he} L_{\varepsilon\tau}{}^{e} L_{\beta\alpha}{}^{h}$. In the sequel, we denote K^L by K.

We put $\overline{W}_{\beta\alpha} = L_{\beta}{}^{\varepsilon}{}_{e} L_{\varepsilon\alpha}{}^{e}$ and $\overline{Z}_{\gamma\beta\alpha}{}^{\omega} = L_{\gamma\alpha}{}^{e} L_{\beta}{}^{\omega}{}_{e} - L_{\beta\alpha}{}^{e} L_{\gamma}{}^{\omega}{}_{e}$. By (3.22), (3.25), (3.28) and (3.29), it is easy to see that

(3.35) $\phi_j{}^e N_{ei} = -\phi_i{}^e N_{je}\,,$

(3.36) $\bar{\phi}_{\beta}{}^{\tau} Q_{\tau i} = -\phi_i{}^e Q_{\beta e}\,,$

(3.37) $\bar{\phi}_{\beta}{}^{\tau} \overline{W}_{\tau\alpha} = -\bar{\phi}_{\alpha}{}^{\tau} \overline{W}_{\beta\tau}\,,$

(3.38) $\overline{Z}_{\varepsilon\beta\alpha}{}^{\varepsilon} = \overline{W}_{\beta\alpha}\,,$

(3.39) $\overline{Z}_{\gamma\beta\alpha}{}^{\omega} + \overline{Z}_{\beta\alpha\gamma}{}^{\omega} + \overline{Z}_{\alpha\gamma\beta}{}^{\omega} = 0\,,$

(3.40) $\overline{Z}_{\gamma\beta\alpha}{}^{\omega} = -\overline{Z}_{\beta\gamma\alpha}{}^{\omega}\,,$

(3.41) $\overline{Z}_{\gamma\beta\alpha\omega} = \overline{Z}_{\alpha\omega\gamma\beta}\,,$

(3.42)
$$\begin{aligned}&''\nabla_{\delta} \overline{Z}_{\gamma\beta\alpha}{}^{\omega} + {}''\nabla_{\gamma} \overline{Z}_{\beta\delta\alpha}{}^{\omega} + {}''\nabla_{\beta} \overline{Z}_{\delta\gamma\alpha}{}^{\omega}\\ &= ({}''\nabla_{\delta} L_{\gamma\alpha}{}^{e} - {}''\nabla_{\gamma} L_{\delta\alpha}{}^{e}) L_{\beta}{}^{\omega}{}_{e} + ({}''\nabla_{\gamma} L_{\beta\alpha}{}^{e} - {}''\nabla_{\beta} L_{\gamma\alpha}{}^{e}) L_{\delta}{}^{\omega}{}_{e}\\ &\quad + ({}''\nabla_{\beta} L_{\delta\alpha}{}^{e} - {}''\nabla_{\delta} L_{\beta\alpha}{}^{e}) L_{\gamma}{}^{\omega}{}_{e} + L_{\gamma\alpha}{}^{e}({}''\nabla_{\delta} L_{\beta}{}^{\omega}{}_{e} - {}''\nabla_{\beta} L_{\delta}{}^{\omega}{}_{e})\\ &\quad + L_{\beta\alpha}{}^{e}({}''\nabla_{\gamma} L_{\delta}{}^{\omega}{}_{e} - {}''\nabla_{\delta} L_{\gamma}{}^{\omega}{}_{e}) + L_{\delta\alpha}{}^{e}({}''\nabla_{\beta} L_{\gamma}{}^{\omega}{}_{e} - {}''\nabla_{\gamma} L_{\beta}{}^{\omega}{}_{e})\,,\end{aligned}$$

(3.43) $\bar{\phi}_{\gamma}{}^{\varepsilon} \bar{\phi}_{\beta}{}^{\tau} \overline{Z}_{\varepsilon\tau\alpha}{}^{\omega} = \overline{Z}_{\gamma\beta\alpha}{}^{\omega}\,,$

(3.44) $\frac{1}{2} \bar{\phi}^{\varepsilon\tau} \overline{Z}_{\varepsilon\tau\alpha}{}^{\omega} = \bar{\phi}^{\varepsilon\tau} \overline{Z}_{\varepsilon\alpha\tau}{}^{\omega} = -\bar{\phi}_{\alpha}{}^{\varepsilon} \overline{W}_{\varepsilon}{}^{\omega}\,,$

(3.45) $\overline{Z}_{\gamma\beta\alpha}{}^{\varepsilon} \bar{\eta}_{\varepsilon} = 0\,,$

where $\overline{Z}_{\gamma\beta\alpha\omega} = \bar{g}_{\varepsilon\omega} \overline{Z}_{\gamma\beta\alpha}{}^{\varepsilon}$. The tensor $\overline{Z}_{\gamma\beta\alpha}{}^{\omega}$ vanishes identically if and only if the fibred Sasakian space has isometric fibres.

4. Fibred Sasakian spaces of constant $\widetilde{\phi}$-holomorphic sectional curvature. Let $\widetilde{M}$ be a fibred Sasakian space of constant $\widetilde{\phi}$-holomorphic sectional curvature $\widetilde{c}$. The curvature tensor $\widetilde{K}_{KJI}{}^{H}$ has the form

$$\begin{aligned}\widetilde{K}_{KJI}{}^{H} = {}& \frac{\widetilde{c}+3}{4}(\widetilde{g}_{JI} \widetilde{\delta}_{K}{}^{H} - \widetilde{g}_{KI} \widetilde{\delta}_{J}{}^{H})\\ &+ \frac{\widetilde{c}-1}{4}(\widetilde{g}_{KI} \widetilde{\eta}_{J} \widetilde{\xi}^{H} - \widetilde{g}_{JI} \widetilde{\eta}_{K} \widetilde{\xi}^{H} + \widetilde{\eta}_{K} \widetilde{\eta}_{I} \widetilde{\delta}_{J}{}^{H} - \widetilde{\eta}_{J} \widetilde{\eta}_{I} \widetilde{\delta}_{K}{}^{H}\\ &\qquad - \widetilde{\phi}_{KI} \widetilde{\phi}_{J}{}^{H} + \widetilde{\phi}_{JI} \widetilde{\phi}_{K}{}^{H} - 2\widetilde{\phi}_{KJ} \widetilde{\phi}_{I}{}^{H})\,.\end{aligned}$$

