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SOME REMARKS ABOUT MYCIELSKI IDEALS

BY

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1. Introduction and definitions. Our set theoretic notation and terminology is standard (see e.g. [4], [5]). Let $\mathbf{c}$ denote $|\mathcal{P}(\omega)|$ (= the cardinality of $\mathcal{P}(\omega)$). Let X be a subset of ω. The set $\{Y \subset X \mid |Y| = \omega\}$ is denoted by $[X]^\omega$. ${}^\omega X$ (${}^{\omega>}X$) denotes the family of ω-sequences (finite sequences) of elements in X, respectively. $\forall^\infty n \in X$ $(\ldots)$ means that $\{n \in X \mid \textit{not}\ \ldots\}$ is finite. $\exists^\infty n \in X$ $(\ldots)$ means that $\{n \in X \mid \ldots\}$ is infinite. For $f, g \in {}^\omega\omega$, g *dominates* f (denoted by $f \prec g$) if $\forall^\infty n < \omega$ $(f(n) < g(n))$. For $F \subset {}^\omega\omega$, F is called a *dominating family* of ${}^\omega\omega$ if $\forall g \in {}^\omega\omega\ \exists f \in F$ $(g \prec f)$, and an *unbounded family* of ${}^\omega\omega$ if $\forall g \in {}^\omega\omega\ \exists f \in F$ $(\textit{not} f \prec g)$. Denote by $\mathbf{d}$ ($\mathbf{b}$) the least cardinality of a dominating (unbounded) family of ${}^\omega\omega$, respectively.

Let $1 < \mathcal{X} \le \omega$. For $X \subset \omega$ and $A \subset {}^\omega\mathcal{X}$, $\Gamma_{\mathcal{X}}(A, X)$ denotes the infinite game between two players, I and II. At each step $n < \omega$, player I chooses $k_n < \mathcal{X}$ if $n \in \omega \setminus X$ and player II chooses $k_n < \mathcal{X}$ if $n \in X$. Player I wins if $\langle k_n \mid n < \omega\rangle \in A$ and player II wins in the opposite case. A *strategy* is a function $\sigma : {}^{<\omega}\mathcal{X} \to \mathcal{X}$. $\mathrm{STR}_{\mathcal{X}}$ denotes the set of strategies. For $\tau, \sigma \in \mathrm{STR}_{\mathcal{X}}$ and $X \subset \omega$, $\tau *_X \sigma$ denotes the resulting ω-sequence of the game $\Gamma_{\mathcal{X}}(A, X)$ when player I follows the strategy τ and II follows σ, i.e.

$$\tau *_X \sigma(n) = \begin{cases} \tau(\tau *_X \sigma\restriction n) & \text{if } n \in \omega \setminus X, \\ \sigma(\tau *_X \sigma\restriction n) & \text{if } n \in X. \end{cases}$$

For $f : \omega \to \mathcal{X}$, we identify f with $\sigma_f \in \mathrm{STR}_{\mathcal{X}}$ which is defined by

$$\sigma_f(s) = f(\mathrm{length}(s)), \quad \text{for any } s \in {}^{<\omega}\mathcal{X}\,.$$

Note that f (i.e. σ_f) is a strategy which does not depend on the previous movements of the players. For $\sigma \in \mathrm{STR}_{\mathcal{X}}$ and $X \subset \omega$, $\mathrm{STR}_{\mathcal{X}} *_X \sigma$ denotes the set of all results of the game determined by X, in which the second player uses strategy σ, i.e.

$$\mathrm{STR}_{\mathcal{X}} *_X \sigma = \{\tau *_X \sigma \mid \tau \in \mathrm{STR}_{\mathcal{X}}\}\,.$$

The following fact is easily checked.

FACT 1.1. *For any $\sigma \in \mathrm{STR}_{\mathcal{X}}$, $X \subset \omega$ and $f \in {}^\omega\mathcal{X}$, the following are equivalent.*

(a) $f \in \mathrm{STR}_{\mathcal{X}} *_X \sigma$.
(b) $f \in \{g *_X \sigma \mid g \in {}^{\omega}\mathcal{X}\}$.
(c) $f = f *_X \sigma$.
(d) $\forall n \in X\ (\sigma(f \restriction n) = f(n))$.

