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LITTLEWOOD–PALEY THEORY ON SOLENOIDS

BY

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1. Introduction. Suppose that G is a locally compact abelian group with dual group X. We say that a family $(\Delta_j)_{j\in I}$ of measurable subsets of X (with respect to Haar measure on X) is a *decomposition* of X if

(i) the Δ_j's are pairwise disjoint;
(ii) $X\backslash(\bigcup_{j\in I}\Delta_j)$ is locally negligible.

If Δ is a measurable subset of X, we let S_Δ denote the *partial sum operator* defined on $L^2(G)\cap L^p(G)$ by

$$(S_\Delta f)^\wedge = 1_\Delta \widehat{f}, \tag{1.1}$$

where, whenever A is a set, 1_A is the indicator function of the set A. If $1<p<\infty$ and S_Δ is bounded from $L^2(G)\cap L^p(G)$ into $L^p(G)$, we use the same symbol to denote the bounded extension of S_Δ to all $L^p(G)$. We say that the decomposition $(\Delta_j)_{j\in I}$ has the LP (*Littlewood–Paley*) property if for every $p\in[1,\infty]$ there are constants α_p and β_p such that

$$\alpha_p\|f\|_p \le \Big\|\Big(\sum_{j\in I}|S_{\Delta_j}f|^2\Big)^{1/2}\Big\|_p \le \beta_p\|f\|_p$$

for all $f\in L^p(G)$. (See [9, Chap. 1] for various equivalent definitions and properties of decompositions with the LP property.) If the decomposition $(\Delta_j)_{j\in I}$ has the LP property, we will simply say that $(\Delta_j)_{j\in I}$ is an *LP decomposition* of X.

In this paper, we consider an arbitrary noncyclic subgroup X of $\mathbb{Q}$ and its compact dual group G. We describe an LP decomposition of X where each set in the decomposition is finite. To establish the LP property of the decomposition, we use results from Littlewood–Paley theory on the real line and for martingale differences. A crucial tool in our proofs is a martingale inequality of Stein [15]. We give two different proofs of the latter inequality,

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showing its connection with Doob's well-known results on maximal functions associated with martingales.

2. Preliminaries

(2.1) *The $\boldsymbol{a}$-adic solenoid and its character group.* Up to an isomorphism, any noncyclic subgroup of $\mathbb{Q}$ can be described as follows. Let $\boldsymbol{a} = (a_1, a_2, \ldots)$ be a fixed sequence of positive integers all greater than 1. Let $A_0 = 1$, $A_1 = a_0, \ldots, A_n = a_0 a_1 \ldots a_{n-1}, \ldots$ Let $\mathbb{Q}_{\boldsymbol{a}}$ be the set of rational numbers l/A_k, $l \in \mathbb{Z}$, $k = 0, 1, \ldots$ Then $\mathbb{Q}_{\boldsymbol{a}}$ is noncyclic, and, as shown in [1], any noncyclic subgroup of $\mathbb{Q}$ is of this form.

The character group of $\mathbb{Q}_{\boldsymbol{a}}$ is a compact solenoidal group denoted by $\Sigma_{\boldsymbol{a}}$. The groups $\Sigma_{\boldsymbol{a}}$ and $\Delta_{\boldsymbol{a}}$ below are described in detail in [14, Section 10]. However, our notation and facts concerning these groups are taken from [13]. The group $\Sigma_{\boldsymbol{a}}$ can be realized as the set $[-1/2, 1/2] \times \Delta_{\boldsymbol{a}}$, where $\Delta_{\boldsymbol{a}}$ is the group of $\boldsymbol{a}$-adic integers. The latter group consists of all sequences $\boldsymbol{x} = (x_0, x_1, \ldots)$ with $x_j \in \{0, 1, \ldots, a_j - 1\}$. Addition in $\Delta_{\boldsymbol{a}}$ is defined coordinatewise and carrying quotients. For $n = 1, 2, \ldots$, let

$$\Lambda_n = \{(0, \boldsymbol{x}) \in \Sigma_{\boldsymbol{a}} : x_0 = x_1 = \ldots = x_{n-1} = 0\}\,, \tag{2.2}$$

and let λ_n denote the normalized Haar measure on Λ_n. The measure λ_n is a singular Borel measure on $\Sigma_{\boldsymbol{a}}$ whose Fourier transform is equal to the indicator function of $(1/A_n)\mathbb{Z}$:

$$\widehat{\lambda}_n = 1_{(1/A_n)\mathbb{Z}}\,. \tag{2.3}$$

The quotient group $\Sigma_{\boldsymbol{a}}/\Lambda_n$ is topologically isomorphic to the circle group $\mathbb{T}$ (parametrized by $[-1/2, 1/2]$). The mapping

$$\pi_n : (t, \boldsymbol{x}) \mapsto \chi_{1/A_n}((t, \boldsymbol{x}))\,,$$

where χ_{1/A_n} is the character corresponding to $1/A_n$, is a homomorphism of $\Sigma_{\boldsymbol{a}}$ onto $\mathbb{T}$ with kernel Λ_n. Moreover, if $f \in L^1(\Sigma_{\boldsymbol{a}})$ and f is constant on the cosets of Λ_n, then there is a function $g \in L^1(\mathbb{T})$ such that $f = g \circ \pi_n$ and

$$\int_{\Sigma_{\boldsymbol{a}}} f\, d\lambda = \int_{\Sigma_{\boldsymbol{a}}} g \circ \pi_n\, d\lambda = \int_{\mathbb{T}} g\, dx \tag{2.4}$$

where λ is the normalized Haar measure on $\Sigma_{\boldsymbol{a}}$.