Transvecting the above equation with $B^{K}{}_{D} B^{J}{}_{C} B^{I}{}_{B} B_{H}{}^{A}$ and using (3.6), (3.9)–(3.12) and (3.15), we see that the above equation is equivalent to the following:

(4.1) $K_{kji}{}^{h} + \frac{1}{4}(\widetilde{c}+3)(g_{ki} \delta_{j}{}^{h} - g_{ji} \delta_{k}{}^{h} + \phi_{ki} \phi_{j}{}^{h} - \phi_{ji} \phi_{k}{}^{h} + 2\phi_{kj} \phi_{i}{}^{h}) = 0\,,$

(4.2) ${}'\nabla_{k} L_{\alpha}{}^{\omega}{}_{j} - {}'\nabla_{j} L_{\alpha}{}^{\omega}{}_{k} + L_{\varepsilon}{}^{\omega}{}_{k} L_{\alpha}{}^{\varepsilon}{}_{j} - L_{\varepsilon}{}^{\omega}{}_{j} L_{\alpha}{}^{\varepsilon}{}_{k} - \frac{1}{2}(\widetilde{c}+3) \phi_{kj} \bar{\phi}_{\alpha}{}^{\omega} = 0\,,$

(4.3) $L_{\gamma\varepsilon}{}^{h} L_{\beta}{}^{\varepsilon}{}_{i} - L_{\beta\varepsilon}{}^{h} L_{\gamma}{}^{\varepsilon}{}_{i} - \frac{1}{2}(\widetilde{c}+3) \bar{\phi}_{\gamma\beta} \phi_{i}{}^{h} = 0\,,$

(4.4) ${}'\nabla_{j} L_{\gamma\alpha}{}^{h} - L_{\gamma}{}^{\varepsilon}{}_{j} L_{\varepsilon\alpha}{}^{h} - \frac{1}{4}(\widetilde{c}+3)(\bar{g}_{\gamma\alpha} \delta_{j}{}^{h} - \bar{\eta}_{\gamma} \bar{\eta}_{\alpha} \delta_{j}{}^{h} + \bar{\phi}_{\gamma\alpha} \phi_{j}{}^{h}) = 0\,,$

$$(4.5)\qquad {}''\nabla_\gamma L_{\beta\alpha}{}^h - {}''\nabla_\beta L_{\gamma\alpha}{}^h = 0\,,$$

$$(4.6)\qquad \overline{K}_{\gamma\beta\alpha}{}^\omega + \overline{Z}_{\gamma\beta\alpha}{}^\omega + \tfrac{1}{4}(\widetilde{c}+3)(\overline{g}_{\gamma\alpha}\overline{\delta}_\beta{}^\omega - \overline{g}_{\beta\alpha}\overline{\delta}_\gamma{}^\omega)$$
$$+\tfrac{1}{4}(\widetilde{c}-1)(\overline{\eta}_\beta\overline{\eta}_\alpha\overline{\delta}_\gamma{}^\omega - \overline{\eta}_\gamma\overline{\eta}_\alpha\overline{\delta}_\beta{}^\omega + \overline{g}_{\beta\alpha}\overline{\eta}_\gamma\overline{\xi}^\omega - \overline{g}_{\gamma\alpha}\overline{\eta}_\beta\overline{\xi}^\omega$$
$$+\overline{\phi}_{\gamma\alpha}\overline{\phi}_\beta{}^\omega - \overline{\phi}_{\beta\alpha}\overline{\phi}_\gamma{}^\omega + 2\overline{\phi}_{\gamma\beta}\overline{\phi}_\alpha{}^\omega) = 0\,.$$

From (3.27) and (4.4), it is easy to see that

$$(4.7)\qquad \overline{W}_{\gamma\alpha} = -\tfrac{1}{4}n(\widetilde{c}+3)(\overline{g}_{\gamma\alpha} - \overline{\eta}_\gamma\overline{\eta}_\alpha)\,,$$

$$(4.8)\qquad N = -\tfrac{1}{4}n(s-1)(\widetilde{c}+3)\,.$$

Also, by contraction of (4.6) in the indices γ and ω and owing to (4.7), we find

$$\overline{K}_{\beta\alpha} = \tfrac{1}{4}\{(n+s+1)\widetilde{c} + 3n + 3s - 5\}\overline{g}_{\beta\alpha}$$
$$- \tfrac{1}{4}\{(n+s+1)\widetilde{c} + 3n - s - 1\}\overline{\eta}_\beta\overline{\eta}_\alpha\,.$$

Furthermore, transvecting this with $\overline{g}^{\beta\alpha}$, we get

$$\overline{K} = \tfrac{1}{4}(s-1)\{(n+s+1)\widetilde{c} + 3n + 3s - 1\}\,,$$

which implies that

$$\overline{K}_{\beta\alpha} = \left(\frac{\overline{K}}{s-1} - 1\right)\overline{g}_{\beta\alpha} - \left(\frac{\overline{K}}{s-1} - s\right)\overline{\eta}_\beta\overline{\eta}_\alpha\,.$$

Hence, we have

THEOREM 4.1. *If* $(\widetilde{M},\widetilde{g})$ *is a fibred Sasakian space of constant* $\widetilde{\phi}$*-holomorphic sectional curvature* $\widetilde{c}$, *then*

(1) $\widetilde{c} \le -3$,

(2) *the base space* M *is of constant holomorphic sectional curvature* $\widetilde{c}+3$, *and*

(3) *each fibre* F (*with* $\dim F \ge 3$) *is an* $\overline{\eta}$*-Einstein space.*

In the case of $\widetilde{c} = -3$, from (4.8) and Theorem A we deduce

COROLLARY 4.2. *If* $(\widetilde{M},\widetilde{g})$ *is a fibred Sasakian space of constant* $\widetilde{\phi}$*-holomorphic sectional curvature* -3, *then*

(1) *the base space* M *is locally Euclidean*, *and*

(2) *each fibre* F (*with* $\dim F \ge 3$) *is a Sasakian space of constant* $\overline{\phi}$*-holomorphic sectional curvature* -3.