A strategy σ is called a *winning strategy* for player II in the game $\Gamma_{\mathcal{X}}(A, X)$ if $(\mathrm{STR}_{\mathcal{X}} *_X \sigma) \cap A = \emptyset$. Denote by $V_{\mathrm{II}}(\mathcal{X}, X)$ the family of all sets $A \subset {}^{\omega}\mathcal{X}$ for which player II has a winning strategy in $\Gamma_{\mathcal{X}}(A, X)$ and $V^*_{\mathrm{II}}(\mathcal{X}, X)$ the family of all sets $A \subset {}^{\omega}\mathcal{X}$ for which player II has in $\Gamma_{\mathcal{X}}(A, X)$ a winning strategy which does not depend on the movements of player I, i.e.

$$V_{\mathrm{II}}(\mathcal{X}, X) = \{A \subset {}^{\omega}\mathcal{X} \mid \exists \sigma \in \mathrm{STR}_{\mathcal{X}}\ ((\mathrm{STR}_{\mathcal{X}} *_X \sigma) \cap A = \emptyset)\},$$
$$V^*_{\mathrm{II}}(\mathcal{X}, X) = \{A \subset {}^{\omega}\mathcal{X} \mid \exists f \in {}^{\omega}\mathcal{X}\ ((\mathrm{STR}_{\mathcal{X}} *_X f) \cap A = \emptyset)\}.$$

A family $\mathcal{K} \subset [\omega]^{\omega}$ is said to be a *normal system* if for any $X \in \mathcal{K}$ there exist $X_1, X_2 \in \mathcal{K}$ such that $X_1, X_2 \subset X$ and $X_1 \cap X_2 = \emptyset$.

For any normal system $\mathcal{K}$, let

$$\mathcal{M}_{\mathcal{X},\mathcal{K}} = \bigcap_{X \in \mathcal{K}} V_{\mathrm{II}}(\mathcal{X}, X) = \{A \subset {}^{\omega}\mathcal{X} \mid \forall X \in \mathcal{K}\ \exists \sigma \in \mathrm{STR}_{\mathcal{X}}\ ((\mathrm{STR}_{\mathcal{X}} *_X \sigma) \cap A = \emptyset)\},$$

and

$$\mathcal{M}^*_{\mathcal{X},\mathcal{K}} = \bigcap_{X \in \mathcal{K}} V^*_{\mathrm{II}}(\mathcal{X}, X) = \{A \subset {}^{\omega}\mathcal{X} \mid \forall X \in \mathcal{K}\ \exists f \in {}^{\omega}\mathcal{X}\ ((\mathrm{STR}_{\mathcal{X}} *_X f) \cap A = \emptyset)\}.$$

These are σ-ideals (called *Mycielski ideals*), introduced by Mycielski [6], and generalized by Rosłanowski [9, 10] and studied in [1, 3, 8–10]. The ideals $\mathcal{M}_{\mathcal{X},[\omega]^{\omega}}$ and $\mathcal{M}^*_{\mathcal{X},[\omega]^{\omega}}$ will be denoted by $\mathcal{C}_{\mathcal{X}}$ and $\mathcal{P}_{\mathcal{X}}$, respectively.

We shall consider ${}^{\omega}\mathcal{X}$ with the product measure and the product topology. The σ-ideals of null sets and meager sets are denoted by $\mathbf{L}_{\mathcal{X}}$ and $\mathbf{K}_{\mathcal{X}}$, respectively.

2. Orthogonality. Throughout this section, we assume that $1 < \mathcal{X} < \omega$. Two ideals $\mathcal{I}, \mathcal{J}$ of $\mathcal{P}({}^{\omega}\mathcal{X})$ are called *orthogonal* if there exist sets $A \in \mathcal{I}$ and $B \in \mathcal{J}$ such that $A \cup B = {}^{\omega}\mathcal{X}$. We study conditions on a normal system $\mathcal{K}$ which imply the orthogonality of $\mathcal{M}_{\mathcal{X},\mathcal{K}}$ and $\mathbf{L}_{\mathcal{X}}$. For each $X \in [\omega]^{\omega}$, let e_X denote the order isomorphism from ω to X. Rosłanowski [10] proved the following two results:

THEOREM 2.1. *If a normal system $\mathcal{K}$ satisfies*

$$\forall Y \in [\omega]^{\omega}\ \exists X \in \mathcal{K}\ \forall^{\infty} n < \omega\ (|[e_Y(n), e_Y(n+1)) \cap X| \le 1), \tag{2.1}$$

then $\mathcal{M}_{\mathcal{X},\mathcal{K}}$ and $\mathbf{L}_{\mathcal{X}}$ are not orthogonal. ∎

THEOREM 2.2. *There exists a normal system $\mathcal{K}$ (with cardinality $\mathbf{c}$) such that $\{e_X \mid X \in \mathcal{K}\}$ is unbounded in ${}^\omega\omega$ and $\mathcal{M}_{\mathcal{X},\mathcal{K}}$ and $\mathbf{L}_\mathcal{X}$ are orthogonal.* ■

He called a normal system $\mathcal{K}$ which satisfies the condition (2.1) *dominating*. This condition is a little stronger than the condition that $\{e_X \mid X \in \mathcal{K}\}$ is a dominating family of ${}^\omega\omega$. In fact, it is easy to check that, for any $\mathcal{U} \subset [\omega]^\omega$, $\{e_X \mid X \in \mathcal{U}\}$ is a dominating family of ${}^\omega\omega$ if and only if for each $Y \in [\omega]^\omega$ there exists an $X \in \mathcal{U}$ such that $\forall^\infty n < \omega$ $(|[e_Y(n), e_Y(n+1)) \cap X| \le n)$. Using this and the fact that a small set $(I_n, S_n)_{n<\omega}$ can be choosen which satisfies $|S_n| \cdot \mathcal{X}^{-|I_n|} < \mathcal{X}^{-2n}$ for any $n < \omega$, a slight modification of Rosłanowski's proof of Theorem 2.1 yields a proof of