(2.5) *Littlewood–Paley decompositions of $\mathbb{Q}_{\boldsymbol{a}}$.* As a subgroup of $\mathbb{R}$, the group $\mathbb{Q}_{\boldsymbol{a}}$ inherits LP decompositions from $\mathbb{R}$. This will be the first type of decompositions that we will describe. The second type is the one associated with martingale differences. The third decomposition that we describe combines the structures of the previous two, and consists of finite blocks. That these decompositions have the LP property will be shown in §4.

(2.6) *Dyadic decomposition.* For each $j \in \mathbb{Z}$, let Δ_j be the following subinterval of $\mathbb{R}$:

$$\Delta_j = \begin{cases} [2^{j-1}, 2^j] & \text{if } j > 0; \\ [-1, 1] & \text{if } j = 0; \\ [-2^{|j|}, -2^{|j|-1}] & \text{if } j < 0. \end{cases}$$

Define the *dyadic decomposition* of $\mathbb{Q}_{\boldsymbol{a}}$ by setting $\mathcal{C}_j = \Delta_j \cap \mathbb{Q}_{\boldsymbol{a}}$ for $j \in \mathbb{Z}$. Note that each $\mathcal{C}_j$ is an infinite subset of $\mathbb{Q}_{\boldsymbol{a}}$.

(2.7) *Decomposition associated with martingale differences.* For $j = 0$, let $\mathcal{D}_0 = \mathcal{X}_0 = \mathbb{Z}$; for $j \geq 1$, let

$$\mathcal{D}_j = \mathcal{X}_j \backslash \mathcal{X}_{j-1} = \frac{1}{A_j}\mathbb{Z} \backslash \frac{1}{A_{j-1}}\mathbb{Z};$$

and for $j < 0$, let $\mathcal{X}_j = \emptyset$.

A detailed study of decompositions of this type is found in [9, Chap. 5].

(2.8) *Finite-block-decomposition of* $\mathbb{Q}_{\boldsymbol{a}}$. Let $\mathcal{X}_j$ be as in (2.7). For $j = 0$, let $\mathcal{B}_0 = \{-1, 0, 1\}$; for $j \geq 1$, let

$$\mathcal{B}_j = ((\mathcal{X}_j \backslash \mathcal{X}_{j-1}) \cap [0, 2^j]\,) \cup (\mathcal{X}_j \cap [2^j, 2^{j+1}]\,)\,;$$

and for $j < 0$, let $\mathcal{B}_j = -\mathcal{B}_{|j|}$.

Since each $\mathcal{X}_j$ is isomorphic to $\mathbb{Z}$, it follows easily that each $\mathcal{B}_j$ is finite. It is also easy to see that the $\mathcal{B}_j$'s are mutually disjoint. We now claim that $\mathbb{Q}_{\boldsymbol{a}} = \bigcup_{j=-\infty}^{\infty} \mathcal{B}_j$. It is enough to show that $\mathbb{Q}_{\boldsymbol{a}} \cap [0, \infty] \subseteq \bigcup_{j=0}^{\infty} \mathcal{B}_j$. Suppose that $x \in [0, \infty] \cap \mathbb{Q}_{\boldsymbol{a}} \cap \mathcal{B}_0^{\mathrm{c}}$, where the superscript "c" denotes set-theoretic complement. Let j_0 be the integer such that $x \in \Delta_{j_0+1} \cap \mathbb{Q}_{\boldsymbol{a}} = [2^{j_0}, 2^{j_0+1}] \cap \mathbb{Q}_{\boldsymbol{a}}$. If $x \in \mathcal{X}_{j_0}$, then $x \in \Delta_{j_0+1} \cap \mathcal{X}_{j_0}$; and so $x \in \mathcal{B}_{j_0}$. If $x \notin \mathcal{X}_{j_0}$, then, because the $\mathcal{X}_j$'s are increasing, it follows that $x \notin \mathcal{X}_j$ for all $j \leq j_0$. Let m be the first integer greater than j_0 such that $x \in (\mathcal{X}_m \backslash \mathcal{X}_{m-1}) \cap [0, 2^m]$. Clearly, $x \in \mathcal{B}_m$, and this proves our claim. Hence $(\mathcal{B}_j)_{j=-\infty}^{\infty}$ is a decomposition of $\mathbb{Q}_{\boldsymbol{a}}$.

Similar blocks have been used by Hewitt and Ritter [12] to study almost everywhere convergence of Fourier series on $\Sigma_{\boldsymbol{a}}$.

3. A maximal inequality of Stein. A crucial tool in Littlewood–Paley theory is an inequality of Stein [15, Theorem 8, p. 103] contained in the following theorem.

(3.1) THEOREM. *Let (Ω, Σ, μ) be a probability space with an increasing sequence of sub-σ-fields $\Sigma_1 \subseteq \Sigma_2 \subseteq \ldots$, and let $\boldsymbol{E}_n$ be the conditional expectation onto Σ_n. If $1 \leq q \leq p < \infty$, or $1 < p \leq q \leq \infty$, then for all sequences of Σ-integrable functions $(f_n)_{n=1}^{\infty}$, we have*

$$\Big\| \Big(\sum_{n=1}^{\infty} |\boldsymbol{E}_n(f_n)|^q \Big)^{1/q} \Big\|_p \leq A_{p,q} \Big\| \Big(\sum_{n=1}^{\infty} |f_n|^q \Big)^{1/q} \Big\|_p \qquad (q < \infty)\,, \tag{3.2}$$

and

$$\|\sup_{n\geq 1}|\boldsymbol{E}_n(f_n)|\|_p \leq p'\|\sup_{n\geq 1}|f_n|\|_p \qquad (q=\infty)\,, \tag{3.3}$$

where

$$A_{p,q} = \begin{cases} (p/q)^{1/q} & \textit{if } p\geq q,\\ (p'/q')^{1/q'} & \textit{if } p<q, \end{cases} \tag{3.4}$$

$p'=p/(p-1)$ *if* $1<p<\infty$, $p'=1$ *if* $p=\infty$, *and* $q'=q/(q-1)$.