5. Fibred Sasakian spaces with vanishing contact Bochner curvature tensor. In this section, we consider a fibred Sasakian space $\widetilde{M}^m$ with vanishing contact Bochner curvature tensor. Then the curvature tensor of $\widetilde{M}$ is given by

$$\widetilde{K}_{KJI}{}^H = -\frac{1}{m+3}(\widetilde{K}_{KI}\widetilde{\delta}_J{}^H - \widetilde{K}_{JI}\widetilde{\delta}_K{}^H + \widetilde{g}_{KI}\widetilde{K}_J{}^H - \widetilde{g}_{JI}\widetilde{K}_K{}^H + \widetilde{S}_{KI}\widetilde{\phi}_J{}^H$$

$$
\begin{aligned}
&\qquad - \widetilde{S}_{JI}\widetilde{\phi}_K{}^H + \widetilde{\phi}_{KI}\widetilde{S}_J{}^H - \widetilde{\phi}_{JI}\widetilde{S}_K{}^H + 2\widetilde{S}_{KJ}\widetilde{\phi}_I{}^H + 2\widetilde{\phi}_{KJ}\widetilde{S}_I{}^H \\
&\qquad - \widetilde{K}_{KI}\widetilde{\eta}_J\widetilde{\xi}^H + \widetilde{K}_{JI}\widetilde{\eta}_K\widetilde{\xi}^H - \widetilde{\eta}_K\widetilde{\eta}_I\widetilde{K}_J{}^H + \widetilde{\eta}_J\widetilde{\eta}_I\widetilde{K}_K{}^H) \\
&- \frac{\widetilde{k}+m-1}{m+3}(\widetilde{\phi}_{KI}\widetilde{\phi}_J{}^H - \widetilde{\phi}_{JI}\widetilde{\phi}_K{}^H + 2\widetilde{\phi}_{KJ}\widetilde{\phi}_I{}^H) \\
&- \frac{\widetilde{k}-4}{m+3}(\widetilde{g}_{KI}\widetilde{\delta}_J{}^H - \widetilde{g}_{JI}\widetilde{\delta}_K{}^H) \\
&+ \frac{\widetilde{k}}{m+3}(\widetilde{g}_{KI}\widetilde{\eta}_J\widetilde{\xi}^H + \widetilde{\eta}_K\widetilde{\eta}_I\widetilde{\delta}_J{}^H - \widetilde{g}_{JI}\widetilde{\eta}_K\widetilde{\xi}^H - \widetilde{\eta}_J\widetilde{\eta}_I\widetilde{\delta}_K{}^H)\,,
\end{aligned}
$$

where we put $\widetilde{k} = \frac{\widetilde{K}+m-1}{m+1}$.

Transvecting the above equation with $B^K{}_D B^J{}_C B^I{}_B B_H{}^A$ and applying (3.6), (3.8)–(3.12) and (3.15), we see that the above equation is equivalent to the following equations:

$$
\begin{aligned}
(5.1)\quad & K_{kji}{}^h + \frac{1}{m+3}(K_{ki}\delta_j{}^h - K_{ji}\delta_k{}^h + g_{ki}K_j{}^h - g_{ji}K_k{}^h + S_{ki}\phi_j{}^h \\
&\qquad - S_{ji}\phi_k{}^h + \phi_{ki}S_j{}^h - \phi_{ji}S_k{}^h + 2S_{kj}\phi_i{}^h + 2\phi_{kj}S_i{}^h) \\
&- \frac{\widetilde{k}}{m+3}(g_{ki}\delta_j{}^h - g_{ji}\delta_k{}^h + \phi_{ki}\phi_j{}^h - \phi_{ji}\phi_k{}^h + 2\phi_{kj}\phi_i{}^h) \\
&- \frac{1}{m+3}(N_{ki}\delta_j{}^h - N_{ji}\delta_k{}^h + g_{ki}N_j{}^h - g_{ji}N_k{}^h \\
&\qquad + \phi_k{}^r N_{ri}\phi_j{}^h - \phi_j{}^r N_{ri}\phi_k{}^h + \phi_{ki}\phi_j{}^r N_r{}^h - \phi_{ji}\phi_k{}^r N_r{}^h \\
&\qquad + 2\phi_k{}^r N_{rj}\phi_i{}^h + 2\phi_{kj}\phi_i{}^r N_r{}^h) = 0\,,
\end{aligned}
$$

$$
(5.2)\quad Q_{\gamma i}\delta_j{}^h - g_{ji}Q_\gamma{}^h + \bar{\phi}_\gamma{}^\tau Q_{\tau i}\phi_j{}^h - \phi_{ji}\bar{\phi}_\gamma{}^\tau Q_\tau{}^h + 2\bar{\phi}_\gamma{}^\tau Q_{\tau j}\phi_i{}^h = 0\,,
$$

$$
\begin{aligned}
(5.3)\quad & L_\gamma{}^\varepsilon{}_i L_{\varepsilon\beta}{}^h - L_\beta{}^\varepsilon{}_i L_{\varepsilon\gamma}{}^h \\
&+ \frac{2}{m+3}[\overline{S}_{\gamma\beta}\phi_i{}^h + \bar{\phi}_{\gamma\beta}\{S_i{}^h - \phi_i{}^r N_r{}^h - (\widetilde{k}-2)\phi_i{}^h\}] = 0\,,
\end{aligned}
$$

$$
\begin{aligned}
(5.4)\quad & -{}'\nabla_k L_\alpha{}^\omega{}_j + {}'\nabla_j L_\alpha{}^\omega{}_k - L_\varepsilon{}^\omega{}_k L_\alpha{}^\varepsilon{}_j + L_\varepsilon{}^\omega{}_j L_\alpha{}^\varepsilon{}_k \\
&+ \frac{2}{m+3}[\{S_{kj} - \phi_k{}^r N_{rj} - (\widetilde{k}-2)\phi_{kj}\}\bar{\phi}_\alpha{}^\omega + \phi_{kj}\overline{S}_\alpha{}^\omega] = 0\,,
\end{aligned}
$$