THEOREM 2.3. *For any normal system $\mathcal{K}$, if $\{e_X \mid X \in \mathcal{K}\}$ is a dominating family of ${}^\omega\omega$, then $\mathcal{M}_{\mathcal{X},\mathcal{K}}$ and $\mathbf{L}_\mathcal{X}$ are not orthogonal.* ■

The following theorem and corollary show that unboundedness is not a sufficient condition for non-orthogonality.

THEOREM 2.4. *Let κ be an uncountable cardinal and P the notion of forcing adjoining κ Cohen reals. Then, in V^P, $\mathcal{M}^*_{\mathcal{X},\mathcal{K}}$ and $\mathbf{L}_\mathcal{X}$ are orthogonal, for any normal system $\mathcal{K} \subset [\omega]^\omega$ with cardinality $< \kappa$.*

Proof. Let $\mathcal{K} \in V^P$ be a normal system with cardinality $< \kappa$. Since $|\mathcal{K}| < \kappa$, we may assume that $\mathcal{K} \in V$. From now on, we work in V^P.

CLAIM 1. *There exists a sequence $\langle S_n \mid n < \omega\rangle$ such that*

(1) $\forall n < m < \omega$ $(S_n \subset \omega$ & $|S_n| \ge n$ & $S_n \cap S_m = \emptyset)$,
(2) $\forall X \in \mathcal{K}\ \exists^\infty n < \omega$ $(S_n \subset X)$.

Proof of Claim 1. Take a Cohen generic subset $U \subset \omega$ over V. For each $n < \omega$, set $S_n = [e_U(n^2), e_U((n+1)^2)) \cap U$. Then $\langle S_n \mid n < \omega\rangle$ is as required. ■

Take a sequence $\langle S_n \mid n < \omega\rangle$ which satisfies (1), (2) of Claim 1. Set

$$A = \{f \in {}^\omega\mathcal{X} \mid \exists^\infty n < \omega\ (f{\restriction}S_n \equiv 0)\} \in \mathbf{L}_\mathcal{X}\,.$$

Since $\mathrm{STR}_\mathcal{X} *_X \mathrm{Const}_0 \subset A$ for all $X \in \mathcal{K}$, we conclude that ${}^\omega\mathcal{X} \setminus A \in \mathcal{M}^*_{\mathcal{X},\mathcal{K}}$. ■

COROLLARY 2.5. *It is consistent with* $\mathbf{b} < \mathbf{d} = \mathbf{c}$ *that* "for any normal system $\mathcal{K}$ with cardinality $< \mathbf{c}$, $\mathcal{M}_{\mathcal{K},\mathcal{X}}$ and $\mathbf{L}_\mathcal{X}$ are orthogonal". ■

Relating to orthogonality, Balcerzak and Rosłanowski [1] proved that

THEOREM 2.6. *For each $A \in \mathbf{K}_\mathcal{X}$, there exists a normal system $\mathcal{K}$ such that $A \in \mathcal{M}^*_{\mathcal{X},\mathcal{K}}$.*

They asked whether a measure analogue of Theorem 2.6 holds. I.e., does, for each $A \in \mathbf{L}_{\mathcal{X}}$, exist a normal system $\mathcal{K}$ such that $A \in \mathcal{M}_{\mathcal{X},\mathcal{K}}$? The following example gives a negative answer to this question.

EXAMPLE 2.7. *Let s be the unique $t < \omega$ such that $2t \le \mathcal{X} < 2(t+1)$ and set*

$$A = \{f \in {}^{\omega}\mathcal{X} \mid \exists^{\infty} n < \omega\ (|\{k < n \mid f(k) \ge s\}| < n/4)\}\,.$$

Then A is a Lebesgue measure zero set and, for any normal system $\mathcal{K} \subset [\omega]^{\omega}$, $A \notin \mathcal{M}_{\mathcal{X},\mathcal{K}}$.

Proof. In order to show that $A \notin \mathcal{M}_{\mathcal{X},\mathcal{K}}$ for all normal systems $\mathcal{K} \subset [\omega]^{\omega}$, we need the following lemma.

LEMMA 2.8. *Let X be a subset of ω such that $A \in V_{\mathrm{II}}(\mathcal{X}, X)$. Then*

$$\forall^{\infty} n < \omega\ (|X \cap n| \ge n/4)\,.$$

Proof. Take $\tau \in \mathrm{STR}_{\mathcal{X}}$ such that $(\mathrm{STR}_{\mathcal{X}} *_X \tau) \cap A = \emptyset$. Set $f = \mathrm{Const}_0 *_X \tau$. Since $f \notin A$, we have $\forall^{\infty} n < \omega\ (|\{k < n \mid f(k) \ge s\}| \ge n/4)$. The assertion follows from this and the fact that $\forall k \in \omega \setminus X\ (f(k) = 0 < s)$. ∎

By Lemma 2.8, for any disjoint subsets X_i (for $i < 5$) of ω, there is some $i < 5$ such that $A \notin V_{\mathrm{II}}(\mathcal{X}, X_i)$. So, $A \notin \mathcal{M}_{\mathcal{X},\mathcal{K}}$ for all normal $\mathcal{K}$.