Remarks. As it appears in Stein [15], Theorem (3.1) is stated for the value $q=2$ only—and this is all we need in this paper. Stein's proof is based on an interpolation theorem of Benedek and Panzone [2] for operators on spaces of vector-valued functions. Also, the proof of Stein does not give the values of the constants $A_{p,q}$ as we do in (3.4). However, the proof does yield the correct asymptotic values of the constants as $p\to 1$ and as $p\to\infty$. Consequently, it shows that the result fails when $p=1$, or $p=\infty$.

We also point out that the case $q=1$ is a special case of an inequality of Burkholder, Davis and Gundy [5].

We will give two proofs of Theorem (3.1). The first one is very direct and elementary. The second one links the result to Doob's famous martingale inequalities. (For the case $q=1$, this proof was also given in [11].) In fact, it shows that certain cases of Theorem (3.1) are equivalent to Doob's results. We end the section by deriving the weak type counterpart of Theorem (3.1). The result is motivated by Doob's weak type $(1,1)$ inequalities. It will not be needed in the later sections of the paper.

First proof of Theorem (3.1). If $p=q=\infty$, then (3.3) follows at once from the inequality $|\boldsymbol{E}_n(f_n)|\leq \boldsymbol{E}_n(|f_n|)$ (see [9, Lemma 5.1.4.iii, p. 78]). We consider next the case when $1=q\leq p<\infty$. Again, because $|\boldsymbol{E}_n(f_n)|\leq \boldsymbol{E}_n(|f_n|)$, we may assume without loss of generality that $f_n\geq 0$. Further, by letting N tend to infinity, we may assume that $f_n=0$ for all $n\geq N$. Thus, if

$$\Big\|\sum_{n=1}^{N}|f_n|\Big\|_p$$

is finite, then so is

$$\Big\|\sum_{n=1}^{N}|\boldsymbol{E}_n(f_n)|\Big\|_p.$$

To avoid a trivial case, we suppose throughout that the last quantity is nonzero. For $1\leq n\leq N$, let

$$S_n=\sum_{m=1}^{n}|\boldsymbol{E}_m(f_m)|\,,$$

and let $S_0 = 0$. Then

$$\begin{aligned}
\Big\| \sum_{n=1}^{N} |\boldsymbol{E}_n(f_n)| \Big\|_p^p &= \int_\Omega S_N^p \, d\mu \\
&= \int_\Omega \sum_{n=1}^{N} (S_n^p - S_{n-1}^p) \, d\mu && \text{(telescoping sum)} \\
&\le p \int_\Omega \sum_{n=1}^{N} (S_n - S_{n-1}) S_n^{p-1} \, d\mu && \text{(the mean value theorem)} \\
&= p \int_\Omega \sum_{n=1}^{N} \boldsymbol{E}_n(f_n) S_n^{p-1} \, d\mu \\
&= p \int_\Omega \sum_{n=1}^{N} f_n \boldsymbol{E}_n(S_n^{p-1}) \, d\mu && (\boldsymbol{E}_n \text{ is self-adjoint}) \\
&= p \int_\Omega \sum_{n=1}^{N} f_n S_n^{p-1} \, d\mu && (S_n^{p-1} \text{ is } \Sigma_n\text{-measurable}) \\
&\le p \int_\Omega \sum_{n=1}^{N} f_n S_N^{p-1} \, d\mu && (S_n^{p-1} \text{ is increasing in } n) \\
&\le p \Big\| \sum_{n=1}^{N} f_n \Big\|_p \|S_N^{p-1}\|_{p/(p-1)} && \text{(Hölder's inequality)} \\
&= p \Big\| \sum_{n=1}^{N} f_n \Big\|_p \Big\| \sum_{n=1}^{N} \boldsymbol{E}_n(f_n) \Big\|_p^{p-1} .
\end{aligned}$$

The result follows now upon dividing both sides by $\| \sum_{n=1}^N \boldsymbol{E}_n(f_n)\|_p^{p-1}$.

To treat the case $1 \le q \le p < \infty$, we note that

$$\Big\| \Big(\sum_{n=1}^{\infty} |\boldsymbol{E}_n(f_n)|^q \Big)^{1/q} \Big\|_p \le \Big\| \sum_{n=1}^{\infty} \boldsymbol{E}_n(|f_n|^q) \Big\|_{p/q}^{1/q} ,$$

by Jensen's inequality for conditional expectation [8, p. 33]. Now we apply the case when $q = 1$, and get

$$\Big\| \Big(\sum_{n=1}^{\infty} |\boldsymbol{E}_n(f_n)|^q \Big)^{1/q} \Big\|_p \le \Big\| \sum_{n=1}^{\infty} \boldsymbol{E}_n(|f_n|^q) \Big\|_{p/q}^{1/q} \le (p/q)^{1/q} \Big\| \sum_{n=1}^{\infty} |f_n|^q \Big\|_p .$$

