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MORE TOPOLOGICAL CARDINAL INEQUALITIES

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A new topological cardinal invariant is defined; it may be considered as a weaker form of the Lindelöf degree.

NOTATIONS. If X is a set, $|X|$ denotes the cardinality of X. For any cardinal number κ, κ^+ stands for the cardinal successor of κ and κ is the set of all ordinal numbers smaller than κ. If X is a set and κ is a cardinal number, $[X]^{\le\kappa}$ is the set of all subsets of X whose cardinality is not greater than κ.

Let X be a Hausdorff space. If $A \subset X$, $\overline{A}$ denotes the closure of A. By $L(X)$, $\omega L(X)$, $c(X)$, $\chi(X)$, $\psi(X)$ we denote the Lindelöf degree, the weak Lindelöf degree, the cellularity, the character and pseudo-character of X respectively. X is a *Urysohn space* if any two distinct points have disjoint closed neighborhoods.

DEFINITION. $\omega L_c(X)$ is the infimum of all infinite cardinal numbers α such that for every closed subset F of X and every open (in X) cover of F, say $\mathcal{C}$, there is a $\mathcal{C}_* \subset \mathcal{C}$, $|\mathcal{C}_*| \le \alpha$, such that $\overline{\bigcup \mathcal{C}_*} \supset F$.

The following inequalities are immediate.

1) $\omega L(X) \le \omega L_c(X) \le L(X)$;
2) $\omega L(X) = \omega L_c(X)$ if X is a normal space;
3) $\omega L_c(X) \le c(X)$;
4) $\omega L_c(X) = \aleph_0$ and $L(X) \ge \aleph_1$ if X is a non-Lindelöf S-space.

THEOREM 1. *If X is a Urysohn space, then* $|X| \le 2^{\omega L_c(X)\chi(X)}$.

Proof. Bella and Cammaroto [2] introduced the notion of θ-closure of a subset A of X and proved that

$$|[A]_\theta| \le |A|^{\chi(X)}$$

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(where $[A]_\theta$ denotes the θ-closure of A in X). According to these authors, if $B \subset X$ and $y \in X$, y is said to be a θ-*adherent point* of B if every closed neighborhood of y meets B; B is said to be θ-*closed* if every θ-adherent point of B belongs to B, and $[B]_\theta$ is the smallest θ-closed set which contains B.

For each $x \in X$ let $\mathcal{V}_x$ denote a fundamental system of open neighborhoods of x, with $|\mathcal{V}_x| \le \chi(X)$. Define $\kappa = \omega L_c(X)\chi(X)$ and construct an increasing family $(A_\alpha)_{\alpha<\kappa^+}$ of θ-closed sets such that

1) $|A_\alpha| \le 2^\kappa$, $\forall \alpha < \kappa^+$ and A_0 is any fixed θ-closed set;

2) $A_\beta = [\bigsqcup_{\alpha<\beta} A_\alpha]_\theta$ if β is a limit ordinal;

3) for each $\alpha < \kappa^+$, if $\mathcal{C} \in [\bigsqcup\{\mathcal{V}_x \mid x \in A_\alpha\}]^{\le \omega L_c(X)}$ and $X \setminus \overline{\bigsqcup \mathcal{C}} \ne \emptyset$, then $A_{\alpha+1} \setminus \overline{\bigsqcup \mathcal{C}} \ne \emptyset$.