We must show that A has Lebesgue measure zero. Let μ denote the Lebesgue measure on ${}^{\omega}\mathcal{X}$. For each $n < \omega$, define $B_n = \{f \in {}^{\omega}\mathcal{X} \mid |\{k < n \mid f(k) \ge s\}| < n/4\}$. Since $A = \bigcap_{m<\omega} \bigcup_{m \le n < \omega} B_n$, we have $\mu(A) \le \lim_{m<\omega}(\sum_{m \le n < \omega} \mu(B_n))$. So, it suffices to show

$$\sum_{n<\omega} \mu(B_n) < \omega\,. \tag{C.1}$$

LEMMA 2.9. $\binom{4(n+1)}{n} \le \left(\frac{4^4}{3^3}\right)^n$ *for all $1 \le n < \omega$.*

Proof. Since $\binom{8}{1} = 8 \le 4^4/3^3$, it suffices to show that

$$\binom{4(n+1)}{n} \le \frac{4^4}{3^3} \cdot \binom{4n}{n-1} \quad \text{for } n \ge 2\,.$$

Indeed,

$$\binom{4(n+1)}{n} = \frac{4}{3} \cdot \frac{(4n+3)(4n+2)(4n+1)}{n(3n+4)(3n+2)} \binom{4n}{n-1} \le \frac{4^4}{3^3} \cdot \binom{4n}{n-1}\,. \ ∎$$

By Lemma 2.9, for any $0 < m < \omega$,

$$\left(\frac{2}{3}\right)^m \left(\frac{1}{2}\right)^{3m} \sum_{k \le m} \binom{4(m+1)}{k} \le (m+1)\left(\frac{2 \cdot 4^4}{3 \cdot 8 \cdot 3^3}\right)^m = (m+1)\left(\frac{64}{81}\right)^m.$$

Using this, we have

$$\sum_{0<m<\omega} \left(\frac{2}{3}\right)^m \left(\frac{1}{2}\right)^{3m} \sum_{k\le m} \binom{4(m+1)}{k} < \omega\,.$$

(C.1) follows from this and from

$$\mu(B_n) = \mathcal{X}^{-n} \sum_{X\in[n]^{<n/4}} ((\mathcal{X}-s)^{|X|}\cdot s^{|n\setminus X|}) \le \sum_{X\in[n]^{<n/4}} \left(\frac{2}{3}\right)^{|X|} \left(\frac{1}{2}\right)^{n-|X|}$$
$$\le \left(\frac{2}{3}\right)^{n/4} \left(\frac{1}{2}\right)^{3n/4} \sum_{k<n/4} \binom{n}{k}, \quad \text{for any } n<\omega\,. \ \blacksquare$$

Remark. In [3], the definition of the ideals $\mathcal{P}_{\mathcal{X}}$ was generalized to all functions $\mathcal{X} \in {}^{\omega}(\omega \setminus 2)$. A similar generalization is possible for the ideals $\mathcal{M}_{\mathcal{X},\mathcal{K}}$ and $\mathcal{M}^*_{\mathcal{X},\mathcal{K}}$, for each $\mathcal{X} : \omega \to (\omega+1\setminus 2)$. By modifying the construction of A in Example 2.7 a little, for each $\mathcal{X} \in {}^{\omega}(\omega\setminus 2)$ we can construct a Lebesgue measure zero subset A of $\prod_{n<\omega} \mathcal{X}(n)$ such that $A \notin \mathcal{M}_{\mathcal{X},\mathcal{K}}$ for any normal system $\mathcal{K}$.

3. Cardinal coefficients. In this section, we study the cardinal coefficients of the ideals $\mathcal{C}_{\mathcal{X}}$ and $\mathcal{P}_{\mathcal{X}}$. For an ideal $\mathcal{I}$ of $\mathcal{P}({}^{\omega}\mathcal{X})$, define

$$\begin{aligned}
\mathrm{cof}(\mathcal{I}) &= \min\{|\mathcal{S}| \mid \mathcal{S}\subset\mathcal{I} \ \&\ \forall A\in\mathcal{I}\ \exists B\in\mathcal{S}\ (A\subset B)\}\,,\\
\mathrm{non}(\mathcal{I}) &= \min\{|A| \mid A\subset {}^{\omega}\mathcal{X} \ \&\ A\notin\mathcal{I}\}\,,\\
\mathrm{cov}(\mathcal{I}) &= \min\Big\{|\mathcal{S}| \,\Big|\, \mathcal{S}\subset\mathcal{I} \ \&\ \bigcup\mathcal{S} = {}^{\omega}\mathcal{X}\Big\}\,,\\
\mathrm{add}(\mathcal{I}) &= \min\Big\{|\mathcal{S}| \,\Big|\, \mathcal{S}\subset\mathcal{I} \ \&\ \bigcup\mathcal{S} \notin \mathcal{I}\Big\}\,.
\end{aligned}$$

The following facts are well-known.