The result for $1 < p \le q \le \infty$ follows by duality. We present the proof for the case $1 < p < q = \infty$. The other cases are dealt with similarly. We have

$$
\begin{aligned}
&\|\sup_{1\le n\le N}|\boldsymbol{E}_n(f_n)|\|_p\\
&\qquad= \sup\Big\{\int_\Omega \sum_{n=1}^N \boldsymbol{E}_n(f_n)g_n\,d\mu : \Big\|\sum_{n=1}^N |g_n|\Big\|_{p'}\le 1\Big\} \qquad (1/p+1/p'=1)\\
&\qquad= \sup\Big\{\int_\Omega \sum_{n=1}^N f_n\boldsymbol{E}_n(g_n)\,d\mu : \Big\|\sum_{n=1}^N |g_n|\Big\|_{p'}\le 1\Big\}\\
&\qquad\le \|\sup_{1\le n\le N}|f_n|\|_p\Big\|\sum_{n=1}^N|\boldsymbol{E}_n(g_n)|\Big\|_{p'}\\
&\qquad\le A_{p',1}\|\sup_{1\le n\le N}|f_n|\|_p\Big\|\sum_{n=1}^N|g_n|\Big\|_{p'} \qquad (\text{by the case } 1\le q\le p<\infty)\\
&\qquad\le A_{p',1}\|\sup_{1\le n\le N}|f_n|\|_p
\end{aligned}
$$

([7, Chap. IV, Section 1]). Since N is arbitrary, the desired result follows. ∎

(3.5) Second proof of Theorem (3.1). If $1<p<\infty$, and (Ω,Σ,μ), Σ_n, and $\boldsymbol{E}_n$ $(n=1,2,\ldots)$ are as in Theorem (3.1), then Doob's inequality [8, Theorem (3.4), p. 317] asserts that

$$\|\sup_{1\le n\le N}|\boldsymbol{E}_n(f)|\|_p\le A_p\|f\|_p \tag{3.6}$$

for all f in $L^p(\mu)$. The relationship between this inequality and Theorem (3.1) above is described as follows. Fix a positive integer N, and let $X_{N,p}$ be the direct sum of N copies of $L^p(\mu)$ with norm $\|(f_1,\ldots,f_N)\|_{X_{N,p}}=\|\sup_{1\le n\le N}|f_n|\|_p$. The dual space $(X_{N,p})^*$ of $X_{N,p}$ is isometrically isomorphic to $\{(g_1,\ldots,g_N) : \|\sum_{n=1}^N|g_n|\|_{p'}<\infty\}$ ([7, Chap. IV, Section 1]). Consider the linear operator $\Lambda : L^p(\mu)\to X_{N,p}$ given by $\Lambda(f)=(\boldsymbol{E}_1(f),\ldots,\boldsymbol{E}_N(f))$. In this setting, Doob's inequality implies that the operator Λ is bounded from $L^p(\mu)$ into $X_{N,p}$ with norm independent of N. It follows that its adjoint $\Lambda^* : (X_{N,p})^*\to L^{p'}(\mu)$ is also bounded with norm independent of N. But the adjoint operator is given by $\Lambda^*((g_1,\ldots,g_N))=\sum_{n=1}^N\boldsymbol{E}_n(g_n)$. So we have

$$\Big\|\sum_{n=1}^N\boldsymbol{E}_n(g_n)\Big\|_{p'}\le A_p\Big\|\sum_{n=1}^N|g_n|\Big\|_{p'}\,, \tag{3.7}$$

and since $|\boldsymbol{E}_n(g_n)|\le\boldsymbol{E}_n(|g_n|)$, we get (3.2) with $q=1$. This was the crucial part of the first proof of Theorem (3.1), as the rest of the proof followed from Jensen's inequality and duality. Hence, Theorem (3.1) is only a modest extension of Doob's inequality. The novelty is in its proof which does not use the stopping time argument. ∎

As is well-known, Doob's inequality has a weak type counterpart ([8, Theorem (3.2), p. 314]). We will next present the corresponding extension in this direction. The proof that we present is motivated by the complex interpolation method of Calderón [6]. Unlike the preceding proof this one does not follow directly from Doob's classical result. The result will not be needed in the sequel; we include it here because of its close connection to Theorem (3.1).

The following notation will be used. If f is a measurable function on Ω, we let f^* denote its decreasing rearrangement, and define the Lorentz $L_{p,q}$ quasi-norm of f by

$$\|f\|_{p,q} = \Big(\int_0^\infty (x^{1/p} f^*(x))^q \frac{dx}{x}\Big)^{1/q} \qquad (0 < p < \infty,\ 0 < q < \infty),$$

$$\|f\|_{p,q} = \sup_{x>0} x^{1/p} f^*(x) \qquad (0 < p < \infty,\ q = \infty).$$

(See [3, pp. 39, 216 ff].)

(3.8) Theorem. *Let (Ω, Σ, μ) be a probability space with an increasing sequence of sub-σ-fields $\Sigma_1 \subseteq \Sigma_2 \subseteq \ldots$, and let $\boldsymbol{E}_n$ be the conditional expectation onto Σ_n. If $1 \le q \le \infty$, then for all sequences of Σ-integrable functions $(f_n)_{n=1}^\infty$, we have*

$$\Big\|\Big(\sum_{n=1}^\infty |\boldsymbol{E}_n(f_n)|^q\Big)^{1/q}\Big\|_{1,q} \le \Big\|\Big(\sum_{n=1}^\infty |f_n|^q\Big)^{1/q}\Big\|_1 \qquad (q < \infty), \tag{3.9}$$

and

$$\|\sup_{n\ge1} |\boldsymbol{E}_n(f_n)|\|_{1,\infty} \le \|\sup_{n\ge1} |f_n|\|_1 \qquad (q = \infty). \tag{3.10}$$

Proof. The case $q = 1$ is obvious. The case $q = \infty$ follows from Doob's inequality [8, Theorem (3.2), p. 314]

$$\|\sup_{n\ge1} |\boldsymbol{E}_n(g)|\|_{1,\infty} \le \|g\|_1$$

applied to the function $g = \sup_{n\ge1} |f_n|$.