$$
\begin{aligned}
(5.5)\quad & -{}'\nabla_j L_{\gamma\alpha}{}^h + L_\gamma{}^\varepsilon{}_j L_{\varepsilon\alpha}{}^h \\
&+ \frac{1}{m+3}\{\overline{K}_{\gamma\alpha}\delta_j{}^h + \overline{S}_{\gamma\alpha}\phi_j{}^h + (\overline{g}_{\gamma\alpha} - \overline{\eta}_\gamma\overline{\eta}_\alpha)(K_j{}^h - N_j{}^h) \\
&\qquad + \bar{\phi}_{\gamma\alpha}(S_j{}^h - \phi_j{}^r N_r{}^h)\} \\
&+ \frac{\widetilde{k}-m+n-1}{m+3}\overline{\eta}_\gamma\overline{\eta}_\alpha\delta_j{}^h - \frac{\widetilde{k}-2}{m+3}(\bar{\phi}_{\gamma\alpha}\phi_j{}^h + \overline{g}_{\gamma\alpha}\delta_j{}^h) = 0\,,
\end{aligned}
$$

$$
(5.6)\quad {}''\nabla_\gamma L_{\beta\alpha}{}^h - {}''\nabla_\beta L_{\gamma\alpha}{}^h
$$

$$+\frac{1}{m+3}\{(\overline{g}_{\gamma\alpha}-\overline{\eta}_\gamma\overline{\eta}_\alpha)Q_\beta{}^h-(\overline{g}_{\beta\alpha}-\overline{\eta}_\beta\overline{\eta}_\alpha)Q_\gamma{}^h$$
$$+\bar{\phi}_{\gamma\alpha}\phi_\beta{}^\varepsilon Q_\varepsilon{}^h-\bar{\phi}_{\beta\alpha}\phi_\gamma{}^\varepsilon Q_\varepsilon{}^h+2\bar{\phi}_{\gamma\beta}\phi_\alpha{}^\varepsilon Q_\varepsilon{}^h\}=0\,,$$

(5.7)
$$\overline{K}_{\gamma\beta\alpha}{}^\omega+\overline{Z}_{\gamma\beta\alpha}{}^\omega$$
$$+\frac{1}{m+3}(\overline{K}_{\gamma\alpha}\bar{\delta}_\beta{}^\omega-\overline{K}_{\beta\alpha}\bar{\delta}_\gamma{}^\omega+\overline{g}_{\gamma\alpha}\overline{K}_\beta{}^\omega-\overline{g}_{\beta\alpha}\overline{K}_\gamma{}^\omega+\overline{S}_{\gamma\alpha}\bar{\phi}_\beta{}^\omega$$
$$-\overline{S}_{\beta\alpha}\bar{\phi}_\gamma{}^\omega+\bar{\phi}_{\gamma\alpha}\overline{S}_\beta{}^\omega-\bar{\phi}_{\beta\alpha}\overline{S}_\gamma{}^\omega+2\overline{S}_{\gamma\beta}\bar{\phi}_\alpha{}^\omega+2\bar{\phi}_{\gamma\beta}\overline{S}_\alpha{}^\omega$$
$$-\overline{K}_{\gamma\alpha}\overline{\eta}_\beta\bar{\xi}^\omega+\overline{K}_{\beta\alpha}\overline{\eta}_\gamma\bar{\xi}^\omega-\overline{\eta}_\gamma\overline{\eta}_\alpha\overline{K}_\beta{}^\omega+\overline{\eta}_\beta\overline{\eta}_\alpha\overline{K}_\gamma{}^\omega)$$
$$+\frac{\widetilde{k}+n}{m+3}(\overline{\eta}_\gamma\overline{\eta}_\alpha\bar{\delta}_\beta{}^\omega-\overline{\eta}_\beta\overline{\eta}_\alpha\bar{\delta}_\gamma{}^\omega+\overline{g}_{\gamma\alpha}\overline{\eta}_\beta\bar{\xi}^\omega-\overline{g}_{\beta\alpha}\overline{\eta}_\gamma\bar{\xi}^\omega)$$
$$-\frac{\widetilde{k}+m-1}{m+3}(\bar{\phi}_{\gamma\alpha}\bar{\phi}_\beta{}^\omega-\bar{\phi}_{\beta\alpha}\bar{\phi}_\gamma{}^\omega+2\bar{\phi}_{\gamma\beta}\bar{\phi}_\alpha{}^\omega)$$
$$-\frac{\widetilde{k}-4}{m+3}(\overline{g}_{\gamma\alpha}\bar{\delta}_\beta{}^\omega-\overline{g}_{\beta\alpha}\bar{\delta}_\gamma{}^\omega)=0\,.$$

Contracting (5.5) with $\overline{g}^{\gamma\alpha}$, we easily get

(5.8)
$$(s-1)K_j{}^h+(n+4)N_j{}^h+\{\overline{K}-(s-1)(\widetilde{k}-1)\}\delta_j{}^h=0\,,$$

and consequently,

(5.9)
$$(s+1)(s-1)K+(n+2)\{n\overline{K}+(n+2s+2)N+n(s-1)\}=0\,.$$

Substituting this into (5.1), we have

(5.10)
$$K_{kji}{}^h=-\frac{1}{n+s+3}(K_{ki}\delta_j{}^h-K_{ji}\delta_k{}^h+g_{ki}K_j{}^h$$
$$+g_{ji}K_k{}^h+S_{ki}\phi_j{}^h-S_{ji}\phi_k{}^h+\phi_{ki}S_j{}^h$$
$$-\phi_{ji}S_k{}^h+2S_{kj}\phi_i{}^h+2\phi_{kj}S_i{}^h)$$
$$+\frac{(n-s+1)K}{n(n+2)(n+s+3)}$$
$$\times(g_{ki}\delta_j{}^h-g_{ji}\delta_k{}^h+\phi_{ki}\phi_j{}^h-\phi_{ji}\phi_k{}^h+2\phi_{kj}\phi_i{}^h)$$
$$+\frac{1}{n+s+3}\bigg\{\bigg(N_{ki}-\frac{N}{n}g_{ki}\bigg)\delta_j{}^h-\bigg(N_{ji}-\frac{N}{n}g_{ji}\bigg)\delta_k{}^h$$
$$+g_{ki}\bigg(N_j{}^h-\frac{N}{n}\delta_j{}^h\bigg)-g_{ji}\bigg(N_k{}^h-\frac{N}{n}\delta_k{}^h\bigg)$$
$$+\phi_k{}^s\bigg(N_{si}-\frac{N}{n}g_{si}\bigg)\phi_j{}^h-\phi_j{}^s\bigg(N_{si}-\frac{N}{n}g_{si}\bigg)\phi_k{}^h$$
$$+\phi_{ki}\phi_j{}^s\bigg(N_s{}^h-\frac{N}{n}\delta_s{}^h\bigg)-\phi_{ji}\phi_k{}^s\bigg(N_s{}^h-\frac{N}{n}\delta_s{}^h\bigg)$$

$$+ 2\phi_k{}^s\left(N_{sj} - \frac{N}{n}g_{sj}\right)\phi_i{}^h + 2\phi_{kj}\phi_i{}^s\left(N_s{}^h - \frac{N}{n}\delta_s{}^h\right)\Big\}.$$