Fact 3.1. *Let $\mathcal{I},\mathcal{J}$ be σ-ideals of $\mathcal{P}({}^{\omega}\mathcal{X})$ such that ${}^{\omega}\mathcal{X}\notin\mathcal{I}$ and $\{f\}\in\mathcal{I}$, for all $f\in{}^{\omega}\mathcal{X}$. Then*

(1) $\mathrm{non}(\mathcal{I}),\mathrm{cov}(\mathcal{I}) \le \mathrm{cof}(\mathcal{I})$.

(2) $\omega_1 \le \mathrm{add}(\mathcal{I}) \le \mathrm{non}(\mathcal{I}),\mathrm{cov}(\mathcal{I})$.

(3) *If $\mathcal{I}$ and $\mathcal{J}$ are orthogonal and translation invariant, then* $\mathrm{cov}(\mathcal{I}) \le \mathrm{non}(\mathcal{J})$.

Fact 3.2. *The cardinal coefficients of the ideals $\mathbf{K}_{\mathcal{X}}$ and $\mathbf{L}_{\mathcal{X}}$ do not depend on the choice of $\mathcal{X}$, i.e. for any $1<\mathcal{X},\mathcal{Y}\le\omega$,* $\mathrm{cof}(\mathbf{K}_{\mathcal{X}}) = \mathrm{cof}(\mathbf{K}_{\mathcal{Y}})$, $\mathrm{cof}(\mathbf{L}_{\mathcal{X}}) = \mathrm{cof}(\mathbf{L}_{\mathcal{Y}})$, ...

For the ideals $\mathcal{C}_{\mathcal{X}}$ and $\mathcal{P}_{\mathcal{X}}$, the following theorems are known.

Theorem 3.3 [8, 10]. (1) $\mathrm{non}(\mathcal{C}_{\mathcal{X}}) = \mathrm{non}(\mathcal{P}_{\mathcal{X}}) = \mathbf{c}$.

(2) $\mathrm{add}(\mathcal{C}_\omega) = \mathrm{add}(\mathcal{P}_\omega) = \mathrm{cov}(\mathcal{C}_\omega) = \mathrm{cov}(\mathcal{P}_\omega) = \omega_1$.

(3) $\mathrm{cov}(\mathcal{P}_{\mathcal{X}}) = \mathrm{add}(\mathcal{P}_{\mathcal{X}})$.
(4) $\mathrm{cof}(\mathcal{P}_\omega) > \mathbf{c}$ *and* $\mathrm{cof}(\mathcal{C}_\omega) > \mathbf{c}$.
(5) *If* $\mathrm{cov}(\mathbf{K}) = \mathbf{c}$, *then* $\mathrm{cof}(\mathcal{P}_{\mathcal{X}}) > \mathbf{c}$ *and* $\mathrm{cof}(\mathcal{C}_{\mathcal{X}}) > \mathbf{c}$.
(6) $\mathrm{cov}(\mathcal{P}_2) \le \mathrm{cof}(\mathbf{L})^+$.

THEOREM 3.4 (I. Recław, see [8]). *The proper forcing axiom* (**PFA**) *implies that* $\mathrm{cov}(\mathcal{P}_2) > \omega_1$.

THEOREM 3.5 [3]. *It is consistent with Martin's Axiom and* $\mathbf{c} = \omega_2$ *that* $\mathrm{cov}(\mathcal{P}_2) = \omega_1$.

We shall show the following.

THEOREM 3.6. $\mathrm{cof}(\mathcal{P}_{\mathcal{X}}) > \mathbf{c}$ *and* $\mathrm{cof}(\mathcal{C}_{\mathcal{X}}) > \mathbf{c}$, *for any* $1 < \mathcal{X} < \omega$.

THEOREM 3.7. **PFA** *implies that* $\mathrm{add}(\mathcal{C}_2) > \omega_1$.

THEOREM 3.8. $\mathrm{add}(\mathcal{C}_2) \le \mathrm{cof}(\mathbf{L})$.

The case of $\mathcal{X} = 2$ in Theorem 3.6 gives an affirmative answer to Problem 5.3.18(c) of [9].

Proof of Theorem 3.6. It suffices to show:

(∗) For any $\{A_\alpha \mid \alpha < \mathbf{c}\} \subset \mathcal{C}_{\mathcal{X}}$, there exists $B \in \mathcal{P}_{\mathcal{X}}$ such that $\forall \alpha < \mathbf{c}$ $(B \not\subset A_\alpha)$.

In order to show (∗), we need several definitions and two lemmas.