To deal with the case $1 < q < \infty$, we make use of the interpolation argument. Let

$$g = \Big(\sum_{n=1}^\infty |f_n|^q\Big)^{1/q},$$

and let $h_n = |f_n|^q/g^{q-1}$. Then $|f_n| = g^{1-s}h_n^s$, where $s = 1/q$. Let

$$G = \sup_{n\ge 1} |\boldsymbol{E}_n(g)|\,, \qquad H = \sum_{n=1}^{\infty} h_n\,.$$

Then

$$\begin{aligned} F \equiv \Big(\sum_{n=1}^{\infty} |\boldsymbol{E}_n(f_n)|^q\Big)^{1/q} &\le \Big(\sum_{n=1}^{\infty} (\boldsymbol{E}_n(g^{1-s}h_n^s))^q\Big)^{1/q} \\ &\le \Big(\sum_{n=1}^{\infty} (\boldsymbol{E}_n(g)^{q-1}\boldsymbol{E}_n(h_n))\Big)^{1/q} \le G^{1-s}H^s\,. \end{aligned}$$

The penultimate inequality follows from Hölder's inequality for conditional expectations: $\boldsymbol{E}_n(g^{1-s}h_n^s) \le \boldsymbol{E}_n(g)^{1-s}\boldsymbol{E}_n(h_n)^s$. Now, by [6, 13.4], we have $F^* \le (G^*)^{1-s}(H^*)^s$. Therefore,

$$\begin{aligned} \Big\|\Big(\sum_{n=1}^{\infty} |\boldsymbol{E}_n(f_n)|^q\Big)^{1/q}\Big\|_{1,q} &\le \|(G^*)^{1-s}(H^*)^s\|_{1,q}^q \\ &= \int_0^\infty x^{q-1}(G^*(x))^{q-1}H^*(x)\,dx \\ &\le \sup_{x>0}(xG^*(x)) \int_0^\infty H^*(x)\,dx = \|G\|_{1,\infty}^{q-1}\|H\|_1\,. \end{aligned}$$

Now, by Doob's weak type maximal inequality,

$$\|G\|_{1,\infty} \le \|g\|_1 = \Big\|\Big(\sum_{n=1}^{\infty} |f_n|^q\Big)^{1/q}\Big\|_1\,.$$

Also,

$$\|H\|_1 = \Big\|\sum_{n=1}^{\infty} h_n\Big\|_1 = \Big\|\Big(\sum_{n=1}^{\infty} |f_n|^q\Big)^{1/q}\Big\|_1^{q-1},$$

and the result follows. ∎

(3.11) Remarks. Theorems (3.1) and (3.8) can be viewed as extensions of Doob's classical inequalities from the domain of scalar-valued functions to domains of vector-valued functions. In the case of a single operator, say $\boldsymbol{E}_{n_0}$, and $q = 2$, the extension reduces to an inequality of the kind

$$\Big\|\Big(\sum_{n=1}^{\infty} |\boldsymbol{E}_{n_0}(f_n)|^2\Big)^{1/2}\Big\|_p \le A_{p,2}\Big\|\Big(\sum_{n=1}^{\infty} |f_n|^2\Big)^{1/2}\Big\|_p\,,$$

and a corresponding weak type inequality. This latter type of extensions is immediate from a theorem of Marcinkiewicz and Zygmund ([9, p. 203], and [10, Theorem (2.7), p. 484]). The extension of the Marcinkiewicz–Zygmund result to a sequence of operators fails in general, as illustrated by a simple example ([10, Examples (2.12.a)]) of a sequence of translation operators on $L^p(\mathbb{R})$. Theorems (3.1) and (3.8) thus provide interesting examples of situations in which the Marcinkiewicz–Zygmund result extends to sequences of operators. Another example of this type of extensions is provided by the vector version of M. Riesz's theorem on conjugate functions. (See [9, 6.5.2, p. 118, and Theorem (4.14) below.) However, as noted at the outset of the proof of this theorem in Edwards and Gaudry, *loc. cit.*, the result itself can be reduced to an application of the Marcinkiewicz–Zygmund to a single operator, namely, the projection of the Fourier transform on $]0,\infty]$. This reduction to a single operator is not possible in our results.

4. Littlewood–Paley theory on the solenoid. In this section, we establish the LP properties of the decompositions (2.6)–(2.8). A few more ingredients are needed for our proofs. Our next topic is the homomorphism theorem for multipliers ([4, Theorems (2.1) and (2.6)]).

If ϕ is a piecewise continuous function on $\mathbb{R}$, we will write ϕ^* for the normalized function defined on $\mathbb{R}$ by $\phi^*(x) = \phi(x)$ if ϕ is continuous at x, and $\phi^*(x) = (\phi(x-) + \phi(x+))/2$ otherwise. The function ϕ^* is normalized in the sense that if $(k_n)_{n=1}^{\infty}$ denotes any summability kernel on $\mathbb{R}$ (e.g. Fejér's kernel), then $k_n * \phi^*(x) \to \phi^*(x)$ for all x in $\mathbb{R}$, as $n \to \infty$. As a consequence of [4, Theorems (2.1) bis and (2.6)] we have the following result.

(4.1) THEOREM. *Let ϱ be a homomorphism from $\mathbb{Q}_{\boldsymbol{a}}$ into $\mathbb{R}$. Suppose that $1 < p < \infty$ and that ϕ is a normalized function on $\mathbb{R}$ which is an $L^p(\mathbb{R})$-multiplier. Denote the norm of the multiplier operator by $\|\phi\|_{M_p(\mathbb{R})}$. Then $\phi\circ\varrho$ is an $L^p(\Sigma_{\boldsymbol{a}})$-multiplier with multiplier norm $\|\phi\circ\varrho\|_{M_p(\mathbb{Q}_{\boldsymbol{a}})} \le \|\phi\|_{M_p(\mathbb{R})}$.*

The following theorem is clearly motivated by the classical LP decomposition of $\mathbb{R}$.