By contraction of (5.10) in k and h, we obtain

$$N_{ji} - \frac{N}{n}g_{ji} = -\frac{s-1}{n+4}\left(K_{ji} - \frac{K}{n}g_{ji}\right). \tag{5.11}$$

Substituting (5.11) into (5.10), we have

$$\begin{aligned} K_{kji}{}^h = &- \frac{1}{n+4}(K_{ki}\delta_j{}^h - K_{ji}\delta_k{}^h + g_{ki}K_j{}^h - g_{ji}K_k{}^h + S_{ki}\phi_j{}^h - S_{ji}\phi_k{}^h \\ &\qquad + \phi_{ki}S_j{}^h - \phi_{ji}S_k{}^h + 2S_{kj}\phi_i{}^h + 2\phi_{kj}S_i{}^h) \\ &+ \frac{K}{(n+2)(n+4)}(g_{ki}\delta_j{}^h - g_{ji}\delta_k{}^h + \phi_{ki}\phi_j{}^h - \phi_{ji}\phi_k{}^h + 2\phi_{kj}\phi_i{}^h). \end{aligned}$$

Hence we get

LEMMA 5.1. *Let $\widetilde{M}$ be a fibred Sasakian space. If the contact Bochner curvature tensor of $\widetilde{M}$ vanishes, then the base space M is a Kählerian space with vanishing Bochner curvature tensor.*

Next, by contraction of (5.2) in h and j, we find

$$Q_{\gamma i} = 0.$$

Substituting this into (5.6), we get

$$''\nabla_\gamma L_{\beta\alpha}{}^h - {}''\nabla_\beta L_{\gamma\alpha}{}^h = 0. \tag{5.12}$$

By transvection of (5.7) in γ and ω, it is clear that

$$\begin{aligned} \overline{K}_{\beta\alpha} = &\frac{1}{n(s-1)}\{n\overline{K} + (n+s+3)N - n(s-1)\}\overline{g}_{\beta\alpha} \\ &- \frac{1}{n(s-1)}\{n\overline{K} + (n+s+3)N - ns(s-1)\}\overline{\eta}_\beta\overline{\eta}_\alpha \\ &- \frac{1}{n}(n+s+3)\overline{W}_{\beta\alpha}. \end{aligned} \tag{5.13}$$

Applying $''\nabla^\alpha$ to (5.13) and making use of (2.3), (3.22), (3.25), (3.26) and (5.12), we obtain

$$''\nabla_\beta\{n\overline{K} + (n+s+3)N\} = 0, \tag{5.14}$$

provided $s > 3$. In the sequel, we assume that $s > 3$.

Substituting (5.13) into (5.7), we have

$$\begin{aligned} &\overline{K}_{\gamma\beta\alpha}{}^\omega + \overline{Z}_{\gamma\beta\alpha}{}^\omega \\ &\quad + \frac{n\overline{K} + (n+2s+2)N + n(s-1)}{n(s+1)(s-1)}(\overline{g}_{\gamma\alpha}\overline{\delta}_\beta{}^\omega - \overline{g}_{\beta\alpha}\overline{\delta}_\gamma{}^\omega) \end{aligned} \tag{5.15}$$

$$+\frac{n\overline{K}+(n+2s+2)N-ns(s-1)}{n(s+1)(s-1)}\overline{H}_{\gamma\beta\alpha}{}^{\omega}-\frac{1}{n}\bar{I}_{\gamma\beta\alpha}{}^{\omega}=0\,,$$

where we put

$$\begin{aligned}\overline{H}_{\gamma\beta\alpha}{}^{\omega}&=\overline{\eta}_\beta\overline{\eta}_\alpha\overline{\delta}_\gamma{}^\omega-\overline{\eta}_\gamma\overline{\eta}_\alpha\overline{\delta}_\beta{}^\omega+\overline{g}_{\beta\alpha}\overline{\eta}_\gamma\overline{\xi}^\omega-\overline{g}_{\gamma\alpha}\overline{\eta}_\beta\overline{\xi}^\omega\\&\quad+\bar{\phi}_{\gamma\alpha}\bar{\phi}_\beta{}^\omega-\bar{\phi}_{\beta\alpha}\bar{\phi}_\gamma{}^\omega+2\bar{\phi}_{\gamma\beta}\bar{\phi}_\alpha{}^\omega,\\ \bar{I}_{\gamma\beta\alpha}{}^{\omega}&=\overline{W}_{\gamma\alpha}\overline{\delta}_\beta{}^\omega-\overline{W}_{\beta\alpha}\overline{\delta}_\gamma{}^\omega+\overline{g}_{\gamma\alpha}\overline{W}_\beta{}^\omega-\overline{g}_{\beta\alpha}\overline{W}_\gamma{}^\omega+\bar{\phi}_\gamma{}^\tau\overline{W}_{\tau\alpha}\bar{\phi}_\beta{}^\omega\\&\quad-\bar{\phi}_\beta{}^\tau\overline{W}_{\tau\alpha}\bar{\phi}_\gamma{}^\omega+\bar{\phi}_{\gamma\alpha}\bar{\phi}_\beta{}^\tau\overline{W}_\tau{}^\omega-\bar{\phi}_{\beta\alpha}\bar{\phi}_\gamma{}^\tau\overline{W}_\tau{}^\omega+2\bar{\phi}_\gamma{}^\tau\overline{W}_{\tau\beta}\bar{\phi}_\alpha{}^\omega\\&\quad+2\bar{\phi}_{\gamma\beta}\bar{\phi}_\alpha{}^\tau\overline{W}_\tau{}^\omega-\overline{W}_{\gamma\alpha}\overline{\eta}_\beta\overline{\xi}^\omega+\overline{W}_{\beta\alpha}\overline{\eta}_\gamma\overline{\xi}^\omega\\&\quad-\overline{\eta}_\gamma\overline{\eta}_\alpha\overline{W}_\beta{}^\omega+\overline{\eta}_\beta\overline{\eta}_\alpha\overline{W}_\gamma{}^\omega\,.\end{aligned}$$