For any $A \subset {}^\omega\mathcal{X}$ and $X \subset \omega$, the set $\{f{\restriction}X \mid f \in A\}$ is denoted by $A|X$. Note that $\mathcal{P}_{\mathcal{X}} = \{A \subset {}^\omega\mathcal{X} \mid \forall X \in [\omega]^\omega\ (A|X \ne {}^X\mathcal{X})\}$.

Take a nonempty $A \in \mathcal{P}_{\mathcal{X}}$ such that

$$\forall c \in A\ \forall d \in {}^\omega\mathcal{X}\ (\forall^\infty n < \omega\ (c(n) = d(n)) \Rightarrow d \in A)\,. \tag{3.1}$$

For each $X \in [\omega]^\omega$, take a sequence $\langle c_{\alpha,X} \mid \alpha < \mathbf{c}\rangle$ such that

$$c_{\alpha,X} \in {}^X\mathcal{X} \setminus A|X \quad \text{and} \quad c_{\alpha,X} \ne c_{\beta,X} \text{ if } \alpha \ne \beta\,.$$

LEMMA 3.9. *Suppose that* $\mathcal{F} \subset \mathbf{c} \times [\omega]^\omega$ *and* $Y \in [\omega]^\omega$ *satisfy*

$$\forall (\alpha, X) \in \mathcal{F}\ (X \setminus Y \textit{ is finite})\ \&\ |\mathcal{F}| < \mathbf{c}\,.$$

Then there exists $g \in {}^Y\mathcal{X}$ *such that*

$$\forall (\alpha, X) \in \mathcal{F}\ (c_{\alpha,X}{\restriction}(X \cap Y) \not\subset g)\,.$$

Proof. For each $(\alpha, X) \in \mathcal{F}$, let $d_{\alpha,X} = c_{\alpha,X}{\restriction}(X \cap Y)$. By (3.1),

$$d_{\alpha,X} \notin A|(X \cap Y) \quad \text{for all } (\alpha, X) \in \mathcal{F}\,.$$

Then, since $\forall (\alpha, X) \in \mathcal{F}\ (\{f \in {}^Y\mathcal{X} \mid d_{\alpha,X} \subset f\}) \cap A|Y = \emptyset)$, we have

$$\Big(\bigcup_{(\alpha,X)\in\mathcal{F}} \{f \in {}^Y\mathcal{X} \mid d_{\alpha,X} \subset f\}\Big) \cap A|Y = \emptyset\,.$$

Since $A|Y \ne \emptyset$, we can take $g \in A|Y$. This g is as required. ■

Recall that $X,Y \in [\omega]^\omega$ are *almost disjoint* if $X \cap Y$ is finite. A family $\mathcal{F} \subset [\omega]^\omega$ is said to be *pairwise almost disjoint* if any two distinct elements of $\mathcal{F}$ are almost disjoint. A *MAD-family* is a maximal family (with the inclusion order) which is pairwise almost disjoint. Take a MAD-family $\mathcal{W} \subset [\omega]^\omega$ such that $|\mathcal{W}| = \mathbf{c}$. Take an enumeration $\langle U_\alpha \mid \alpha < \mathbf{c}\rangle$ of $\bigcup_{X\in\mathcal{W}}[X]^\omega$.

To prove Theorem 3.6, let $\{A_\alpha \mid \alpha < \mathbf{c}\} \subset \mathcal{C}_\mathcal{X}$. Take $\langle \tau_{\alpha,X} \mid \alpha < \mathbf{c}$ & $X \in [\omega]^\omega\rangle$ such that

$$\tau_{\alpha,X} \in \mathrm{STR}_\mathcal{X} \quad \text{and} \quad ({}^\omega\mathcal{X} *_X \tau_{\alpha,X}) \cap A_\alpha = \emptyset \quad \text{for all } \alpha < \mathbf{c}, X \in [\omega]^\omega\,.$$

Lemma 3.10. *There exist sequences $\langle h_\alpha \mid \alpha < \mathbf{c}\rangle$ and $\langle e_\alpha \mid \alpha < \mathbf{c}\rangle$ which satisfy the following*

(1) $h_\alpha \in {}^\omega\mathcal{X} \setminus A_\alpha$ *and* $e_\alpha \in {}^{U_\alpha}\mathcal{X}$.
(2) $e_\alpha \not\subset h_\beta$, *for any* $\alpha, \beta < \mathbf{c}$.

Proof. We shall show, by induction on $\alpha < \mathbf{c}$, that there exist $h_\alpha \in {}^\omega\mathcal{X} \setminus A_\alpha$ and $e_\alpha \in \{c_{\eta,U_\alpha} \mid \eta < \mathbf{c}\}$ which satisfy $e_\xi \not\subset h_\alpha$ and $e_\alpha \not\subset h_\xi$ for all $\xi \le \alpha$.