(4.2) THEOREM. *The dyadic decomposition of $\mathbb{Q}_{\boldsymbol{a}}$ has the LP property. That is, for each $p \in [1,\infty]$ there are constants α_p and β_p such that*

$$\alpha_p\|f\|_p \le \Big\|\Big(\sum_{j\in\mathbb{Z}} |S_{\mathcal{C}_j} f|^2\Big)^{1/2}\Big\|_p \le \beta_p\|f\|_p \tag{4.3}$$

for all $f \in L^p(\Sigma_{\boldsymbol{a}})$.

Proof. A simple approximation argument allows us to consider (and we do throughout the proof) only $f \in L^p(\Sigma_{\boldsymbol{a}}) \cap L^2(\Sigma_{\boldsymbol{a}})$. We also note that to prove (4.3) it is enough to establish the right side inequality:

$$\Big\|\Big(\sum_{j\in\mathbb{Z}} |S_{\mathcal{C}_j} f|^2\Big)^{1/2}\Big\|_p \le \beta_p\|f\|_p\,.$$

(See [9, 1.2.6, ii, p. 9].) Equivalently, we will show that the series $\sum_{j\in\mathbb{Z}} S_{\mathcal{C}_j} f$ converges unconditionally in $L^p(\Sigma_{\boldsymbol{a}})$ ([9, 1.2.9, p. 15]), which is also equivalent to the fact that any function m on $\mathbb{Q}_{\boldsymbol{a}}$ that takes values in $\{-1,1\}$ and is constant on the $\mathcal{C}_j$'s is a bounded multiplier on $L^p(\Sigma_{\boldsymbol{a}})$ with norm depending only on p.

The proof is done in two basic steps. We want to apply Theorem (4.1). However, since 1_{Δ_j} is not normalized, we are led to treat two cases separately: $\widehat{f}$ is supported on $\mathbb{Z}$; and $\widehat{f}$ is supported away from $\mathbb{Z}$. The general case follows then, since any function in $L^p(\Sigma_{\boldsymbol{a}})$ can be written as the sum of two functions of the kind that we just described.

Suppose that m is a bounded measurable function on $\mathbb{R}$, and denote its restriction to $\mathbb{Z}$ by the same symbol. Let $(\Delta_j\cap\mathbb{Z})_{j\in\mathbb{Z}}$ denote the dyadic decomposition of $\mathbb{Z}$. The strong Marcinkiewicz property of the decomposition $(\Delta_j)_{j\in\mathbb{Z}}$ (respectively, $(\Delta_j\cap\mathbb{Z})_{j\in\mathbb{Z}}$) of $\mathbb{R}$ (respectively, of $\mathbb{Z}$) asserts that, for each $p\in[1,\infty]$, there is a constant c_p depending only on p such that

$$\|m\|_{M_p(\mathbb{R})} \le c_p \sup_j \operatorname*{Var}_{\Delta_j}(m) \tag{4.5}$$

$$\text{(respectively, } \|m\|_{M_p(\mathbb{Z})} \le c_p \sup_j \operatorname*{Var}_{\Delta_j\cap\mathbb{Z}}(m)\text{)}\,. \tag{4.6}$$

(See [9, Theorems 8.2.1, 8.3.1].) Suppose that $f\in L^p(\Sigma_{\boldsymbol{a}})$ and $\widehat{f}=0$ on $\mathbb{Q}_{\boldsymbol{a}}\backslash\mathbb{Z}$. Then, by (2.3) and Fourier inversion, we have $f=f*\lambda_0$, and so f is constant on the cosets on Λ_0. Let $g\in L^1(\mathbb{T})$ be such that $f=g\circ\pi_0$. It is easy to see from (2.4) that

$$\widehat{f}(l)=\widehat{g}(l) \quad \text{for all } l\in\mathbb{Z}\,.$$

Given a bounded function m on $\mathbb{Z}$, let $B=\sup_j \operatorname{Var}_{\Delta_j\cap\mathbb{Z}}(m)$, where

$$\operatorname*{Var}_{\Delta_j\cap\mathbb{Z}}(m)=\sum_{n\in\Delta_j\cap\mathbb{Z}} |m(n+1)-m(n)|\,.$$

Using (2.4) and (4.6), we find that

$$\|(\widehat{f}m)^\vee\|_p = \|(\widehat{g}m)^\vee\|_{L^p(\mathbb{T})} \le Bc_p\|g\|_{L^p(\mathbb{T})} = Bc_p\|f\|_p\,. \tag{4.7}$$

(To avoid confusion, we will use the symbol $\|\cdot\|_{L^p(\mathbb{T})}$ to denote the L^p-norm on $\mathbb{T}$.)

Let ψ be any function on $\mathbb{Q}_{\boldsymbol{a}}$ such that ψ is constant on the dyadic intervals and ψ takes values in $\{-1,1\}$. Let ψ^* be the piecewise-linear, normalized function (in the sense of Theorem (4.1)) on $\mathbb{R}$ such that the restriction of ψ^* to the interior of Δ_j coincides with ψ on $\mathbb{Q}_{\boldsymbol{a}}$. Let f be an arbitrary function in $L^p(\Sigma_{\boldsymbol{a}})\cap L^2(\Sigma_{\boldsymbol{a}})$. Clearly,

$$\sup_j \operatorname*{Var}_{\Delta_j}(\psi^*) = \sup_j \operatorname*{Var}_{\Delta_j\cap\mathbb{Z}}(\psi) \le 2\,, \tag{4.8}$$

and

$$\begin{aligned}(\widehat{f}\psi)^\vee &= ((\widehat{f}1_{\mathbb{Q}_{\boldsymbol{a}}\setminus\mathbb{Z}})\psi^*\circ\varrho)^\vee + ((\widehat{f}1_{\mathbb{Z}})\psi)^\vee \\ &= ((f - f*\lambda_0)^\wedge\psi^*\circ\varrho)^\vee + ((f*\lambda_0)^\wedge\psi)^\vee\,,\end{aligned}$$

where ϱ is the identity homomorphism from $\mathbb{Q}_{\boldsymbol{a}}$ into $\mathbb{R}$. Using Theorem (4.1), (4.6), (4.7), and (4.8), we find that

$$\begin{aligned}\text{(4.9)}\qquad \|(\widehat{f}\psi)^\vee\|_p &\le \|((f - f*\lambda_0)^\wedge\psi^*\circ\varrho)^\vee\|_p + \|((f*\lambda_0)^\wedge\psi)^\vee\|_p \\ &\le 2c_p(\|f - f*\lambda_0\|_p + \|f*\lambda_0\|_p) \le 6c_p\|f\|_p\,.\end{aligned}$$