Applying $''\nabla_\delta$ to (5.15) and using (5.14), we find

$$\begin{aligned}(5.16)\quad &''\nabla_\delta\overline{K}_{\gamma\beta\alpha}{}^\omega+''\nabla_\delta\overline{Z}_{\gamma\beta\alpha}{}^\omega\\&+\frac{1}{n(s+1)}(\overline{g}_{\gamma\alpha}\overline{\delta}_\beta{}^\omega-\overline{g}_{\beta\alpha}\overline{\delta}_\gamma{}^\omega+\overline{H}_{\gamma\beta\alpha}{}^\omega)''\nabla_\delta N\\&+\frac{n\overline{K}+(n+2s+2)N-ns(s-1)}{n(s+1)(s-1)}''\nabla_\delta\overline{H}_{\gamma\beta\alpha}{}^\omega-\frac{1}{n}''\nabla_\delta\bar{I}_{\gamma\beta\alpha}{}^\omega=0\,.\end{aligned}$$

Furthermore, by contraction of (5.16) in δ and ω, we have

$$\begin{aligned}(5.17)\quad &''\nabla_\gamma\overline{K}_{\beta\alpha}-''\nabla_\beta\overline{K}_{\gamma\alpha}\\&-\frac{s-1}{2n(s+1)}\{(\overline{g}_{\gamma\alpha}-\overline{\eta}_\gamma\overline{\eta}_\alpha)\overline{\delta}_\beta{}^\varepsilon-(\overline{g}_{\beta\alpha}-\overline{\eta}_\beta\overline{\eta}_\alpha)\overline{\delta}_\gamma{}^\varepsilon\\&\qquad+\bar{\phi}_{\gamma\alpha}\bar{\phi}_\beta{}^\varepsilon-\bar{\phi}_{\beta\alpha}\bar{\phi}_\gamma{}^\varepsilon+2\bar{\phi}_{\gamma\beta}\bar{\phi}_\alpha{}^\varepsilon\}''\nabla_\varepsilon N\\&+\frac{n\overline{K}+(n+s+3)N-ns(s-1)}{n(s+1)}(\bar{\phi}_{\gamma\alpha}\overline{\eta}_\beta-\bar{\phi}_{\beta\alpha}\overline{\eta}_\gamma+2\bar{\phi}_{\gamma\beta}\overline{\eta}_\alpha)\\&+\frac{1}{n}\{(n+1)(''\nabla_\gamma\overline{W}_{\beta\alpha}-''\nabla_\beta\overline{W}_{\gamma\alpha})\\&\quad+\bar{\phi}_\gamma{}^\varepsilon\bar{\phi}_\beta{}^\tau(''\nabla_\varepsilon\overline{W}_{\tau\alpha}-''\nabla_\tau\overline{W}_{\varepsilon\alpha})\\&\quad-2\bar{\phi}_\alpha{}^\varepsilon\bar{\phi}_\gamma{}^\tau{}''\nabla_\varepsilon\overline{W}_{\tau\beta}-s\overline{\eta}_\beta\bar{\phi}_\gamma{}^\tau\overline{W}_{\tau\alpha}+(s+2)\overline{\eta}_\gamma\bar{\phi}_\beta{}^\tau\overline{W}_{\tau\alpha}\\&\quad-2(s+1)\overline{\eta}_\alpha\bar{\phi}_\gamma{}^\tau\overline{W}_{\tau\beta}\}=0\,,\end{aligned}$$

where we have used (3.26), (3.38), (3.42), (5.12) and Bianchi's identity.

Also, by interchanging indices as $\delta\to\gamma\to\beta$ in (5.16) and adding all together, owing to (3.42), (5.12) and Bianchi's identity, we obtain

$$\begin{aligned}(s-1)\{&(\overline{g}_{\gamma\alpha}\overline{\delta}_\beta{}^\omega-\overline{g}_{\beta\alpha}\overline{\delta}_\gamma{}^\omega+\overline{H}_{\gamma\beta\alpha}{}^\omega)''\nabla_\delta N\\&+(\overline{g}_{\beta\alpha}\overline{\delta}_\delta{}^\omega-\overline{g}_{\delta\alpha}\overline{\delta}_\beta{}^\omega+\overline{H}_{\beta\delta\alpha}{}^\omega)''\nabla_\gamma N\\&+(\overline{g}_{\delta\alpha}\overline{\delta}_\gamma{}^\omega-\overline{g}_{\gamma\alpha}\overline{\delta}_\delta{}^\omega+\overline{H}_{\delta\gamma\alpha}{}^\omega)''\nabla_\beta N\}\end{aligned}$$

$$+\{n\overline{K}+(n+2s+2)N-ns(s-1)\}$$
$$\times({}''\nabla_\delta \overline{H}_{\gamma\beta\alpha}{}^\omega + {}''\nabla_\gamma \overline{H}_{\beta\delta\alpha}{}^\omega + {}''\nabla_\beta \overline{H}_{\delta\gamma\alpha}{}^\omega)$$
$$-(s+1)(s-1)({}''\nabla_\delta \bar{I}_{\gamma\beta\alpha}{}^\omega + {}''\nabla_\gamma \bar{I}_{\beta\delta\alpha}{}^\omega + {}''\nabla_\beta \bar{I}_{\delta\gamma\alpha}{}^\omega) = 0\,.$$