So, let $\alpha < \mathbf{c}$. Take $X \in \mathcal{W}$ such that $\forall \xi < \alpha$ ($U_\xi \cap X$ is finite). Set $Y = \omega \setminus X$. Then by Lemma 3.9, there exists $g \in {}^Y\mathcal{X}$ such that $e_\xi{\restriction}(U_\xi \cap Y) \not\subset g$ for all $\xi < \alpha$. Set $h_\alpha = g *_X \tau_{\alpha,X}$ ($\in {}^\omega\mathcal{X} \setminus A_\alpha$). Since $g \subset h_\alpha$, we have $e_\xi \not\subset h_\alpha$, for all $\xi < \alpha$. Take $e_\alpha \in \{c_{\eta,U_\alpha} \mid \eta < \mathbf{c}\}$ such that $e_\alpha \notin \{h_\xi{\restriction}U_\alpha \mid \xi \le \alpha\}$. ∎

Let $\langle h_\alpha \mid \alpha < \mathbf{c}\rangle$ and $\langle e_\alpha \mid \alpha < \mathbf{c}\rangle$ be sequences which satisfy (1) and (2) of Lemma 3.10. Set $B = \{h_\alpha \mid \alpha < \mathbf{c}\}$. Since $\forall \alpha < \mathbf{c}$ ($h_\alpha \notin A_\alpha$), we have $B \not\subset A_\alpha$ for all $\alpha < \mathbf{c}$. To show $B \in \mathcal{P}_\mathcal{X}$, let $X \in [\omega]^\omega$. Take $Y \in \mathcal{W}$ such that $X \cap Y$ is infinite and $\alpha < \mathbf{c}$ such that $U_\alpha = X \cap Y$. Then, since $e_\alpha \notin B|U_\alpha$, it follows that $B|X \neq {}^X\mathcal{X}$. ∎

Proof of Theorem 3.7. In order to show Theorem 3.7, we need to modify the notion of covering systems in [10].

Definition. Let κ be a cardinal, $U \in [\omega]^\omega$, and $h : U \to \omega \setminus \{0\}$. A double indexed sequence $\langle f_{\alpha,X} \mid \alpha < \kappa$ & $X \in [U]^\omega\rangle$ is called a *κ-covering system* for h if it satisfies

(1) $f_{\alpha,X} \in \prod_{n\in X} h(n)$,
(2) $\forall g \in \prod_{n\in U} h(n)\ \exists \alpha < \kappa\ \forall X \in [U]^\omega\ (f_{\alpha,X} \not\subset g)$.

Lemma 3.11 (**PFA**)

(C) *There does not exist an ω_1-covering system for h, for any $h : U \to \omega \setminus \{0\}$ and $U \in [\omega]^\omega$.*

Proof. Let $U \in [\omega]^\omega$ and $h : U \to \omega \setminus \{0\}$.

Suppose that a sequence $F = \langle f_{\alpha,X} \mid \alpha < \omega_1 \ \&\ X \in [U]^\omega \rangle$ satisfies the condition (1) in the definition of covering systems. (We show that F does not satisfy (2).)

Define the forcing notion P $(= P_F)$ by

$$P = \Big\{ p \ \Big|\ \exists X \in [U]^\omega \ \Big(p \in \prod_{n \in U \setminus X} h(n) \Big) \Big\}, \qquad p \le q \ \text{ iff } \ p \supset q\,.$$

Since the partial ordering $\le_n$ (for $n < \omega$) on P defined by

$$\begin{aligned} p \le_n q \ \text{ iff } \ p \le q \text{ and } &\text{“the set of the first } n \text{ elements of } U \setminus \mathrm{dom}(p)\text{”} \\ = &\text{“the set of the first } n \text{ elements of } U \setminus \mathrm{dom}(q)\text{”} \end{aligned}$$

satisfies Axiom A of Baumgartner, P is proper. For each $\alpha < \omega_1$, set

$$D_\alpha = \{ p \in P \mid \exists X \in [U]^\omega \ (f_{\alpha,X} \subset p) \}\,.$$

Since $\forall \alpha < \omega_1$ (D_α is dense in P), by **PFA**, there exists a $\{D_\alpha \mid \alpha < \omega_1\}$-generic filter $\mathcal{G}$ on P. Since $\mathcal{G}$ is a filter, we can take $g \in \prod_{n \in U} h(n)$ such that $\bigcup \mathcal{G} \subset g$. Then $\forall \alpha < \omega_1 \ \exists X \in [U]^\omega \ (f_{\alpha,X} \subset g)$. ■

By Lemma 3.11, it suffices to show that (C) implies $\mathrm{add}(\mathcal{C}_2) > \omega_1$. To show this, assume that (C) holds and let $\{A_\alpha \mid \alpha < \omega_1\} \subset \mathcal{C}_2$.