Since ψ is an arbitrary change of signs in the series $\sum_{j\in\mathbb{Z}} S_{\mathcal{C}_j}f$, it follows from (4.9) that the series is unconditionally convergent. ∎

We now consider the decomposition (2.7). If $f\in L^1(\Sigma_{\boldsymbol{a}})$ and $j\ge 0$, it is obvious that $S_{\mathcal{X}_j}f = f*\lambda_j$. The sequence $(f*\lambda_j)_{j\ge 0}$ is a martingale relative to the sequence of σ-algebras $(\mathfrak{B}_j)_{j\ge 0}$, where $\mathfrak{B}_j$ consists of all the Borel subsets of $\Sigma_{\boldsymbol{a}}$ of the form $A+\Lambda_j$, where $A\subseteq\Sigma_{\boldsymbol{a}}$. In fact, $f*\lambda_j$ is the conditional expectation of f relative to $\mathfrak{B}_j$. (See [9, Theorem 5.4.1].)

The fact that the decomposition $(\mathcal{D}_j)_{j=0}^\infty$ has the LP property follows from a well-known property of the martingale difference $(f*\lambda_j - f*\lambda_{j-1})_{j=1}^\infty$. (See [9, Theorem 5.3.8].)

(4.10) Theorem. *If $1<p<\infty$, there are constants A_p and B_p such that*

$$\text{(4.11)}\qquad A_p\|f\|_p \le \Big\|\Big(\sum_{j=0}^\infty |S_{\mathcal{D}_j}f|^2\Big)^{1/2}\Big\|_p \le B_p\|f\|_p$$

for all $f\in L^p(\Sigma_{\boldsymbol{a}})$.

Two more results are needed before handling the case of the decomposition (2.8). The first one is a simple application of Theorem (3.1) with $q=2$.

(4.12) Theorem. *Let p be any number in $[1,\infty]$, and let N be an arbitrary positive integer. There is a constant A_p, depending only on p, such that*

$$\text{(4.13)}\qquad \Big\|\Big(\sum_{j=0}^N |f_j*\lambda_j|^2\Big)^{1/2}\Big\|_p \le A_p\Big\|\Big(\sum_{j=0}^N |f_j|^2\Big)^{1/2}\Big\|_p$$

for all $f_0, f_1, \dots, f_N$ in $L^p(\Sigma_{\boldsymbol{a}})$.

Next we present a vector version of M. Riesz's theorem on $\Sigma_{\boldsymbol{a}}$. The proof follows the same lines as those of the proof on $\mathbb{R}$ ([9, Theorem 6.5.2]). We will briefly sketch the details.

(4.14) THEOREM. *Let (I_j) be a countable collection of open subintervals of $\mathbb{R}$. To each p in $[1,\infty]$ corresponds a number D_p such that*

$$\Big\|\Big(\sum_j |S_{I_j\cap\mathbb{Q}_{\boldsymbol{a}}} f_j|^2\Big)^{1/2}\Big\|_p \le D_p \Big\|\Big(\sum_j |f_j|^2\Big)^{1/2}\Big\|_p \tag{4.15}$$

for all $f_j \in L^p(\Sigma_{\boldsymbol{a}})$, where the constant D_p depends only on p and not on (I_j).

Proof. It is enough to consider a finite collection of trigonometric polynomials $f_1,\dots,f_N$ on $\Sigma_{\boldsymbol{a}}$. For $j=1,\dots,N$, let $\alpha_j = \min\{\chi \in I_j\cap\mathbb{Q}_{\boldsymbol{a}} : \widehat{f_\nu}(\chi)\ne 0 \text{ for some } \nu=1,\dots,N\}$, and let $\beta_j = \max\{\chi\in I_j\cap\mathbb{Q}_{\boldsymbol{a}} : \widehat{f_\nu}(\chi)\ne 0$ for some $\nu = 1,\dots,N\}$. An easy consequence of M. Riesz's theorem on $L^p(\Sigma_{\boldsymbol{a}})$ [13, Theorem (7.2)] is that the operator $S_{[-\infty,0]\,\cap\mathbb{Q}_{\boldsymbol{a}}}$ is bounded from $L^p(\Sigma_{\boldsymbol{a}})$ into $L^p(\Sigma_{\boldsymbol{a}})$. Let M_p denote the norm of the operator $S_{[-\infty,0]\,\cap\mathbb{Q}_{\boldsymbol{a}}}$. The theorem of Marcinkiewicz and Zygmund [9, p. 203] now implies that

$$\Big\|\Big(\sum_{j=1}^N |S_{[-\infty,0]\,\cap\mathbb{Q}_{\boldsymbol{a}}} g_j|^2\Big)^{1/2}\Big\|_p \le M_p \Big\|\Big(\sum_{j=1}^N |g_j|^2\Big)^{1/2}\Big\|_p \tag{4.16}$$