The proof uses the classical Pol–Shapirovskiĭ's technique. Finally, put $A = \bigsqcup_{\alpha<\kappa^+} A_\alpha$, hence $|A| \le 2^\kappa$. We prove that A is θ-closed and it is equal to X. Indeed, if $z \in X$ is θ-adherent to A, then $\overline{V} \cap A \ne \emptyset$, $\forall V \in \mathcal{V}_z$. For each $V \in \mathcal{V}_z$ fix a smallest $\alpha_V < \kappa^+$ so that $\overline{V} \cap A_{\alpha_V} \ne \emptyset$. If $\beta = \sup\{\alpha_V \mid V \in \mathcal{V}_z\}$, then $\beta < \kappa^+$ and $\overline{V} \cap A_\beta \ne \emptyset$, $\forall V \in \mathcal{V}_z$, which implies $z \in A_\beta$ (because A_β is θ-closed). If $y \in X \setminus A$, there is a $W \in \mathcal{V}_y$ so that $\overline{W} \cap A = \emptyset$ (because A is θ-closed); for each $x \in A$, fix $V_x \in \mathcal{V}_x$ so that $V_x \subset X \setminus \overline{W}$. Then $\{V_x \mid x \in A\}$ is an open (in X) cover of the closed set A and there is an $A_* \subset A$ with $|A_*| \le \omega L_c(X)$ and $\overline{\bigsqcup_{x\in A_*} V_x} \supset A$. But $A_* \subset A_\beta$ for a suitable $\beta < \kappa^+$, hence $A_{\beta+1} \setminus \overline{\bigsqcup_{x\in A_*} V_x}$ would be non-empty (contradiction). ∎

COROLLARY ([3], p. 38). *If X is a normal space, then* $|X| \le 2^{\omega L_c(X)\chi(X)}$.

THEOREM 2. *If X is a regular space with a dense subset of isolated points, then*

$$|X| \le 2^{\omega L_c(X)\psi(X)t(X)}$$

where $t(X)$ denotes the tightness of X.

As a matter of fact a more general result may be proved. If X is a Hausdorff space let $\psi_c(X)$ denote the smallest infinite cardinal number α such that for each $x \in X$, there is a collection $\mathcal{V}$ of *closed* neighborhoods, with $|\mathcal{V}| \le \alpha$, whose intersection is $\{x\}$. (If X is regular, then $\psi(X) = \psi_c(X)$.)

THEOREM 3. *If X is a Hausdorff space with a dense subset of isolated points, then* $|X| \le 2^{\omega L_c(X)\psi_c(X)t(X)}$.

Proof. First of all, for each $x \in X$, fix $\mathcal{V}_x$, a collection of open neighborhoods of x, such that $|\mathcal{V}_x| \le \psi_c(X)$ and $\bigcap\{\overline{V} \mid V \in \mathcal{V}_x\} = \{x\}$. If $A \subset X$, then $|\overline{A}| \le |A|^{t(X)} \cdot 2^{\psi_c(X)t(X)} \le |A|^{t(X)\psi_c(X)}$. Define $\kappa = \omega L_c(X)\psi_c(X)t(X)$.

We now construct an increasing family $(A_\alpha)_{\alpha<\kappa^+}$ of closed subsets of X satisfying

1) $|A_\alpha| \le 2^\kappa,\ \forall\alpha<\kappa^+$;

2) $A_\beta = \overline{\bigsqcup_{\alpha<\beta} A_\alpha}$ if β is a limit ordinal;

3) for any $\alpha<\kappa^+$ and $\mathcal{C} \in [\bigsqcup\{\mathcal{V}_x \mid x \in A_\alpha\}]^{\le\omega L_c(X)}$, if $X \setminus \overline{\bigsqcup\mathcal{C}} \ne \emptyset$, then $A_{\alpha+1} \setminus \overline{\bigsqcup\mathcal{C}} \ne \emptyset$.