To show that $\bigcup_{\alpha<\omega_1} A_\alpha \in \mathcal{C}_2$, let $X \in [\omega]^\omega$. For each $\alpha < \omega_1$ and $Y \in [X]^\omega$, take $\tau_{\alpha,Y} \in \mathrm{STR}_2$ such that $({}^\omega 2 *_Y \tau_{\alpha,Y}) \cap A_\alpha = \emptyset$. Set $f_{\alpha,Y} = \langle \tau_{\alpha,Y} \restriction {}^n 2 \mid n \in Y \rangle$. Using (C), take $g = \langle g_n \mid n \in X \rangle$ such that $\forall \alpha < \omega_1$ $\exists Y \in [X]^\omega \ (f_{\alpha,Y} \subset g)$. Define $\tau \in \mathrm{STR}_2$ by

$$\tau(s) = \begin{cases} g_n(s) & \text{if } \mathrm{length}(s) = n \in X, \\ 0 & \text{otherwise.} \end{cases}$$

To show that $({}^\omega 2 *_X \tau) \cap A_\alpha = \emptyset$ for all $\alpha < \omega_1$, let $\alpha < \omega_1$. Take $Y \in [X]^\omega$ such that $f_{\alpha,Y} \subset g$. Then $\tau_{\alpha,Y} \restriction (\bigcup_{n \in Y} {}^n 2) \subset \tau$. So, ${}^\omega 2 *_X \tau \subset {}^\omega 2 *_Y \tau = {}^\omega 2 *_Y \tau_{\alpha,Y}$. Since $({}^\omega 2 *_Y \tau_{\alpha,Y}) \cap A_\alpha = \emptyset$, we conclude $({}^\omega 2 *_X \tau) \cap A_\alpha = \emptyset$. ■

Proof of Theorem 3.8. Define $h : \omega \to \omega$ by

$$h(0) = 0, \quad h(n+1) = 2^n (h(n) + 1)\,.$$

For each $n < \omega$, set

$$A_n = \{ u \mid u : {}^{h(n+1)} 2 \to 2 \}\,.$$

Set

$$\mathcal{T} = \{ \langle S_n \mid n < \omega \rangle \mid \forall n < \omega \ (S_n \subset A_n \ \&\ |S_n| \le 2^n) \}\,.$$

Using Bartoszyński's Characterization Theorem [2], take $\mathcal{B} \subset \mathcal{T}$ such that

$$|\mathcal{B}| = \mathrm{cof}(\mathbf{L}) \quad \text{and} \quad \forall g \in \prod_{n<\omega} A_n \ \exists S \in \mathcal{B} \ \forall n < \omega \ (g(n) \in S(n))\,.$$

A tree $T \subset {}^{\omega>}2$ is called a *thin tree* if it satisfies

$$\forall n < \omega \ (T_n \text{ has at most one branch which ramifies})\,.$$

Note that if T is a thin tree then $\{f \in {}^{\omega}2 \mid \forall n < \omega \ (f{\restriction}n \in T)\} \in \mathcal{C}_2$. For each $S \in \mathcal{B}$, take a thin tree T_S such that

$$(**)\ \ \forall s \in T_S \cap {}^{h(n)+1}2\ \forall \varrho \in S(n)\ \exists t \in T_S \cap {}^{h(n+1)}2\ (s \subset t \ \&\ t^\wedge\langle \varrho(t)\rangle \in T_S)\,,$$

and set

$$A_S = \{d \in {}^{\omega}2 \mid \forall n < \omega \ (d{\restriction}n \in T_S)\}\,.$$

Since $A_S \in \mathcal{C}_2$ for all $S \in \mathcal{B}$, we can complete the proof by showing that $\bigcup_{S\in\mathcal{B}} A_S \notin \mathcal{C}_2$. Let $X = \{h(n+1) \mid n < \omega\}$. We claim that, for any $\sigma \in \mathrm{STR}_2$, σ is not a winning strategy for player II in the game $\Gamma_2(\bigcup_{S\in\mathcal{B}} A_S, X)$. To show this, set $g = \langle \sigma{\restriction}({}^{h(n+1)}2) \mid n < \omega\rangle \in \mathcal{T}$. Take $S \in \mathcal{B}$ such that $\forall n < \omega \ (g(n) \in S(n))$. By induction on $n < \omega$, define $s_n \in T_S \cap {}^{h(n)}2$ as follows. For $n = 0$, take an arbitrary $s_0 \in T_S \cap {}^{h(0)}2$. Assume that s_n was defined. Then, using $(**)$, take $s_{n+1} \in T_X \cap {}^{h(n+1)}2$ such that $s_n{}^\wedge\langle\sigma(s_n)\rangle \subset s_{n+1}$ and $s_{n+1}{}^\wedge\langle\sigma(s_{n+1})\rangle \in T_S$.

Set $f = \bigcup_{n<\omega} s_n \in A_S$. Since $\forall k \in X \ (\sigma(f{\restriction}k) = f(k))$, $f \in \mathrm{STR}_2 *_X \sigma$. Thus, $(\mathrm{STR}_2 *_X \sigma) \cap A_S \neq \emptyset$. So, $\bigcup_{S\in\mathcal{B}} A_S \notin \mathcal{C}_2$. ∎

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