By contracting in δ and ω, from (3.38), (3.42), (5.12) and (5.14) we find

$$\begin{aligned}(5.18)\qquad &(s+2)({}''\nabla_\gamma \overline{W}_{\beta\alpha} - {}''\nabla_\beta \overline{W}_{\gamma\alpha}) + s\overline{\eta}_\beta \bar{\phi}_\gamma{}^\tau \overline{W}_{\tau\alpha}\\ &-(s+2)\overline{\eta}_\gamma \bar{\phi}_\beta{}^\tau \overline{W}_{\tau\alpha} + 2(s+1)\overline{\eta}_\alpha \bar{\phi}_\gamma{}^\tau \overline{W}_{\tau\beta}\\ &-\bar{\phi}_\gamma{}^\varepsilon \bar{\phi}_\beta{}^\tau({}''\nabla_\varepsilon \overline{W}_{\tau\alpha} - {}''\nabla_\tau \overline{W}_{\varepsilon\alpha}) + 2\bar{\phi}_\alpha{}^\varepsilon \bar{\phi}_\gamma{}^\tau {}''\nabla_\varepsilon \overline{W}_{\tau\beta}\\ &-\tfrac{1}{2}\{(\overline{g}_{\gamma\alpha} - \overline{\eta}_\gamma\overline{\eta}_\alpha)\bar{\delta}_\beta{}^\varepsilon - (\overline{g}_{\beta\alpha} - \overline{\eta}_\beta\overline{\eta}_\alpha)\bar{\delta}_\delta{}^\varepsilon\\ &\qquad -\bar{\phi}_{\gamma\alpha}\bar{\phi}_\beta{}^\varepsilon + \bar{\phi}_{\beta\alpha}\bar{\phi}_\gamma{}^\varepsilon - 2\bar{\phi}_{\gamma\beta}\bar{\phi}_\alpha{}^\varepsilon\}{}''\nabla_\varepsilon N = 0\,.\end{aligned}$$

If we transvect (5.18) with $\overline{g}^{\gamma\alpha}$ and use (3.38), (3.42) and (5.12), then we get

$$(5.19)\qquad {}''\nabla_\beta N = 0\,.$$

Substituting (5.19) into (5.18), we have

$$\begin{aligned}&(s+2)({}''\nabla_\gamma \overline{W}_{\beta\alpha} - {}''\nabla_\beta \overline{W}_{\gamma\alpha}) + s\overline{\eta}_\beta \bar{\phi}_\gamma{}^\tau \overline{W}_{\tau\alpha}\\ &-(s+2)\overline{\eta}_\gamma \bar{\phi}_\beta{}^\tau \overline{W}_{\tau\alpha} + 2(s+1)\overline{\eta}_\alpha \bar{\phi}_\gamma{}^\tau \overline{W}_{\tau\beta}\\ &-\bar{\phi}_\gamma{}^\varepsilon \bar{\phi}_\beta{}^\tau({}''\nabla_\varepsilon \overline{W}_{\tau\alpha} - {}''\nabla_\tau \overline{W}_{\varepsilon\alpha}) + 2\bar{\phi}_\alpha{}^\varepsilon \bar{\phi}_\gamma{}^\tau {}''\nabla_\varepsilon \overline{W}_{\tau\beta} = 0\,,\end{aligned}$$

which implies that

$$(5.20)\qquad {}''\nabla_\gamma \overline{W}_{\beta\alpha} = \overline{\eta}_\beta \bar{\phi}_\alpha{}^\tau \overline{W}_{\tau\gamma} + \overline{\eta}_\alpha \bar{\phi}_\beta{}^\tau \overline{W}_{\tau\gamma}\,.$$

Because of (2.2), (2.3), (5.19) and (5.20), equation (5.17) can be rewritten as follows:

$$\begin{aligned}&{}''\nabla_\gamma \overline{K}_{\beta\alpha} - {}''\nabla_\beta \overline{K}_{\gamma\alpha}\\ &+\frac{n\overline{K}+(n+s+3)N-ns(s-1)}{n(s+1)}(\bar{\phi}_{\gamma\alpha}\overline{\eta}_\beta - \bar{\phi}_{\beta\alpha}\overline{\eta}_\gamma + 2\bar{\phi}_{\gamma\beta}\overline{\eta}_\alpha)\\ &-\frac{1}{n}(n+s+3)(\overline{\eta}_\beta \bar{\phi}_\gamma{}^\tau \overline{W}_{\tau\alpha} - \overline{\eta}_\gamma \bar{\phi}_\beta{}^\tau \overline{W}_{\tau\alpha} + 2\overline{\eta}_\alpha \bar{\phi}_\gamma{}^\tau \overline{W}_{\tau\beta}) = 0\,.\end{aligned}$$

Applying $\bar{\xi}^\gamma$ to this, owing to (2.2), (2.3) and (5.13), we find

$$(5.21)\qquad n\overline{K} + (n+s+3)N - ns(s-1) = 0\,,$$

and consequently, from (5.19) we see that the scalar curvature $\overline{K}$ is constant on each fibre F.

By (5.9) and (5.21), we get

$$(5.22)\qquad (n+2)'\nabla_i N + (s+1)'\nabla_i K = 0\,.$$

Also, substituting (5.3) into (5.4), we obtain

$$(5.23)\qquad {}'\nabla_k L_\alpha{}^\omega{}_j - {}'\nabla_j L_\alpha{}^\omega{}_k = 0\,.$$

Applying $'\nabla^j$ to (5.11) and using (3.27) and (5.23), we have

$$(n+4)'\nabla_i N + (s-1)'\nabla_i K = 0\,,$$

from which, together with (5.22), we find

$$'\nabla_i K = 0\,, \tag{5.24}$$

that is, the scalar curvature K is constant on the base space M. Since N is a nonnegative constant, from (5.9), (5.19), (5.21), (5.22) and (5.24) we find

LEMMA 5.2. *Let $\widetilde{M}$ be a fibred Sasakian space and* $\dim F > 3$. *If the contact Bochner curvature tensor of $\widetilde{M}$ vanishes, then the scalar curvatures K and $\overline{K}$ are constant. Moreover, $K \le -n(n+2)$ and $\overline{K} \le s(s-1)$, where equality holds when $\widetilde{M}$ has conformal fibres.*

From Lemmas 5.1, 5.2 and Theorem C, we have

THEOREM 5.3. *Let $\widetilde{M}$ be a fibred Sasakian space and* $\dim F > 3$. *If the contact Bochner curvature tensor of $\widetilde{M}$ vanishes, then either*

(1) *M is a space of constant holomorphic sectional curvature c (≤ -4), or*

(2) *M is locally a product of two spaces of constant holomorphic sectional curvatures c and $-c$, where $|c| > 4$.*

Let $M_1^p(c)$ and $M_2^{n-p}(-c)$ be a space of constant holomorphic sectional curvature c of dimension p and of constant holomorphic sectional curvature $-c$ of dimension $n-p$, respectively. By Theorem 5.3, the base space M^n is locally a product $M_1^p(c) \times M_2^{n-p}(-c)$; if $p = 0$ or $p = n$, then M is considered to be a space of constant holomorphic sectional curvature $-c$ or c, respectively.