Once again the proof uses the Pol–Shapirovskiĭ's technique. Finally, define $A = \bigsqcup_{\alpha<\kappa^+} A_\alpha$, which is closed (because $t(X) \le \kappa$) and $|A| \le 2^\kappa$. We prove that $X = A$; indeed, if $y \in X \setminus A$ is isolated, for each $a \in A$ there is a $W_a \in \mathcal{V}_a$ so that $y \notin \overline{W}_a$. Then $\{W_a \mid a \in A\}$ is an open (in X) cover of the closed set A, hence there is an $A_* \subset A$ with $|A_*| \le \omega L_c(X)$ and $\overline{\bigsqcup_{a\in A_*} W_a} \supset A$. Since $A_* \subset A_\beta$ for a suitable $\beta<\kappa^+$ and $y \notin \overline{\bigsqcup_{a\in A_*} W_a}$, $A_{\beta+1} \setminus \overline{\bigsqcup_{a\in A_*} W_a}$ would be non-empty by 3), which is a contradiction. ■

THEOREM 4. *If X is a Hausdorff countably compact space with a dense subset of points of countable character, then* $|X| \le 2^{\omega L_c(X)\psi_c(X)t(X)}$.

Proof. Proceed as in the proof of Theorem 3. To show that $X = A$, assume on the contrary that there is a $y \in X \setminus A$ of countable character. Let (V_n) be a decreasing fundamental system of open neighborhoods of y such that $V_1 \cap A = \emptyset$. For each $a \in A$, there is a $W_a \in \mathcal{V}_a$ so that $y \notin \overline{W}_a$. Fix n_a such that $V_{n_a} \cap \overline{W}_a = \emptyset$. For each $n = 1,2,\dots$ put $\mathcal{U}_n = \bigsqcup\{W_a \mid n_a \le n\}$; then $\bigsqcup_{n=1}^\infty \mathcal{U}_n \supset A$ and, since A is closed and countably compact, there is an n_* so that $\mathcal{U}_{n_*} \supset A$.

Consider $\mathcal{C} = \{W_a \mid n_a \le n_*\}$, which is an open (in X) cover of A; there is an $A_* \subset A$ with $|A_*| \le \omega L_c(X)$ such that

$$y \notin \overline{\bigsqcup\{W_a \mid a \in A_*\}} \supset A\,,$$

because V_{n_*} does not intersect $\overline{\bigsqcup\{W_a \mid a \in A_*\}}$, which completes the proof. ■

THEOREM 5. *If X is a Hausdorff initially κ-compact space with a dense subset of points of character $\le\kappa$, then* $|X| \le 2^{\omega L_c(X)\psi_c(X)t(X)}$.

EXAMPLES. I. This example appears in [1]. Let κ be any uncountable cardinal, $\mathbb{Q}$ be the set of rational numbers and let A be any countable dense subset of the space of irrational numbers. Define $X = (\mathbb{Q}\times\kappa)\cup A$ and consider the following topology τ on X:

1) each point $(q,\alpha) \in \mathbb{Q}\times\kappa$ has a fundamental system of neighborhoods of type

$$\{(r,\alpha) \mid |r-q| < 1/n\,,\ r \in \mathbb{Q}\} \quad \text{where } n = 1,2,\dots;$$

2) each $a \in A$ has a fundamental system of neighborhoods of type

$$\{b \in A \mid |b-a| < 1/n\} \cup \{(q,\alpha) \in \mathbb{Q} \times \kappa \mid |q-a| < 1/n\}$$

where $n = 1, 2, \ldots$

Then (X, τ) is first countable, Hausdorff, non-Urysohn, $\omega L(X) = \aleph_0$ and $\omega L_c(X) = \kappa = |X|$.

II. Let X be the set $\{0,1\}^{2^{\aleph_0}}$ and let D be a countable dense subset of the topological product space $\{0,1\}^{2^{\aleph_0}}$. A new topology τ on X will be considered:

1) $\{x\}$ is open, $\forall x \in D$;

2) for $x \in X \setminus D$ a neighborhood of x must contain a set $\{x\} \cup (V \cap D)$ where V is a neighborhood of x in the product topology.

Then (X, τ) is Urysohn, $\omega L_c(X) = \aleph_0$, $\aleph(X) = 2^{\aleph_0}$, $\psi(X) = \aleph_0$ and $t(X) = \aleph_0$. This example shows that Theorem 2 cannot be extended to Urysohn spaces.

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