for all $g_j\in L^p(\Sigma_{\boldsymbol{a}})$, $j=1,\dots,N$. For each j, write $I_j=[a_j,b_j]$. We have

$$\begin{aligned} S_{I_j\cap\mathbb{Q}_{\boldsymbol{a}}} f_j &= S_{[-\infty,b_j]\,\cap\mathbb{Q}_{\boldsymbol{a}}} f_j - S_{[-\infty,a_j]\,\cap\mathbb{Q}_{\boldsymbol{a}}} f_j \\ &= \chi_{\beta_j} S_{[-\infty,0]\,\cap\mathbb{Q}_{\boldsymbol{a}}}(\overline{\chi}_{\beta_j} f_j) - \chi_{\alpha_j} S_{[-\infty,0]\,\cap\mathbb{Q}_{\boldsymbol{a}}}(\overline{\chi}_{\alpha_j} f_j)\end{aligned}$$

where we have written χ_γ for the character of $\Sigma_{\boldsymbol{a}}$ corresponding to $\gamma\in\mathbb{Q}_{\boldsymbol{a}}$. Now using (4.16), we find that

$$\begin{aligned}\Big\|\Big(\sum_{j=1}^N |S_{I_j\cap\mathbb{Q}_{\boldsymbol{a}}} f_j|^2\Big)^{1/2}\Big\|_p &\le \Big\|\Big(\sum_{j=1}^N |\chi_{\beta_j} S_{[-\infty,0]\,\cap\mathbb{Q}_{\boldsymbol{a}}}(\overline{\chi}_{\beta_j} f_j)|^2\Big)^{1/2}\Big\|_p \\ &\quad + \Big\|\Big(\sum_{j=1}^N |\chi_{\alpha_j} S_{[-\infty,0]\,\cap\mathbb{Q}_{\boldsymbol{a}}}(\overline{\chi}_{\alpha_j} f_j)|^2\Big)^{1/2}\Big\|_p \\ &\le \Big\|\Big(\sum_{j=1}^N |S_{[-\infty,0]\,\cap\mathbb{Q}_{\boldsymbol{a}}}(\overline{\chi}_{\beta_j} f_j)|^2\Big)^{1/2}\Big\|_p \\ &\quad + \Big\|\Big(\sum_{j=1}^N |S_{[-\infty,0]\,\cap\mathbb{Q}_{\boldsymbol{a}}}(\overline{\chi}_{\alpha_j} f_j)|^2\Big)^{1/2}\Big\|_p \\ &\le 2M_p \Big\|\Big(\sum_{j=1}^N |f_j|^2\Big)^{1/2}\Big\|_p .\end{aligned}$$

This establishes (4.15) for trigonometric polynomials, and by the density of these polynomials in $L^p(\Sigma_{\boldsymbol{a}})$, the theorem follows. ∎

We are now in a position to prove our main result.

(4.17) THEOREM. *Let $(\mathcal{B}_j)_{j\in\mathbb{Z}}$ be the finite-block-decomposition of $\mathbb{Q}_{\boldsymbol{a}}$ given in* (2.8). *Then the decomposition $(\mathcal{B}_j)_{j\in\mathbb{Z}}$ has the LP property.*

Proof. Let $f \in L^p(\Sigma_{\boldsymbol{a}}) \cap L^2(\Sigma_{\boldsymbol{a}})$, and let N be an arbitrary positive integer. By taking Fourier transforms, we can easily show that, for every positive integer j, the following equalities hold a.e. on $\Sigma_{\boldsymbol{a}}$:

$$S_{\mathcal{B}_j} f = S_{[0,2^j]\cap\mathbb{Q}_{\boldsymbol{a}}}(f*\lambda_j - f*\lambda_{j-1}) + S_{[2^j,2^{j+1}]\cap\mathbb{Q}_{\boldsymbol{a}}}(f*\lambda_j) \tag{4.18}$$

and

$$S_{[2^j,2^{j+1}]\cap\mathbb{Q}_{\boldsymbol{a}}}(f*\lambda_j) = (S_{[2^j,2^{j+1}]\cap\mathbb{Q}_{\boldsymbol{a}}} f)*\lambda_j\,. \tag{4.19}$$

Now recall that multipliers commute with each other, and, in particular, with convolution. Use Theorems (4.14), (4.10), (4.12), and get

$$\begin{aligned}\Big\|\Big(\sum_{j=1}^N |S_{\mathcal{B}_j} f|^2\Big)^{1/2}\Big\|_p &\le \Big\|\Big(\sum_{j=1}^N |S_{[0,2^j]\cap\mathbb{Q}_{\boldsymbol{a}}}(f*\lambda_j - f*\lambda_{j-1})|^2\Big)^{1/2}\Big\|_p \\ &\quad + \Big\|\Big(\sum_{j=1}^N |(S_{[2^j,2^{j+1}]\cap\mathbb{Q}_{\boldsymbol{a}}} f)*\lambda_j|^2\Big)^{1/2}\Big\|_p \\ &\le D_p\Big\|\Big(\sum_{j=1}^N |(f*\lambda_j - f*\lambda_{j-1})|^2\Big)^{1/2}\Big\|_p \\ &\quad + A_p\Big\|\Big(\sum_{j=1}^N |S_{[2^j,2^{j+1}]\cap\mathbb{Q}_{\boldsymbol{a}}} f|^2\Big)^{1/2}\Big\|_p \\ &\le (D_pB_p + A_p\beta_p)\|f\|_p\,.\end{aligned}$$

Since N is arbitrary, this shows that

$$\Big\|\Big(\sum_{j=1}^\infty |S_{\mathcal{B}_j} f|^2\Big)^{1/2}\Big\|_p \le \gamma_p\|f\|_p$$

where γ_p depends only on p. A similar argument applies to $j \le 0$ and completes the proof of the theorem. ∎

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