Remark. By Lemma 5.2, we find $|c| \ge 4n/|n-2p|$ if $n \ne 2p$.

We now consider the fibre F of a fibred Sasakian space with vanishing contact Bochner curvature tensor. It is easy to see from (5.13) and (5.20) that

$$''\nabla_\gamma \overline{K}_{\beta\alpha} = -\overline{\eta}_\beta \overline{S}_{\gamma\alpha} - \overline{\eta}_\alpha \overline{S}_{\gamma\beta} + (s-1)(\overline{\eta}_\beta \bar{\phi}_{\gamma\alpha} + \overline{\eta}_\alpha \bar{\phi}_{\gamma\beta})\,.$$

Thus, we find

PROPOSITION 5.4. *Let $\widetilde{M}$ be a fibred Sasakian space. If the contact Bochner curvature tensor of $\widetilde{M}$ vanishes, then the Ricci tensor of each fibre F ($s > 3$) is $\overline{\eta}$-parallel.*

Denoting by $\overline{B}_{\gamma\beta\alpha}{}^\omega$ the contact Bochner curvature tensor of each fibre F, from (5.15) and (5.21) we get

$$\overline{B}_{\gamma\beta\alpha}{}^\omega = -\overline{Z}_{\gamma\beta\alpha}{}^\omega + \frac{N}{(s+1)(s+3)}(\overline{g}_{\gamma\alpha}\bar{\delta}_\beta{}^\omega - \overline{g}_{\beta\alpha}\bar{\delta}_\gamma{}^\omega + \overline{H}_{\gamma\beta\alpha}{}^\omega) - \frac{1}{s+1}\bar{I}_{\gamma\beta\alpha}{}^\omega\,,$$

from which together with (3.25), (3.37), (3.38), (3.44) and (3.45), we have

$$|\overline{B}|^2 = 2|N|^2 - \frac{16(s-1)}{(s+1)^2}|\overline{W}|^2 + \frac{8(s^2+4s+11)}{(s+1)^2(s+3)^2}N^2 ,$$

where we put $|\overline{B}|^2 = \overline{B}_{\gamma\beta\alpha\omega}\overline{B}^{\gamma\beta\alpha\omega}$, $|N|^2 = N_{ji}N^{ji}$ and $|\overline{W}|^2 = \overline{W}_{\beta\alpha}\overline{W}^{\beta\alpha}$.

We put $|\overline{\mathrm{Ric}}|^2 = \overline{K}_{\beta\alpha}\overline{K}^{\beta\alpha}$. From (5.11), (5.13), (5.21) and Theorem 5.3, we obtain

LEMMA 5.5. *Let $\widetilde{M}$ be a fibred Sasakian space with vanishing contact Bochner curvature tensor. Then*

$$\begin{aligned}|\overline{\mathrm{Ric}}|^2 \leq\ & \frac{(s+1)^2}{8(s-1)}\left\{\frac{1}{n} + \frac{4(s^2+4s+11)}{(s+1)^2(s+3)^2}\right\}\overline{K}^2 \\ & - \frac{s(s+1)^2}{4}\left\{\frac{1}{n} - \frac{4(s^3+6s^2-5s-18)}{s(s+1)^2(s+3)^2}\right\}\left\{\overline{K} - \frac{s(s-1)}{2}\right\} \\ & + \frac{1}{128}p\left(1-\frac{p}{n}\right)\left(1+\frac{s+3}{n}\right)^2(s-1)(s+1)^2c^2 .\end{aligned}$$

Equality holds if and only if the contact Bochner curvature tensor $\overline{B}_{\gamma\beta\alpha}{}^{\omega}$ of each fibre F vanishes.

By (5.19), (5.21), Lemma 5.5, and Theorems D and E, we have

THEOREM 5.6. *Let $\widetilde{M}$ be a fibred Sasakian space with vanishing contact Bochner curvature tensor and* $\dim F \geq 7$. *If*

$$\begin{aligned}& \frac{(s+1)^2}{8(s-1)}\left\{\frac{1}{n} + \frac{4(s^2+4s+11)}{(s+1)^2(s+3)^2}\right\}\overline{K}^2 \\ & - \frac{s(s+1)^2}{4}\left\{\frac{1}{n} - \frac{4(s^3+6s^2-5s-18)}{s(s+1)^2(s+3)^2}\right\}\left\{\overline{K} - \frac{s(s-1)}{2}\right\} \\ & + \frac{1}{128}p\left(1-\frac{p}{n}\right)\left(1+\frac{s+3}{n}\right)^2(s-1)(s+1)^2c^2 \\ & \leq |\overline{\mathrm{Ric}}|^2 < \frac{s^3-5s^2+7s+29}{(s+1)^2(s-5)^2}\overline{K}^2 - \frac{2(s^4-10s^3+58s+79)}{(s+1)^2(s-5)^2}\overline{K} \\ & + \frac{(s-1)^2(s^4-7s^3+s^2+47s+54)}{(s+1)^2(s-5)^2} ,\end{aligned}$$

then each fibre F is a space of constant $\bar{\phi}$-holomorphic sectional curvature.

THEOREM 5.7. *Let $\widetilde{M}$ be a fibred Sasakian space with vanishing contact Bochner curvature tensor and* $\dim F = 5$. *If $\overline{K} \neq -4$ and*

$$|\overline{\mathrm{Ric}}|^2 \geq \frac{9}{8}\left(\frac{1}{n}+\frac{7}{72}\right)\overline{K}^2 - 45\left(\frac{1}{n}-\frac{29}{360}\right)(\overline{K}-10) + \frac{9}{8}p\left(1-\frac{p}{n}\right)\left(1+\frac{8}{n}\right)^2 c^2,$$

then each fibre F is a space of constant $\bar{\phi}$-holomorphic sectional curvature.

From Lemma 5.2 and Theorem B, we find

COROLLARY 5.8. *If $\widetilde{M}$ is a fibred Sasakian space with vanishing contact Bochner curvature tensor and conformal fibres of dimension $s > 3$, then the base space M is of constant holomorphic sectional curvature -4 and each fibre F is of constant $\bar{\phi}$-holomorphic sectional curvature* 1.

Remark. By (5.13) and (5.20), $|\overline{W}|$ and $|\overline{\mathrm{Ric}}|$ are constant on each fibre F (with $s > 3$).

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